Elliptic Functions

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1 Meromorphic Functions and Lattices

First of all, let's give a short review about the meromorphic functions, which we have learned in the complex analysis.

Definition 1.1. A function f is called meromorphic on $\mathbb{C}$, when there exists a closed and discrete subset D_f, such that

(i) f: $\mathbb{C} \setminus D_f \to \mathbb{C}$ is holomorphic and

(ii) the points in D_f are poles.

A closed subset D in $\mathbb{C}$ is called discrete if for every $c \in \mathbb{C}$ there exists a neighborhood U of c, such that $D \cap U$ is finite. In other words, it holds that the set $\{z \in D : |z| \leq \rho\}$ is finite for every $\rho > 0$.

We denote the class of all the meromorphic functions on $\mathbb{C}$ by $\mathcal{M}$. It's easy to check that for any meromorphic functions f, g and h, $f + g$ and $f \cdot g$ are also meromorphic. Clearly we have $(f(g + h) = fg + fh$ and $f + 0 = f$, $f \cdot 1 = f$. Regarding the poles, it holds that

$$D_{\alpha f} = D_f(\text{for } \alpha \neq 0), D_{f+g} \subset D_f \cup D_g, D_{fg} \subset D_f \cap D_g$$

. With the identity theorem we know that the zeros of a function $0 \neq f \in \mathcal{M}$ is closed and discrete in $\mathbb{C}$. Thus, $1/f$ is meromorphic as well. As a result, we have the following theorem:

Theorem 1.2. *The class of meromorphic functions on* $\mathbb{C}$ *forms a field over* $\mathbb{C}$.

Now let's turn our eyes to a special type of meromorphic function which is periodic. For $\omega \in \mathbb{C}$ and $D \subset \mathbb{C}$ we write $D + \omega$ of $\mathbb{C}$ as

$$D + \omega := \{d + \omega : d \in D\}$$

Definition 1.3. Let f be a meromorphic function on $\mathbb{C}$. $\omega \in \mathbb{C}$ is called period of f, when the following conditions are satisfied:

(P.1) $D_f + \omega = D_f$ and

(P.2) $f(z + \omega) = f(z)$ for every $z \in \mathbb{C} \setminus D_f$.

Obviously 0 is a period of any meromorphic function f. We denote the set of all the periods of f by $Perf$. In case that f is a constant meromorphic function, we have $Perf = \mathbb{C}$. The following lemma describes the structural property of $Perf$ in the case otherwise:

Lemma 1.4. Let $f \in \mathcal{M}$ be non-constant, then $Perf$ is a closed, discrete subgroup of the additive group $(\mathbb{C}, +)$.

Proof. Assuming that $Perf$ is not discrete or abgeschlossen, then there exits a sequence of distinct $\omega_n \in Perf$ such that $\omega := \lim_{n\to\infty} \omega_n$ exists. Because of $D_f + \omega = D_f$, for any function f, which is holomorphic at c, we know that f is also holomorphic at $c + \omega$. And it holds that $f(c) = f(c + \omega_n)$ for every n. With the identity theorem, we conclude that f is constant which leads to a contradiction. □

The following lemma describes what $Perf$ looks like.

Lemma 1.5 (Fundamental-Lemma)**.** Let $f \in \mathcal{M}$ be non-constant, then exactly one of the following cases holds:

1. $Perf = 0$.
2. There is a uniquely determined, up to sign, $\omega_f \in \mathbb{C} \setminus \{0\}$ such that $Perf = \mathbb{Z}\omega_f := \{m\omega_f : m \in \mathbb{Z}\}$.
3. There exist $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ with the following properties:
   - (i) ω_1 and ω_2 are linearly independent over $\mathbb{R}$.
   - (ii) $Perf = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 := \{m_1\omega_1 + m_2\omega_2 : m_1, m_2 \in \mathbb{Z}\}$.
   - (iii) $\tau := \omega_1/\omega_2$ satisfies $\operatorname{Im}\tau > 0$, $|\operatorname{Re}\tau| \leq \frac{1}{2}$ and $|\tau| \geq 1$.

Obviously, $\omega_1, \omega_2 \in \mathbb{C} \setminus \{1\}$ are linearly independent over $\mathbb{R}$ if and only if ω_1/ω_2 is not real.

Proof. [1] pp. 13-14. □

To further investigate the third case above, we introduce the concept of lattices.

Definition 1.6. Let V be an n-dimensional $\mathbb{R}$-vector space with $n \geq 1$. A lattice of in V is a subset of the form $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + ... + \mathbb{Z}\omega_n$ with linearly independent vectors $\omega_1, ..., \omega_n$ of V. The tupel $(\omega_1, \omega_2, ..., \omega_n)$ is called a basis of Ω.

When $Perf$ satisfies the case 3. in the fundamental-lemma, then $Perf$ is a lattice in the $\mathbb{R}$-vector space $\mathbb{C}$.

Proposition 1.7. Any lattice Ω in $\mathbb{C}$ is closed and discrete in $\mathbb{C}$.

Proof. Let ρ be an arbitrary real number and $M := \{\omega \in \Omega : |\omega| \leq \rho\}$. With a normalization, we assume that $\Omega = \mathbb{Z}\tau + \mathbb{Z}, \operatorname{Im}(\tau) > 0$ without loss of generality([3],pp.262 and [1], pp. 14). Write $\tau = x + iy$ and consider $\omega = m\tau + n \in M$ with $m, n \in \mathbb{Z}$, then we have

$$\tau^2 \geq |m\tau + n|^2 = (mx + n)^2 + m^2y^2 \geq m^2y^2,$$

which leads to $|m| \leq \rho/y$. On the other side,

$$\rho \geq |mx + n| \geq |n| - |mx|,$$

which leads to $|n| \leq \rho(1 + |x|/y)$. Therefore M is finite. □

Now let's take a closer look at the basis of a lattice in $\mathbb{C}$.

Lemma 1.8. Let Ω be a lattice in $\mathbb{C}$ and (ω_1, ω_2) a basis of Ω. For any $\omega_1', \omega_2' \in \mathbb{C}$ it holds that:

1. $\omega_1', \omega_2' \in \Omega$ if and only if there exists $U \in Mat(2, \mathbb{Z})$ such that

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = U \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

2. (ω_1', ω_2') is a basis of Ω, if and only if $U \in GL(2, \mathbb{Z})$.

Definition 1.9. Let Ω be a lattice in $\mathbb{C}$ and (ω_1, ω_2) a basis of Ω. For $u \in \mathbb{C}$ the period parallelogram with respect to ω_1, ω_2 with basis point u is defined as

$$\diamond(u; \omega_1, \omega_2) \coloneqq \{u + \alpha_1\omega_1 + \alpha_2\omega_2 : 0 \leq \alpha_1 < 1, 0 \leq \alpha_2 < 1\}.$$

In the case that $u = 0$ we write

$$\diamond(\omega_1, \omega_2) \coloneqq \diamond(u; \omega_1, \omega_2) = \{\alpha_1\omega_1 + \alpha_2\omega_2 : 0 \leq \alpha_1 < 1, 0 \leq \alpha_2 < 1\}$$

and $\diamond(\omega_1, \omega_2)$ is called the fundamental mesh of the lattice.

There are many period parallelograms in every lattice, and every period parallelogram $P \coloneqq \diamond(u; \omega_1, \omega_2)$ is a fundamental region of $\mathbb{C}$ with respect to Ω in the sense of following proposition:

Proposition 1.10. For any $z \in \mathbb{C}$ there exists exactly an $\omega \in \Omega$ with $z + \omega \in P$. If $z, z + \omega \in P$, then it holds that $\omega = 0$.

Proof. Let u be the basis point of the period parallelogram P. Then for some $\xi_1, \xi_2 \in \mathbb{R}$, it hold that $z = u + \xi_1\omega_1 + \xi_2\omega_2$. We write $\xi_1 = \lfloor \xi_1 \rfloor + \alpha_1$ and $\xi_2 = \lfloor \xi_2 \rfloor + \alpha_2$ where $\lfloor \xi_1 \rfloor, \lfloor \xi_2 \rfloor \in \mathbb{Z}$ and $0 \leq \alpha_1, \alpha_2 < 1$. $\square$

2 Elliptic Functions

Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$.

Definition 2.1. A meromorphic function f on $\mathbb{C}$ is called elliptic or doubly periodic with respect to Ω, if it holds that $\Omega \subset Perf$. In other words, the following conditions hold:

1. $D_f + \omega = D_f$ for any $\omega \in \Omega$,
2. $f(z + \omega) = f(z)$ for any $\omega \in \Omega$ and $z \in \mathbb{C} \setminus D_f$.

We denote the set of all the elliptic functions with respect to the lattice Ω by $\mathcal{K}(\Omega)$.

Proposition 2.2. The elliptic functions $\mathcal{K}(\Omega)$ with respect to Ω forms a subfield of the field $\mathcal{M}$ of all the meromorphic functions on $\mathbb{C}$, which includes the constant functions. Any $f \in \mathcal{K}(\Omega)$ has only finitely many poles in every period parallelogram.

We ignore the proof of this proposition. Instead, we make two remarks regarding the properties of elliptic functions.

Remark 2.3. *Let Ω be a given lattice. Any two complex numbers z and ω are congruent modulo Ω if $z - w \in \Omega$. From Proposition 1.10 we conclude that every point in $\mathbb{C}$ is congruent to a unique point in a given period parallelogram. Hence, f is uniquely determined by its behavior on any period parallelogram.*

Remark 2.4. *From complex analysis we know that, a non-zero function $f \in \mathcal{M}$ if and only if for any $c \in \mathbb{C}$ there exists an $n \in \mathbb{Z}$, a neighborhood U of c and a holomorphic function $g : U \to \mathbb{C}$ such that*

$$f(z) = (z - c)^n \cdot g(z) \qquad \text{for any } z \in U \setminus \{c\} \text{ and } g(c) \neq 0.$$

We call n the order of f at c, denoted by $ord_c f$. We write the Laurent series expansion at c as

$$f(z) = \sum_{i \geq n} a_i (z - c)^i, a^n \neq 0, n \in \mathbb{Z}.$$

Here n is the order of f at c and the residue of f at c is defined as $res_c f := a_{-1}$. Furthermore, for $f \in \mathcal{K}(\Omega)$, ω in the lattice Ω and z in a proper neighborhood of $c + \omega$ we have

$$f(z) = f(z - \omega) = \sum_{i \geq n} a_i (z - \omega - c)^i = \sum_{i \geq n} a_i (z - (c + \omega))^i$$

and it follows that

$$ord_c + \omega f = ord_c f \quad \text{and} \quad res_c + \omega f = res_c f.$$

In particular, for any $\omega \in \Omega$ and c is a pole of f, then $c + \omega$ is also a pole of f.

In 1847, J. Liouville found several important results regarding the elliptic functions. Here we present 4 important theorems by Liouville.

Theorem 2.5. *If $f \in \mathcal{K}(\Omega)$ is holomorphic, then f is constant.*

Proof. Let P be a periodic parallelogram, then the closure of P is compact. Let $f \in \mathcal{K}(\Omega)$ be arbitrary and holomorphic. Since any holomorphic function on a compact domain is bounded, we have, for some positive real number C, $|f(z)| \leq C$ for any $z \in P$. Let $\omega \in \mathbb{C}$ be arbitrary. From Proposition 1.10 we know, that there exists $\omega' \in \Omega$ such that $\omega + \omega' \in P$. Therefore we have

$$|f(\omega)| = |f(\omega + \omega')| \leq C,$$

which means that f is bounded on $\mathbb{C}$. Liouville's theorem in complex analysis states that every bounded entire function is constant. Hence, f is constant. $\square$

Theorem 2.6. *Let $f \in \mathcal{K}(\Omega)$ and P be a periodic parallelogram, then it holds that*

$$\sum_{c \in P} res_c f = 0.$$

Proof. Since the set of poles of f D_f is discrete, there are only finitely many poles in the periodic parallelogram P. Assuming that f has no poles on ∂P, with residue formula ([3], pp 77) we have

$$\begin{aligned}2\pi i \sum_{c\in P} res_c f = &\int_u^{u+\omega_1} f(z)dz + \int_{u+\omega_1}^{u+\omega_1+\omega_2} f(z)dz \\ &+ \int_{u+\omega_1+\omega_2}^{u+\omega_2} f(z)dz + \int_{u+\omega_2}^{u} f(z)dz \\ &= \int_u^{u+\omega_1} (f(z) - f(z+\omega_2))dz + \int_{u+\omega_2}^{u} (f(z) - f(z+\omega_1))dz\end{aligned}$$

Because of the periodicity of f, the right side of the above equation is zero. Therefore $\sum_{c\in P} res_c f = 0$. □

Theorem 2.7. *Let $f \in \mathcal{K}(\Omega)$ be non-constant and P be a period parallelogram, then for any $\omega \in \mathbb{C}$*

$$\sum_{c\in P} ord_c(f - \omega) = 0.$$

Proof. We first show that any non-constant elliptic functions have as many zeros as they have poles, counted with their multiplicities. For any $\omega \in \mathbb{C}$ we have

$$f(z+\omega) = f(z) \Rightarrow f'(z+\omega) = f'(z)$$

. Hence it holds that $f'(z) \in \mathcal{K}(\otimes)$. Thus also $\frac{f'(z)}{f(z)}$ an elliptic function w.r.t. Ω. As argued before, without loss of generality, assume that f has no zeros or poles on ∂P. The argument principle in complex analysis states that

$$\int_{\partial P} \frac{f'(z)}{f(z)} dz = 2\pi i (Z - P)$$

where Z and P denote the number of zeros and pole of f in P, counted with their multiplicities. From Theorem 2.6 we know that $\int_{\partial P} \frac{f'(z)}{f(z)} dz = 0$. Therefore $Z - P = 0$.
For any $\omega \in \mathbb{C}$, $f - \omega$ is elliptic and has as many poles as f. Thus, the equation $f(z) = \omega$ has as many solutions as the poles of f. □

Theorem 2.8. *Let $f \in \mathcal{K}(\Omega)$ be non-zero and P be a period parallelogram, then it holds that*

$$\sum_{c\in P} (ord_c f) \cdot c \in \Omega.$$

Proof. From complex analysis, we know that, for $f \in \mathcal{M}$, the function f'/f has simple poles at the zeros and poles of f, and the residue is the order of the zero of f or the negative of the order of the pole of f ([3], pp.90). Therefore we have

$$\sum_{c\in P} (ord_c f) \cdot c = \sum_{c\in P} res_c \frac{f'}{f} \cdot c.$$

With residue formula([3], pp.77), it holds

$$2\pi i \sum_{c\in P} res_c \frac{f'}{f} \cdot c = \int_{\partial P} z \cdot \frac{f'(z)}{f(z)} dz.$$

We calculate the integral

$$\begin{aligned}\int_{\partial P} z \cdot \frac{f'(z)}{f(z)} dz &= \int_{u}^{u+\omega_1} z\frac{f'(z)}{f(z)} - (z+\omega_2)\frac{f'(z+\omega_2}{f(z+\omega_2}dz \\ &+ \int_{u+\omega_2}^{u} z\frac{f'(z)}{f(z)} - (z+\omega_1)\frac{f'(z+\omega_1}{f(z+\omega_1}dz \\ &= (\omega_1 \int_{u}^{u+\omega_2} \frac{f'(z)}{f(z)} - \omega_2 \int_{u}^{u+\omega_1} \frac{f'(z)}{f(z)})dz.\end{aligned}$$

The periodicity of f with period ω_1 and ω_2 implies that

$$\int_{u}^{u+\omega_j} \frac{f'(z)}{f(z)} dz \in 2\pi i\mathbb{Z} \quad for \quad j = 1, 2.$$

Combining all the equations we get the result. □

3 The Construction of the Weierstrass $\wp$-Function

From now on, we fix a lattice $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Let $P = \diamond(\omega_1, \omega_2)$ its fundamental mesh. We now want to find a non-constant elliptic function with respect to Ω. By Theorem 2.5, we must have at least one pole in P. On the other hand, there does not exist an elliptic function having a unique simple pole in P.[1]

For the sake of constructing the Weierstrass $\wp$ function, we want a function with exactly one double pole on each lattice point. How does a function like this look like? The first terms of the Laurent expansion at each lattice point $\omega \in \Omega$ are given by

$$\frac{A}{(z-\omega)^2} + \frac{B}{(z-\omega)}$$

To simplify, we assume $A = 1, B = 0$. To obtain such an expansion for all ω, the idea is to take a function of the form

$$\sum_{\omega \in \Omega} \frac{1}{(z-\omega)^2}$$

But we have a problem:

Lemma 3.1 (Apostol, Lemma 1.1)**.** For $\alpha \in \mathbb{R}$, the sum

$$\sum_{0 \neq \omega \in \Omega} \frac{1}{|\omega|^\alpha}$$

converges if and only if $\alpha > 2$.

Lemma 3.2 (Apostol, Lemma 1.2)**.** If $R > 0, \alpha \in \mathbb{N}$, then the series

$$\sum_{|\omega| > R} \frac{1}{(z-\omega)^\alpha}$$

converges absolutely and uniformly in the disk $|z| \leq R$ if and only if $\alpha > 2$.

[1] In the theory of Riemann surfaces, an elliptic function having a unique simple pole in P induces a holomorphic map from the complex torus C/Ω to the Riemann sphere $\overline{\mathbb{C}} \cong S^2$ of degree 1. But all maps of degree 1 between compact Riemann surfaces are biholomorphic, thus homeomorphisms. This is impossible.

Therefore, we start with a degree 3 map, which turns out to be the derivative of the Weierstrass $\wp$-function up to a constant.

Theorem 3.3 (Apostol, Thm 1.9). *Let $f : \mathbb{C} \setminus \Omega \to \mathbb{C}$ be defined by*

$$f(z) = \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^3}$$

Then the sum converges absolutely and uniformly in each compact set $K \subseteq \mathbb{C} \setminus \Omega$ and f is an elliptic function with a pole of order 3 at each lattice point ω.

Proof. Fix any $R > 0$. Then the series

$$\sum_{|\omega|>R} \frac{1}{(z-\omega)^3} \tag{3.1}$$

converges uniformly in the disk $|z| \leq R$ by Lemma 3.2, therefore this part of f is holomorphic on $|z| < R$ (as uniform limit of holomorphic functions). The remaining part is meromorphic only having poles of order 3 in the lattice points in the disk. Therefore, f is meromorphic on the open disk $|z| < R$. Since R was arbitrary, this shows that f is meromorphic with poles of order 3 at the lattice points.

The convergence on compact sets $K \subseteq \mathbb{C} \setminus \Omega$ follows similarly by taking R big enough such that K is contained in $|z| \leq R$ and using the uniform convergence for Eq. (3.1).

It remains to proof double periodicity: By absolute convergence, the order of summation does not matter. So since $\omega_1 + \Omega = \Omega$, we obtain

$$f(z+\omega_1) = \sum_{\omega \in \Omega} \frac{1}{(z+\omega_1-\omega)^3} = \sum_{\omega \in \omega_1 + \Omega} \frac{1}{(z+\omega_1-\omega)^3} = f(z)$$

and similarly for ω_2. □

To obtain an elliptic function with a pole of order 2, we remove the term z^{-3} corresponding to $\omega = 0$ and integrate out from 0 to z, where we can exchange integral and sum by absolute convergence:

$$\int_0^z \sum_{0 \neq \omega \in \Omega} \frac{1}{(t-\omega)^3} dt = \sum_{0 \neq \omega \in \Omega} \left(-\frac{1}{2}\right) \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right) \tag{3.2}$$

It is now an easy consequence that Eq. (3.2) must again converge absolutely and uniformly on compact sets![2] To get back to periodicity, we simply add the term $\frac{1}{z^2}$ after removing the constant and we have found the Weierstrass $\wp$-function.

[2]Absolute convergence, since $\sum_{0 \neq \omega} \left| \int_0^z \frac{1}{(\omega-t)^3} dt \right| \leq |z| \sum_{0 \neq \omega} \frac{1}{(\omega - t^*)^3}$ for some t^*, and uniform convergence on $K \subseteq (\mathbb{C} \setminus \Omega) \cup 0$ compact (w.l.o.g. path-connected, $0 \in K$), since

$$\sup_{z \in K} \left| \sum_{|\omega|>R} \int_0^z \frac{1}{(\omega-t)^3} dt \right| \leq const \cdot \sup_{t \in K} \sum_{|\omega|>R} \frac{1}{(\omega-t)^3} \to 0$$

as $R \to \infty$

Theorem 3.4 (Construction Theorem for the $\wp$-function)**.** *The series defining the Weierstrass $\wp$-function*

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \omega \in \Omega} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

converges in each compact set $K \subseteq \mathbb{C} \setminus \Omega$ absolutely and uniformly. We have the following properties:

(i) $\wp$ is an elliptic function (with respect to Ω), holomorphic on $\mathbb{C} \setminus \Omega$, with Laurent expansion

$$\frac{1}{z^2} + a_2 z^2 + \mathcal{O}(z^4),$$

where $a_2 \in \mathbb{C}$.

(ii) $\wp$ is even, having a pole of order 2 at each lattice point

(iii) $\wp'$ is odd, having a pole of order 3 at each lattice point and is given by

$$\wp'(z) = (-2) \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^3}$$

Proof. We already established convergence and the formula of $\wp'$. Since $\Omega = -\Omega$, it is immediate by reordering of the summands that $\wp$ is even and $\wp'$ is odd. The Laurent expansion will be given in the next section.

To show that $\wp$ is elliptic, we still need to prove the double periodicity. Since $\wp'$ is elliptic, for all $z \in \mathbb{C} \setminus \Omega, \omega \in \Omega$:

$$\wp'(z+\omega) = \wp'(z).$$

Therefore, the function

$$\wp(z+\omega) - \wp(z)$$

is constant. Inserting $z = -\frac{\omega}{2}$ gives that the constant is

$$\wp(\frac{\omega}{2}) - \wp(-\frac{\omega}{2}) = 0,$$

since $\wp$ is even. $\square$

4 The Laurent Expansion

Definition 4.1 (Eisenstein series)**.** For all $k \geq 3$, define the Eisenstein series of the lattice Ω to be

$$G_k = \sum_{0 \neq \omega \in \Omega} \frac{1}{\omega^k}$$

Remark 4.2. *Since $\omega \in \mathbb{C}$ is lattice point if and only if $-\omega$ is, we see immediately that $G_k = (-1)^k G_k$, meaning that $G_k = 0$ for odd k.*

Theorem 4.3. *We have the Laurent expansion at 0 given by*

$$\wp(z) = z^{-2} + \sum_{n=2}^{\infty} (2n-1) G_{2n} z^{2n-2}$$

Proof. The trick we use is to write, for $|t| < 1$,

$$\frac{1}{(1-t)^2} = \frac{d}{dt}\left(\frac{1}{1-t}\right) = \sum_{m=1}^{\infty} m t^{m-1}.$$

Let $\gamma := \min\{|\omega| | 0 \neq \omega \in \Omega\}$. Now we can write, if $\omega \neq 0$ and $|z| < \gamma$:

$$\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \frac{1}{\omega^2}\left(\frac{1}{(1-\frac{z}{w})^2} - 1\right) = \sum_{m=2}^{\infty} m \frac{z^{m-1}}{\omega^{m+1}}.$$

Inserting this in the expression of the $\wp$ function gives for $|z| < \gamma$:

$$\wp(z) = z^{-2} + \sum_{0\neq\omega\in\Omega}\left(\sum_{m=2}^{\infty} m \frac{z^{m-1}}{\omega^{m+1}}\right).$$

Since $\left|m\frac{z^{m-1}}{\omega^{m+1}}\right| \leq \gamma m \left(\frac{|z|}{\gamma}\right)^{m-1} |\omega|^{-3}$, we have absolute convergence when summing over ω and m, so we can exchange the summation signs and obtain

$$\begin{aligned}\wp(z) &= z^{-2} + \sum_{m\geq 2} m z^{m-1} \sum_{0\neq\omega\in\Omega} \frac{1}{\omega^{m+1}} \\ &= z^{-2} + \sum_{m\geq 2} m z^{m-1} G_{m+1} \\ &= z^{-2} + \sum_{n=2}^{\infty} (2n-1) G_{2n} z^{2n-2},\end{aligned}$$

where in the last equality we used that $G_m = 0$ for odd m. □

5 The Differential Equation

Theorem 5.1. *The Weierstrass $\wp$-function satisfies the differential equation*

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3,$$

where $g_2 = 60G_4$ and $g_3 = 140G_6$ are constants only depending on Ω, called the Weierstrass-invariants *of the lattice.*

Proof. Starting from the Laurent expansion at 0

$$\wp(z) = z^{-2} + 3G_4 z^2 + 5G_6 z^4 + \mathcal{O}(z^6),$$

we calculate

$$\begin{aligned}\wp^2(z) &= z^{-4} + 6G_4 + 10G_6 z^2 + \mathcal{O}(z^3), \\ \wp^3(z) &= z^{-6} + 9G_4 z^{-2} + 15G_6 + \mathcal{O}(z), \\ \wp'(z) &= (-2)z^{-3} + 6G_4 z + 20G_6 z^3 + \mathcal{O}(z^5), \\ \wp'^2(z) &= 4z^{-6} - 24G_4 z^{-2} - 80G_6 + \mathcal{O}(z).\end{aligned}$$

Therefore,

$$\wp'^2(z) - 4\wp^3(z) + 60G_4 + 140G_6 = \mathcal{O}(z). \tag{5.1}$$

Now Eq. (5.1) belongs to $\mathcal{K}(\Omega)$ and only can have poles where $\wp$ and $\wp'$ also have poles. But since we are $\mathcal{O}(z)$ at 0, there is no pole at 0, meaning that by [Theorem A] Eq. (5.1) is constant. But a constant which is $\mathcal{O}(z)$ must be 0. $\square$

Corollary 5.2 (Uniqueness up to translation)**.** *Fix a domain $G \subseteq \mathbb{C}$. Then every meromorphic, non-constant solution f of the differential equation*

$$f'^2 = 4f^3 - g_2 f - g_3$$

is given by $f(z) = \wp(z + z_0)$ for $z \in G$ and some $z_0 \in \mathbb{C}$. In particular, f is an elliptic function with respect to Ω.

Proof. Let f be a meromorphic, non-constant solution defined on G. First, note that first order polynomials do not solve the equation, so f' is not constant. Since the number of poles and zeros of f and f' are finite in each bounded subset, we can choose $u_0 \in G$, an open disk $U \subseteq G$ around u, such that f is holomorphic and f' never vanishes on U.

Further, we can assume that U is small enough such that

$$f' = \sqrt{4f^3 - g_2 f - g_3} \tag{5.2}$$

for some branch of the square root.

Now by Liouville's third theorem there exists $z_0 \in \mathbb{C}$ such that

$$\wp(u_0 + z_0) = f(u_0).$$

Since both $\wp$ and f satisfy the differential equation, we know that $\wp'(u_0 + z_0) = \pm f'(u_0 + z_0)$. But since $\wp$ is even and $\wp'$ is odd we can assume (by otherwise replacing z_0 by $-z_0 - 2u_0$) that

$$\wp'(u_0 + z_0) = f'(u_0 + z_0).$$

Note that $f(z)$ and $g(z) := \wp(z + z_0)$ both satisfy the initial value problem Eq. (5.2). But since $h(z) = \sqrt{4z^3 - g_2 z - g_3}$ is holomorphic (and thus locally Lipschitz continuous) on $f(U)$, by the Picard-Lindelöf theorem, we must have a unique solution in a neighbourhood of u_0 which means that $f \equiv g$ on G by the identity theorem.

$\square$

References

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