Modular forms

S. Bhat, A. Guadagnini

5.12.2023

1 Ingredients of Modular Forms

1.1 Modular Group and the Upper Half Plane

In the following, we consider the upper half plane of the complex number plane

$$\mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$$

The special linear group of 2×2 matrices with integer entries

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^2 \mid \det(M) = 1 \right\}$$

acts on the upper half plane by means of fractional linear transformations (or Mobius transformations)

$$z \mapsto Mz := \frac{az+b}{cz+d}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$$\Im(Mz) = \frac{\Im(z)}{|cz+d|^2},$$

so the action is well-defined. Also, one quickly computes that $Iz = z$, where I denotes the identity matrix, and that $M(Nz) = (MN)z$ holds for $M, N \in \mathrm{SL}_2(\mathbb{Z})$. The group $\mathrm{SL}_2(\mathbb{Z})$ is also called the modular group.

Examples :

- The matrix $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ acts by translation $Tz = z + 1$.
- The matrix $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ acts by inversion on the unit circle $Sz = -\frac{1}{z}$.

Note : The modular group $\mathrm{SL}_2(\mathbb{Z})$ can be generated by the matrices S and T. A proof of this statement can be found in [3].

1.2 Fundamental Domain

We now describe a fundamental domain for the operation of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half-plane. This is a subset $F \subset \mathbb{H}$ such that no two distinct points from F are equivalent under the operation of $\mathrm{SL}_2(\mathbb{Z})$ and under the operation of $\mathrm{SL}_2(\mathbb{Z})$ every point $z \in \mathbb{H}$ is equivalent to a point in F.

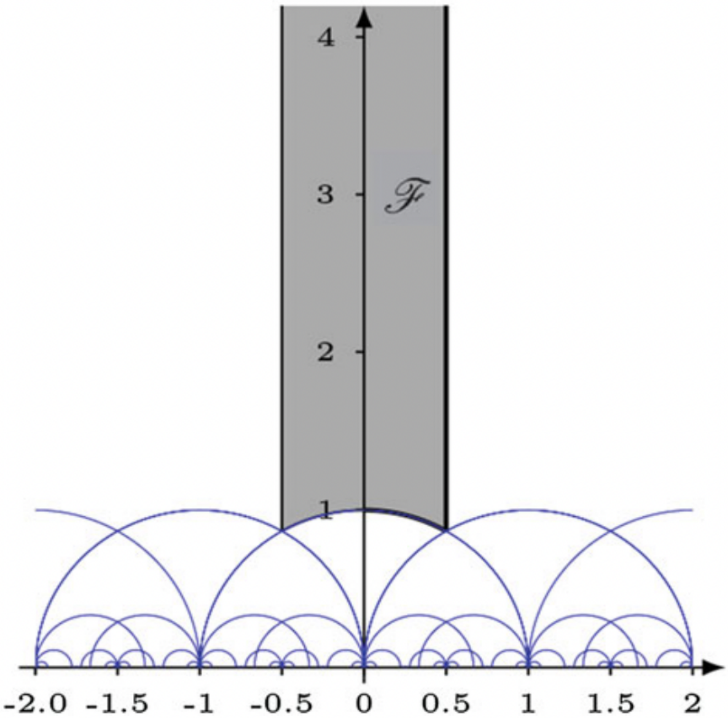
Theorem 1.1. *The set*

$$F = \left\{ z \in \mathbb{H} \middle| \Re(z) \in \left(-\frac{1}{2}, \frac{1}{2} \right], \middle| |z| \geq 1, |z| > 1 \textit{ for } \Re(z) \in \left(-\frac{1}{2}, 0 \right] \right\}$$

is a fundamental domain for the operation of $\mathrm{SL}_2(\mathbb{Z})$ *on the upper half-plane.*

Proof. We first show that for every $z \in \mathbb{H}$ there exists a matrix $M \in \mathrm{SL}_2(\mathbb{Z})$ such that Mz lies in the set

$$F = \left\{ z \in \mathbb{H} || z | \geq 1, \Re(z) \in \left[-\frac{1}{2}, \frac{1}{2} \right] \right\}$$



The lattice $\mathbb{Z}z + \mathbb{Z}$ is discrete in $\mathbb{C}$. So there is a pair $(c, d) \in \mathbb{Z}^2 \backslash \{(0,0)\}$ such that

$$|mz + n| \geq |cz + d| \quad \text{for all } (m, n) \in \mathbb{Z}^2 \backslash \{(0,0)\}$$

If we choose $M_0 = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, then it holds that $\frac{1}{|mz+n|^2} \leq \frac{1}{|cz+d|^2}$ and thus

$$\Im(Mz) \leq \Im(M_0 z) \;\text{ for all } M \in SL_2(\mathbb{Z})$$

We set $z_0 := M_0 z$ and consider $z_0 + n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} z_0$ with $n \in \mathbb{Z}$. By choosing n appropriately, we can assume that $|\Re(z_0)| \leq \frac{1}{2}$, since this does not change the imaginary part of z_0.

If we choose $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} M_0$ in (3.1), it follows that.

$$\Im(z_0) \geq \Im\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z_0\right) = \frac{\Im(z_0)}{|z_0|^2},$$

so $|z_0|^2 \geq 1$. We now note that the points $z \in \mathbb{H}$ with $\Re(z) = \pm\frac{1}{2}$ are equivalent to each other by the action of the matrix $T(z \mapsto z+1)$. The points $z \in \mathbb{H}$ on the left and right half of the arc $|z| = 1$ are transformed into each other via $S\left(z \mapsto -\frac{1}{z}\right)$.

$\square$

2 Vector Space of Modular Forms

Definition 2.1. *Let $k \in \mathbb{Z}$. A function $f : \mathbb{H} \to \mathbb{C}$ is called modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ if the following conditions are satisfied:*

- *f is holomorphic on $\mathbb{H}$;*
- *We have*

$$f(Mz) = (cz+d)^k f(z) \text{ for all } z \in \mathbb{H} \text{ and } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

- *f is holomorphic in ∞.*

If f vanishes in ∞, then f is called a cusp form of weight k for $\mathrm{SL}_2(\mathbb{Z})$.

We now discuss in detail, the last point in the definition, the holomorphicity in ∞. In doing so, first note that invariance under the matrix T implies the 1-periodicity of a modular form f. The mapping $\mathbb{H} \to \{q \in \mathbb{C} : 0 < |q| < 1\}, z \mapsto e^{2\pi i z} =: q$ is surjective and holomorphic. Because of the 1-periodicity of f, the function $g(q) = f\left(\frac{\log q}{2\pi i}\right)$ is well-defined and holomorphic for $0 < |q| < 1$. So g has a Laurent expansion $g(q) = \sum_{n\in\mathbb{Z}} a(n)q^n$. Now the fact that f is holomorphic in ∞ means that $g(q) = \sum_{n=0}^{\infty} a(n)q^n$. Thus f has a Fourier expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a(n)e^{2\pi i n z} \text{ where } a(n) \in \mathbb{C}$$

If f is a cusp form, then $a(0) = 0$. Further, $a(n) = \int_0^1 f(x+iy)e^{-2\pi i n(x+iy)}dx$, where $y > 0$ is arbitrary.

Remark 2.1. *We can make the following remarks :*

- *Let $k \in \mathbb{Z}$ be odd. Then it holds*

$$f(z) = f((-I)z) = (-1)^k f(z) = -f(z),$$

so there are no modular forms of odd weight for $\mathrm{SL}_2(\mathbb{Z})$.

- *Because of* $\mathrm{SL}_2(\mathbb{Z}) = \langle \mathrm{S}, T\rangle$*, it suffices to check the transformation property, for the matrices* S *and* T*.*
- *Let* f *be a modular form of weight* k *for* $\mathrm{SL}_2(\mathbb{Z})$ *and* g *be a modular form of weight* ℓ *for* $\mathrm{SL}_2(\mathbb{Z})$*. Then,* $f \cdot g$ *then has weight* $k + \ell$*.*

Definition 2.2. *For* $k \in \mathbb{Z}$*, let* M_k *denote the* $\mathbb{C}$*-vector space of all modular forms of weight* k *for* $\mathrm{SL}_2(\mathbb{Z})$ *and* S_k *denote the* $\mathbb{C}$*-vector space of all cusp forms of weight* k *for* $\mathrm{SL}_2(\mathbb{Z})$*.*

In the following, we determine the dimension of these vector spaces in some cases. First, we note that $M_k = \{0\}$ holds for $k < 0$. First, we derive the so-called $\frac{k}{12}$-formula.

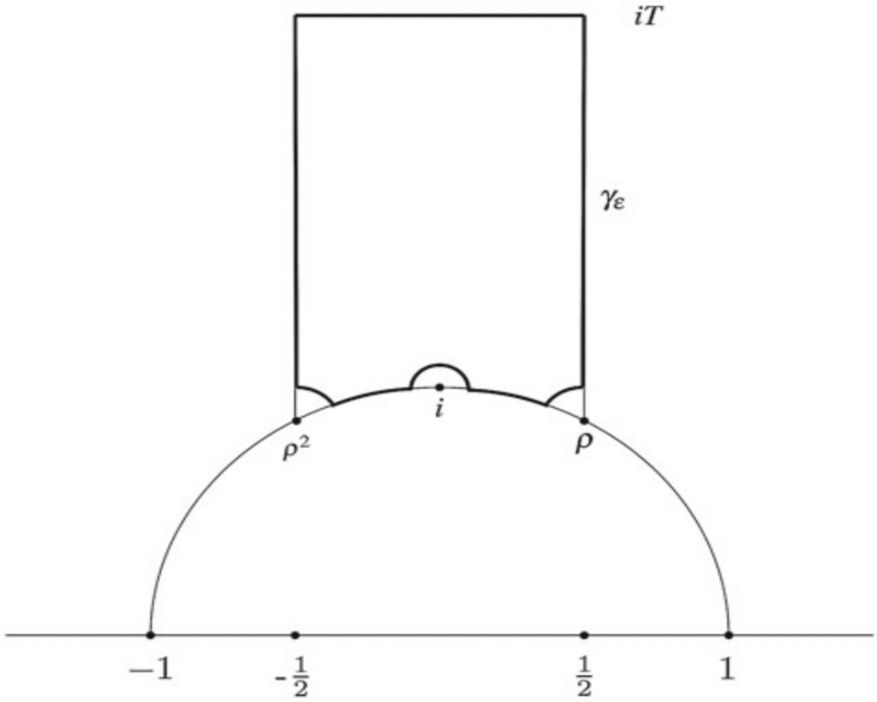
Theorem 2.1. *Let* $k \in \mathbb{Z}$ *and* $f \in M_k, f \not\equiv 0$*, be a nontrivial modular form of weight* k *for* $\mathrm{SL}_2(\mathbb{Z})$*. Then it holds:*

$$\frac{1}{2}\operatorname{ord}_i(f) + \frac{1}{3}\operatorname{ord}_\rho(f) + \operatorname{ord}_\infty(f) + \sum_{\substack{z \\ z \neq \rho, i}} \operatorname{ord}_z(f) = \frac{k}{12}$$

with $\rho = e^{\frac{\pi i}{3}}$*. Here we set*

$$\operatorname{ord}_\infty(f) = \min\{n : a(n) \neq 0\}, \text{ where } f(z) = \sum_{n=0}^{\infty} a(n) q^n$$

Proof. We give a brief sketch of the proof.



One integrates the function $h(z) = \frac{f'(z)}{f(z)}$ along the path γ_ε shown in the figure, which runs counterclockwise along the boundary of the fundamental domain cut off at height $\Im(z) = T$, and uses the residue theorem from complex analysis. Here, one chooses ε such that γ_ε includes all zeros and poles of f except ρ and i in F. Then it holds

$$\frac{1}{2\pi i} \int_{\gamma_\varepsilon} h(z)dz = \sum_{\substack{z\in F\\ z\neq \rho,i}} \operatorname{ord}_z f.$$

The integrals over the individual segments of the path then provide the corresponding contributions in the formula (after letting $\varepsilon \to 0$).

□

With the help of simple considerations one can conclude the following statements from the $\frac{k}{12}$-formula.

Corollary 2.1.1. *It holds:*

- $M_k = \{0\}, k < 0$;
- $M_0 = \mathbb{C}, S_0 = \{0\}$;
- $M_2 = \{0\}$;
- $\dim M_k \leq 1$ *and* $S_k = \{0\}$ *for* $k = 4, 6, 8, 10$.

3 Building Modular Forms and Examples

3.1 Eisenstein Series

We want to introduce the Eisenstein series, which can be seen as building blocks for modular forms. As explained earlier by analyzing the Weierstrass $\wp$-function we are led to the Eisenstein series. The original definition of this depends on a lattice; however, as we have seen, we can normalize lattices in a certain way such that the Eisenstein series becomes a function on a parameter $z \in \mathbb{H}$.

Definition 3.1. *Let* $k \in \mathbb{Z}$ *with* $k > 2$. *We define the Eisenstein series of weight* k *for* $SL_2(\mathbb{Z})$ *by*

$$G_k(z) = \sum_{\substack{c,d\in\mathbb{Z}\\ (c,d)\neq(0,0)}} \frac{1}{(cz+d)^k},\ z \in \mathbb{H}.$$

Remark 3.1. *For odd* k, *the terms* $(cz+d)^k$ *and* $(-cz-d)^k$ *cancel each other out in the sum, so that in this case it holds* $G_k \equiv 0$.

We now demonstrate that Eisenstein series for $k > 2$ are modular forms of weight k for $SL_2(\mathbb{Z})$. Holomorphicity on $\mathbb{H}$ follows from the convergence of the Eisenstein series. Further, we show the transformation property for the matrices S and T, and hence all of $SL_2(\mathbb{Z})$.

$$G_k(Tz) = G_k(z+1) = \sum_{\substack{c,d\in\mathbb{Z}\\ (c,d)\neq(0,0)}} \frac{1}{(cz+c+d)^k} = G_k(z) = (0z+1)^k G_k(z),$$

where for the third equality we performed an index shift. It also holds

$$\begin{aligned} G_k(Sz) = G_k(-\frac{1}{z}) &= \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)\neq(0,0)}} \frac{1}{(c(-\frac{1}{z})+d)^k} \\ &= z^k \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)\neq(0,0)}} \frac{1}{(dz-c)^k} = (1z+0)^k G_k(z). \end{aligned}$$

The following statement about the Fourier expansion of the Eisenstein series implies the holomorphicity in ∞.

Lemma 3.1. *Let $k > 2$ be even. The Eisenstein series G_k has the Fourier expansion*

$$G_k(z) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where $q = e^{2\pi i z}$, $\sigma_l(n) = \sum_{0<d|n} d^l$ denotes the divisor sum function, and $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ denotes the Riemann zeta function.

In particular, G_k has a Fourier expansion which has no terms with a negative index, hence is holomorphic in ∞. Thus, we have shown $G_k \in M_k$.

Definition 3.2. *For $k \in \mathbb{Z}$ even and $k \geq 2$, the normalized Eisenstein series is defined as*

$$E_k(z) = \frac{1}{2\zeta(k)} G_k(z) = 1 + \frac{(2\pi i)^k}{\zeta(k)(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

Using Euler's formula, one can show that the normalized Eisenstein series has Fourier expansion

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where B_k denotes the k-th Bernoulli number defined by $\frac{t}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n$.

Example 3.1. *We have*

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n = 1 + 240q + 2160q^2 + 6720q^3 + \ldots.$$

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n = 1 - 504q - 16632q^2 - 122976q^3 - \ldots.$$

A simple conclusion from Corollary 2.1.1 is now that

$$M_k = \mathbb{C} E_k$$

for $k = 4, 6, 8, 10$.

Example 3.2. *It holds*

$$E_4(z)^2 = E_8(z),$$

since the dimension of M_8 is 1 and both $E_4(z)^2$ and $E_8(z)$ have leading coefficient 1, so they must agree.

3.2 The Delta Function

Let us now construct a first example of a non-trivial cusp form.

Definition 3.3. *The* Δ*-function or discriminant is defined by*

$$\Delta(z) = \frac{E_4^3(z) - E_6^2(z)}{1728} = q - 24q^2 + 252q^3 + \dots.$$

Note that Δ is a cusp form of weight 12. By multiplying the Fourier expansions, we can see that E_4^3 and E_6^2 are both modular forms of weight 12 that have constant term 1. Therefore, $E_4^3 - E_6^2 \in S_{12}$. The term 1728 normalizes Δ so that the Fourier expansion begins with q. In particular, Δ is not the zero function.

Remark 3.2. *The* Δ*-function has the product expansion*

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

Lemma 3.2. *The* Δ*-function has no zeros on* $\mathbb{H}$*. In particular the map*

$$M_k \to S_{k+12},\ f \mapsto \Delta \cdot f$$

is a $\mathbb{C}$*-vector space isomorphism.*

Proof. For $k = 12$, the right-hand side of the $\frac{k}{12}$-formula is equal to 1. Moreover, $ord_\infty(\Delta) = 1$. Since $ord_z(\Delta) \geq 0$ for all $z \in \mathbb{H}$, it follows from the $\frac{k}{12}$-formula that $ord_z(\Delta) = 0$ for all $z \in \mathbb{H}$, hence Δ has no zeros on $\mathbb{H}$. In particular, the map

$$S_{k+12} \to M_k,\ g \mapsto \frac{g}{\Delta}$$

is well-defined and yields the inverse mapping of $f \mapsto \Delta \cdot f$. □

Observing that $S_k = \Delta \cdot M_{k-12}$ and using the $\frac{k}{12}$-formula, the following dimension formulas for even k follow by induction.

Theorem 3.3. *The spaces* M_k *of modular forms for* k *even are finite-dimensional with dimensions given by*

$$dimM_k = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1, & \textit{if } k \not\equiv 2 \ (\textit{mod } 12), \\ \lfloor \frac{k}{12} \rfloor, & \textit{if } k \equiv 2 \ (\textit{mod } 12). \end{cases}$$

Proof. We proceed by induction on k. According to Corollary 2.1.1, this dimension formulas applies for $0 \leq k < 12$. For $k = 12$, the formula holds by invoking (without proof) the identity $M_{12} = \mathbb{C}E_{12} \oplus \mathbb{C}\Delta$. Let $k > 12$. Employing (without proof) the result $\dim M_k = 1 + \dim S_k$ for $k \geq 4$ along with Lemma 3.2, we get

$$\begin{aligned} \dim M_k &= 1 + \dim S_k \\ &= 1 + \dim M_{k-12} \\ &= 1 + \begin{cases} \lfloor \frac{k-12}{12} \rfloor + 1, & \text{if } k - 12 \not\equiv 2 \text{ (mod 12)}, \\ \lfloor \frac{k-12}{12} \rfloor, & \text{if } k - 12 \equiv 2 \text{ (mod 12)}, \end{cases} \\ &= \begin{cases} \lfloor \frac{k}{12} \rfloor + 1, & \text{if } k \not\equiv 2 \text{ (mod 12)}, \\ \lfloor \frac{k}{12} \rfloor, & \text{if } k \equiv 2 \text{ (mod 12)}. \end{cases} \end{aligned}$$

We obtain the claimed formula. □

We are now in a position to give an explicit basis for the space of modular forms of fixed weight.

Theorem 3.4. *Let $k \in \mathbb{Z}$ and let $a, b \in \mathbb{N}_0$ with $4a+6b = k$. The modular forms $E_4^a \cdot E_6^b$ form a basis of the space of modular forms of weight k for $SL_2(\mathbb{Z})$.*

Proof. We prove this statement by induction on the weight k. We have already treated the cases $k \leq 20$ and $k = 14$. Let $a, b \in \mathbb{N}_0$ such that $4a + 6b = k$. Since $g := E_4^a E_6^b$ has constant term equal to 1, g is not a cusp form. Let now $f \in M_k$ be arbitrary. Since $g(\infty) = 1 \neq 0$, there exists an $\alpha \in \mathbb{C}$ such that $f - \alpha g \in S_k$. As we have seen in Lemma 3.2, $f \mapsto \Delta \cdot f$ is an isomorphism and hence there exists an $h \in M_{k-12}$ such that $h\Delta = f - \alpha g$. By induction hypothesis, h is a polynomial in E_4 and E_6 with the corresponding powers. By definition, this is also true for Δ, so it is also true for f. By elementary considerations it follows that there are no non-trivial relations between the products. $\square$

3.3 The j-invariant

In this section, we consider so-called modular functions, modular forms of weight zero. Since there is no holomorphic functions on compact Riemann surfaces (like $SL_2(\mathbb{Z}) \setminus \mathbb{H} \cup \{\infty\}$), we have to admit poles of finite order in ∞.

Definition 3.4. *We define the j-invariant or absolute invariant by*

$$j(z) = \frac{E_4^3(z)}{\Delta(z)}, z \in \mathbb{H}.$$

Remark 3.3.
1. *From the $\frac{k}{12}$-formula it follows that Δ has no zeros on $\mathbb{H}$, hence j is holomorphic on $\mathbb{H}$.*
2. *The j-invariant has weight 0, so $j(Mz) = j(z)$ for all $M \in SL_2(\mathbb{Z})$.*
3. *From the Fourier expansions of E_4^3 and Δ we get the Fourier expansion of j:*

$$\begin{aligned} j(z) &= q^{-1} + 744 + 196.884q + \dots \\ &= q^{-1} + \sum_{m=0}^{\infty} j_m q^m, \end{aligned}$$

that is, j has a pole of order 1 in ∞.

In particular, j is not a modular form in the sense of our original definition, since the Fourier expansion has terms with negative q-exponents. This gives rise to the following generalization of the definition of a modular form.

Definition 3.5. *Let $k \in \mathbb{Z}$. A function $f : \mathbb{H} \to \mathbb{C}$ is called weakly holomorphic modular form of weight k for $SL_2(\mathbb{Z})$, if the following conditions are satisfied:*

1. *f is holomorphic on $\mathbb{H}$;*
2. *The following holds*

$$f(Mz) = (cz + d)^k f(z) \text{ for all } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), z \in \mathbb{H};$$

3. The Fourier expansion of f in ∞ is of the form

$$f(z) = \sum_{n=m}^{\infty} a(n)q^n$$

with $m \in \mathbb{Z}$. That is, f has a pole of finite order in ∞.

We denote the space of weakly holomorphic modular forms of weight k for $SL_2(\mathbb{Z})$ by $M_k^!$.

The next theorem states that the space of modular functions, the weakly holomorphic modular forms of weight 0, is given by complex polynomials in the j-invariant.

Theorem 3.5. *It holds that $M_0^! = \mathbb{C}[j]$.*

4 L-functions of Modular Forms

Let us first recall some important results about the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}, s \in \mathbb{C}, \Re(s) > 1.$$

This series converges normally (locally uniform and absolute) and the function defined by it is holomorphic. The function $\zeta(s) - \frac{1}{s-1}$ has a holomorphic continuation to $\mathbb{C}$.

Theorem 4.1. *The Riemann zeta function has a meromorphic continuation to $\mathbb{C}$. The completed Riemann zeta function $\zeta^*(s) := \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ satisfies the functional equation*

$$\zeta^*(s) = \zeta^*(1-s).$$

Here $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$ is for $\Re(s) > 0$ denotes the gamma function.

Analogous to the Riemann zeta function, one can consider L-functions of cusp forms.

Definition 4.1. *Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n, q = e^{2\pi i z}$, be a cusp form of weight $k > 0$ for $SL_2(\mathbb{Z})$. We call*

$$L(f,s) = \sum_{n=1}^{\infty} a(n)n^{-s}, s \in \mathbb{C},$$

the L-function of f.

Remark 4.1. *One can show that $L(f,s)$ converges absolutely and locally uniformly for $s \in \mathbb{C}$ with $\Re(s) > \frac{k}{2} + 1$.*

Theorem 4.2. *Let $f \in S_k$. Then the associated L-function has a holomorphic continuation to all $\mathbb{C}$. The function*

$$\Lambda(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s)$$

is entire and satisfies the functional equation

$$\Lambda(f,s) = (-1)^{\frac{k}{2}}\Lambda(f,k-s).$$

The function $\Lambda(f,s)$ is bounded on every vertical strip, that is, for every $T > 0$ there exists a $C_T > 0$ such that $|\Lambda(f,s)| \leq C_T$ for all $s \in \mathbb{C}$ with $\Re(s) \leq T$.

References

[1] C. Alfes-Neumann, *Modular Forms*, Springer (2021).

[2] M. Schwagenscheidt, *Vorlesung Modulformen*, lecture notes, Universität Köln (2021).

[3] Koecher M., Kreig A., *Elliptische Funktionen und Modularformen*, Springer Berlin (2007)