ELLIPTIC FUNCTIONS

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The first talk of the seminar "Elliptic Functions and Modular Forms" focuses on elliptic functions in the general sense. One starts defining two notions: the periods of a meromorphic function and lattices in the complex plane. These are the natural framework for introducing the field of elliptic functions in the third part. Various properties of elliptic functions are then investigated through the four theorems of Liouville. Also appealing to the notion of a lattice, the Einsenstein series are defined in the part five. Their absolute convergence yields to a first example of an elliptic function. An appendix finally presents the argument principle and its generalized version. This fundamental theorem of complex analysis is used in different forms in the proofs of Liouville's theorems II, III, and IV.

1. Periods

Definition 1.1 (holomorphic function)**.** *Let $D \subseteq \mathbb{C}$ be open. A function $f : D \to \mathbb{C}$ is called holomorphic on D, if*

$$\forall z_0 \in D : \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists in } \mathbb{C}.$$

The set of all holomorphic functions over D is denoted by $\mathcal{H}(D)$.

Definition 1.2 (discrete subset)**.** *A closed subset $D \subseteq \mathbb{C}$ is called discrete, if*

$$\forall z \in D : \exists r > 0 : B_r(z) \cap D \text{ is finite } (D \text{ has no accumulation point})$$

Definition 1.3 (meromorphic function)**.** *Let $D \subseteq \mathbb{C}$ be open. A function $f : D \to \mathbb{C}$ is called meromorphic on D, if there exists a closed, discrete subset $P_f \subseteq D$ such that*

(1.1) $f : D \setminus P_f \to \mathbb{C}$ *is holomorphic*
(1.2) f *has poles at the points of* P_f

The set of all meromorphic functions over D is denoted by $\mathcal{M}(D)$.

Lemma 1.4. *The meromorphic functions on $\mathbb{C}$ form a field.*

Theorem 1.5 (identity theorem for meromorphic functions)**.** *Let $D \subseteq \mathbb{C}$ be open and let $f, g : D \to \mathbb{C}$ be two meromorphic functions. Then*

$$f = g \iff \{z \in D \mid f(z) = g(z)\} \text{ has an accumulation point in } D \setminus (P_f \cup P_g)$$

Definition 1.6 (period)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be a meromorphic function and let P_f be the set of poles of f. Then, $w \in \mathbb{C}$ is called a period of f, if*

(1.1) $P_f + w = P_f$
(1.2) $\forall z \in \mathbb{C} \setminus P_f : f(z + w) = f(z)$

$\mathrm{Per}(f)$ *denotes the set of all periods of f.*

Lemma 1.7. *For every meromorphic function $f : \mathbb{C} \to \mathbb{C}$, it holds that*

(1.1) 0 *is always a period of f.*
(1.2) *If f is constant, then* $\mathrm{Per}(f) = \mathbb{C}$.
(1.3) $\mathrm{Per}(f)$ *is a subgroup of* $(\mathbb{C}, +, 0)$.

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Theorem 1.8 (structure theorem). *Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant meromorphic function. Then,* $\mathrm{Per}(f)$ *is a closed and discrete subgroup of* $(\mathbb{C}, +, 0)$.
In particular, $\mathrm{Per}(f)$ *has one of the following forms:*

(1.1) $\mathrm{Per}(f) = \{0\}$
(1.2) $\mathrm{Per}(f) = \mathbb{Z}w_1 = \{nw_1 \mid n \in \mathbb{Z}\}$ *for a* $w_1 \in \mathbb{C} \setminus \{0\}$
(1.3) $\mathrm{Per}(f) = \mathbb{Z}w_1 + \mathbb{Z}w_2 = \{n_1w_1 + n_2w_2 \mid n_1, n_2 \in \mathbb{Z}\}$ *for two* $w_1, w_2 \in \mathbb{C} \setminus \{0\}$ *with:*
  - *(a)* w_1 *and* w_2 *are linearly independent over* $\mathbb{R}$
  - *(b)* $|\frac{w_2}{w_1}| \geqslant 1$
  - *(c)* $Im(\frac{w_2}{w_1}) > 0$
  - *(d)* $|Re(\frac{w_2}{w_1})| \leqslant \frac{1}{2}$

Proof. Assume per contradiction that $\mathrm{Per}(f)$ is not discrete.
Then $\exists w \in \mathrm{Per}(f) : \forall r > 0 : B_r(w) \cap \mathrm{Per}(f)$ is infinite. Thus $\mathrm{Per}(f)$ has an accumulation point. Let $z \in \mathbb{C}$ be such that f is holomorphic in z. Then $\forall n \in \mathbb{N} : f(z) = f(z + w_n)$, so f agrees with a constant function on $\mathrm{Per}(f)$. After the identity theorem for holomorphic functions, it follows that f is constant, which contradicts the assumption.

Assume $\mathrm{Per}(f) \supsetneq \{0\}$. Since $\mathrm{Per}(f)$ is closed and discrete, it holds that $\exists w_1 \in \mathrm{Per}(f)$:

$$|w_1| = \inf\{|w| \mid w \in \mathrm{Per}(f) \setminus \{0\}\} > 0. \tag{1}$$

Claim 1: $\mathrm{Per}(f) \cap \mathbb{R}w_1 = \mathbb{Z}w_1$
Since $w_1 \in \mathrm{Per}(f)$, it holds that $\forall n \in \mathbb{Z} : nw_1 \in \mathrm{Per}(f)$ by induction and the group structure of $\mathrm{Per}(f)$. Hence $\mathrm{Per}(f) \cap \mathbb{R}w_1 \supseteq \mathbb{Z}w_1$
Let $w \in \mathrm{Per}(f) \cap \mathbb{R}w_1$ be arbitrary. Then $\exists \alpha \in \mathbb{R} : w = \alpha w_1$. Choose $n \in \mathbb{Z}$ such that $|\alpha - n| \leqslant \frac{1}{2}$. Then $w - nw_1 \in \mathrm{Per}(f)$ and

$$|w - nw_1| = |\alpha w_1 - nw_1| = |(\alpha - n)| \cdot |w_1| < |w_1|$$

After (1), it holds that $w - nw_1 = 0$. Therefore $w = nw_1$ and $\mathrm{Per}(f) \cap \mathbb{R}w_1 \subseteq \mathbb{Z}w_1$
This proves claim 1. Hence, if all points of $\mathrm{Per}(f)$ lie on a line through the origin, so $\exists w \in \mathbb{C} : \mathrm{Per}(f) \subseteq \mathbb{R}w$, then $\exists w_1 \in \mathbb{C} \setminus \{0\} : \mathrm{Per}(f) = \mathbb{Z}w_1$.

Now assume that additionally $\mathrm{Per}(f) \neq \mathbb{Z}w_1$ for some $w_1 \in \mathbb{C} \setminus \{0\}$ from (1). Since $\mathrm{Per}(f)$ is closed and discrete, it holds that $\exists w_2 \in \mathrm{Per}(f) \setminus \mathbb{Z}w_1$:

$$|w_2| = \inf\{|w| \mid w \in \mathrm{Per}(f) \setminus \mathbb{Z}w_1\} > |w_1|. \tag{2}$$

Write $w_1 = r_1 e^{i\phi_1}$ and $w_2 = r_2 e^{i\phi_2}$ in polar form. It holds that

(a) $\frac{w_2}{w_1} \in \mathbb{R} \iff \phi_2 - \phi_1 = 0 \iff \phi_1 = \phi_2 \iff w_1$ and w_2 lie on the same line through the origin, hence w_1 and w_2 are contained in $\mathbb{R}w_1 = \mathbb{R}w_2$, contradicting the assumption. Therefore $\frac{w_2}{w_1} \notin \mathbb{R}$, so w_1 and w_2 must be linearly independent over $\mathbb{R}$.
(b) As $|w_2| \geqslant |w_1|$, it holds that $|\frac{w_2}{w_1}| \geqslant 1$
(c) $\mathrm{Im}(\frac{w_2}{w_1}) > 0 \iff 0 < \phi_2 - \phi_1 < \pi \iff \phi_2 - \pi < \phi_1 < \phi_2$
Notice that $w_1 \in \mathrm{Per}(f) \iff -w_1 \in \mathrm{Per}(f)$, $|w_1| = |-w_1|$ and $-w_1 = r_1 e^{i\phi_1 - \pi}$,
so $\mathrm{Im}(\frac{w_2}{w_1}) > 0$ can be assured by choosing w_1 with a suitable sign.
(d) As $w_2 \pm w_1 \in \mathrm{Per}(f) \setminus \mathbb{Z}w_1$, it follows from (2) that $|w_2 \pm w_1| \geqslant |w_2|$ and thus $|\frac{w_2}{w_1} \pm 1| \geqslant |\frac{w_2}{w_1}|$
Write $\frac{w_2}{w_1} = a + ib$. Then $a^2 + b^2 \leqslant (a \pm 1)^2 + b^2 \iff 0 \leqslant \pm 2a + 1 \iff |\mathrm{Re}(\frac{w_2}{w_1})| = |a| \leqslant \frac{1}{2}$.

Claim 2: $\mathrm{Per}(f) = \mathbb{Z}w_1 + \mathbb{Z}w_2$
Since $w_1, w_2 \in \mathrm{Per}(f)$, it holds that $\forall n_1, n_2 \in \mathbb{Z} : n_1w_1 + n_2w_2 \in \mathrm{Per}(f)$ due to the group structure of $\mathrm{Per}(f)$. Hence $\mathrm{Per}(f) \supseteq \mathbb{Z}w_1 + \mathbb{Z}w_2$.
Let $w \in \mathrm{Per}(f)$ be arbitrary. Since w_1, w_2 is an $\mathbb{R}$-basis of $\mathbb{C}$, there exists $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $w = \alpha_1 w_1 + \alpha_2 w_2$. Choose $n_1, n_2 \in \mathbb{Z}$ such that for $\beta_1 = \alpha_1 - n_1$ and $\beta_2 = \alpha_2 - n_2$, it holds that $|\beta_1| \leqslant \frac{1}{2}$ and $|\beta_2| \leqslant \frac{1}{2}$. Then

$$w' := w - n_1w_1 - n_2w_2 = \beta_1 w_1 + \beta_2 w_2 \in \mathrm{Per}(f)$$

If $\beta_2 = 0$, then $w' = \beta_1 w_1 \in \mathbb{R}w_1$. So the same reasoning as in claim 1 yields $w' = 0$.
If $\beta_2 \neq 0$, then $w' \in \mathrm{Per}(f) \setminus \mathbb{Z}w_1$ and

$$|w'|^2 = |\beta_1 w_1 + \beta_2 w_2|^2 = (\beta_1^2 + 2\beta_1\beta_2 \mathrm{Re}(\frac{w_2}{w_1}) + \beta_1^2 |\frac{w_2}{w_1}|^2) \cdot |w_1|^2$$
$$\leqslant (\beta_1^2 + |\beta_1||\beta_2| + \beta_2^2) \cdot |\frac{w_2}{w_1}|^2 \cdot |w_1|^2 \leqslant \frac{3}{4}|w_1|^2$$

After (2), it holds that $w' = 0$, thus $w = n_1 w_1 + n_2 w_2$ and therefore $\mathrm{Per}(f) \subseteq \mathbb{Z}w_1 + \mathbb{Z}w_2$. This proves claim 2. Hence, all sets of periods are of one of the three forms in the statement. □

2. Lattices in $\mathbb{C}$

Definition 2.1 (lattice)**.** *Let V be a $\mathbb{R}$-vector space of dimension $n \in \mathbb{N}$.*
A subset $\Omega \subseteq V$ is called a lattice in V, if there exists an $\mathbb{R}$-basis $(\omega_1, \dots, \omega_n)$ of V such that

$$\Omega = \mathbb{Z}\omega_1 + \ldots + \mathbb{Z}\omega_n$$

Then, $(\omega_1, \dots, \omega_n)$ is called a basis of Ω.

Lemma 2.2. *Every lattice Ω in $\mathbb{C}$ is closed and discrete in $\mathbb{C}$.*

Lemma 2.3. *Let Ω be a lattice in $\mathbb{C}$ with basis (ω_1, ω_2) and let $\omega_1', \omega_2' \in \mathbb{C}$ be arbitrary. Then*

$$\omega_1', \omega_2' \in \Omega \iff \exists U \in Mat_{2\times 2}(\mathbb{Z}) : \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = U \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

Furthermore, (ω_1', ω_2') is a basis of $\Omega \iff U \in \mathrm{GL}_2(\mathbb{Z})$.

Proof. It holds that $\omega_1', \omega_2' \in \Omega$ iff $\exists a, b, c, d \in \mathbb{Z}$ such that $\omega_1' = a\omega_1 + b\omega_2$ and $\omega_2' = c\omega_1 + d\omega_2$ iff for $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{Z}) : \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = U \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$.

If (ω_1', ω_2') is a basis of Ω, then with the same reasoning as above $\exists V \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ with $\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = V \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}$.

It follows that $\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = VU \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$ and $\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = UV \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}$.

Since (ω_1, ω_2) and (ω_1', ω_2') are linearly independent, it holds that $VU = UV = I_2$ and thus $U \in \mathrm{GL}_2(\mathbb{Z})$.

If $U \in \mathrm{GL}_2(\mathbb{Z})$, then $\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = U \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$ yields that (ω_1', ω_2') is linearly independent over $\mathbb{R}$.

For arbitrary $\omega_1'', \omega_2'' \in \Omega$, there is also a $W \in \mathrm{GL}_2(\mathbb{Z})$ such that $\begin{pmatrix} \omega_1'' \\ \omega_2'' \end{pmatrix} = W \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$.

Hence $\begin{pmatrix} \omega_1'' \\ \omega_2'' \end{pmatrix} = WU^{-1} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}$ and thus ω_1'' and ω_2'' are both linear combination of ω_1' and ω_2' over $\mathbb{Z}$.

Therefore, (ω_1', ω_2') is indeed a basis of Ω. □

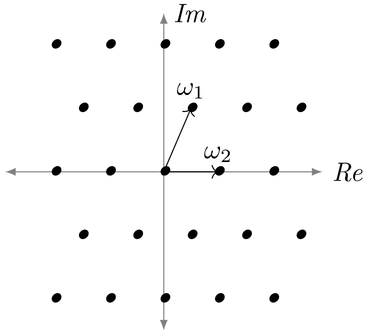
Definition 2.4 (fundamental parallelogram)**.** *Let Ω be a lattice in $\mathbb{C}$, let (ω_2, ω_1) be a basis of Ω and let $u \in \mathbb{C}$ be an arbitrary point. Then*

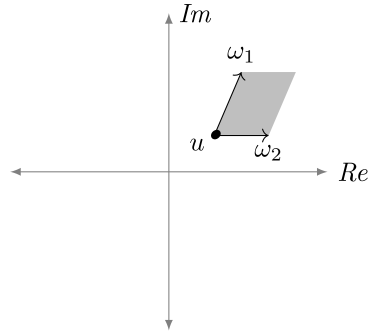
$$\mathrm{Par}(u; \omega_1, \omega_2) := \{u + \alpha_1\omega_2 + \alpha_2\omega_1 \mid \alpha_1, \alpha_2 \in [0, 1)\}$$

is called the fundamental parallelogram w.r.t. the basis (ω_2, ω_1) and the base point u.

Definition 2.5 (volume of a fundamental parallelogram)**.** *Let Ω be a lattice and let $Q = \mathrm{Par}(u; \omega_1, \omega_2)$ be a fundamental parallelogram in Ω.*
Then $\mathrm{vol}(\Omega) := |\mathrm{Im}(\omega_1\overline{\omega_2})|$ is called the volume of Q.

Example 2.6. $\Omega = \mathbb{Z} \cdot (0.6 + 1.4 \cdot i) + \mathbb{Z} \cdot 1.2$ *and* $Q = \mathrm{Par}(1 + 0.8 \cdot i; 0.6 + 1.4 \cdot i, 1.2)$





Lemma 2.7. $\mathrm{vol}(\Omega) = |\mathrm{Im}(\omega_1 \overline{\omega_2})|$ *is independent of the basis* (ω_1, ω_2) *and the base point* u.

Proof. A basic computation yields:
$\mathrm{Im}(\omega_1 \overline{\omega_2}) = \mathrm{Im}((\mathrm{Re}(\omega_1) + i \cdot \mathrm{Im}(\omega_1)) \cdot (\mathrm{Re}(\omega_2) - i \cdot \mathrm{Im}(\omega_2))) = \mathrm{Im}(\omega_1) \cdot \mathrm{Im}(\omega_2) - \mathrm{Re}(\omega_1) \cdot \mathrm{Im}(\omega_2)$.
Hence the volume of $Q = \mathrm{Par}(u; \omega_1, \omega_2)$ is given by

$$\mathrm{vol}(\Omega) = |\mathrm{Im}(\omega_1 \overline{\omega_2})| = \left| \det \begin{pmatrix} \mathrm{Im}(\omega_1) & \mathrm{Re}(\omega_1) \\ \mathrm{Im}(\omega_2) & \mathrm{Re}(\omega_2) \end{pmatrix} \right|$$

If (ω_1', ω_2') is a different basis of Ω, then there is after the lemma above, a matrix $U \in \mathrm{GL}_2(\mathbb{Z})$ such that $\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = U \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$. Therefore

$$|\mathrm{Im}(\omega_1' \overline{\omega_2'})| = \left| \det \begin{pmatrix} \mathrm{Im}(\omega_1') & \mathrm{Re}(\omega_1') \\ \mathrm{Im}(\omega_2') & \mathrm{Re}(\omega_2') \end{pmatrix} \right| = \left| \det \left(U \cdot \begin{pmatrix} \mathrm{Im}(\omega_1) & \mathrm{Re}(\omega_1) \\ \mathrm{Im}(\omega_2) & \mathrm{Re}(\omega_2) \end{pmatrix} \right) \right| = |\det(U)| \cdot |\mathrm{Im}(\omega_1, \overline{\omega_2})|$$

Since $\forall U \in \mathrm{GL}_2(\mathbb{Z}) : |\det(U)| = 1$, the volume of Ω is indeed independent of the basis. □

Lemma 2.8. *Let* Ω *be a lattice and let* $Q = \mathrm{Par}(u; \omega_1, \omega_2)$ *be a fundamental parallelogram in* Ω.
Then $\forall z \in \mathbb{C} : \exists! \omega \in \Omega : z + \omega \in Q$.

Proof. It holds that $\forall z \in \mathbb{C} : \exists \beta_1, \beta_2 \in \mathbb{R} : z = u + \beta_1 \omega_1 + \beta_2 \omega_2$.
Define $n_1 := \lfloor \beta_1 \rfloor$, $n_2 := \lfloor \beta_2 \rfloor$, $\alpha_1 := \beta_1 - n_1 \in [0, 1)$ and $\alpha_2 := \beta_2 - n_2 \in [0, 1)$.
Then $z = u + n_1 \omega_1 + \alpha_1 \omega_1 + n_2 \omega_2 + \alpha_2 \omega_2$, so for $\omega = -n_1 \omega_1 - n_2 \omega_2 \in \Omega$, it holds that

$$z + \omega = z - n_1 \omega_1 - n_2 \omega_2 = u + \alpha_1 \omega_1 + \alpha_2 \omega_2 \in Q.$$

Note that $z \in Q \iff \omega = 0$. □

Theorem 2.9 (fundamental parallelogram as a torus)**.** *Let* Ω *be a lattice in* $\mathbb{C}$ *and let* $Q = \mathrm{Par}(u; \omega_1, \omega_2)$ *be a fundamental parallelogram in* Ω. *Since* Ω *is abelian and a subgroup of* $(\mathbb{C}, +, 0)$, *its factor group* $\mathbb{C}/\Omega$ *is also abelian.* $((a + \Omega) + (b + \Omega) = (a + b) + \Omega = (b + a) + \Omega = (b + \Omega) + (a + \Omega))$.
Let $\pi : \mathbb{C} \to \mathbb{C}/\Omega, a \mapsto a + \Omega$ *be the canonical projection. The restriction* $\pi|_Q : Q \xrightarrow{\sim} \mathbb{C}/\Omega$ *is an isomorphism. By identifying the opposite edges of* Q, $\mathbb{C}/\Omega$ *can be viewed as a torus in* $\mathbb{R}^3$.

3. What are elliptic functions?

In the next sections, $\Omega = \mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2$ will denote a lattice in $\mathbb{C}$. For the sake of brevity, an *elliptic function wrt. the lattice* Ω will thus be called an *elliptic function*. In a similar way, *fundamental parallelograms* (without further specification) will implicitly refer to the lattice Ω.

3.1. **The first properties.**

Definition 3.1. *A meromorphic function $f : \mathbb{C} - P_f \to \mathbb{C}$ is called* elliptic *or* doubly periodic *with respect to Ω, if the latter is contained in the set of periods of f: $\Omega \subset \mathrm{Per}(f)$.*

More explicitly, this implies:

(3.1) $f(z+\omega) = f(z) \quad \forall \omega \in \Omega \quad \forall z \in \mathbb{C} - P_f$

(3.2) $P_f + \omega = P_f \quad \forall \omega \in \Omega$

The condition (3.1) implies in particular the double periodicity of the zero set: $Z_f = Z_f + \omega$ for all $\omega \in \Omega$. To verify the properties (3.1) and (3.2), it is sufficient to verify them for a basis of Ω (e.g. (ω_1, ω_2)). The set of all elliptic functions with respect to Ω will be denoted by $\mathcal{K}(\Omega)$. If one chooses to represent a meromorphic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ as a function on $\mathbb{C}$ assuming values on the extended complex plane $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, one can gather both conditions in a single one of the form:

$$f(z+\omega) = f(z) \quad \forall \omega \in \Omega \quad \forall z \in \overline{\mathbb{C}}\,.$$

The following property of elliptic functions is a consequence of their periodicity. For every function $f : \mathbb{C} \to \overline{\mathbb{C}}$ in $\mathcal{K}(\Omega)$, there exists a unique map $\widehat{f} : \mathbb{C}/\Omega \to \overline{\mathbb{C}}$, so that $\widehat{f}([z]) := f(z)$. In other words, so that the following diagram commutes:

$$\begin{tikzcd} z \arrow[d, mapsto] & \mathbb{C} \arrow[r, "f"] \arrow[d, two heads, "\pi"'] & \overline{\mathbb{C}} \\ {[z]} & \mathbb{C}/\Omega \arrow[ur, dotted, "\widehat{f}"'] & \end{tikzcd}$$

Because f is periodic wrt. Ω, the function $\widehat{f}$ is well-defined (i.e. independent of the representative z). By a slight abuse of notation, the function $\widehat{f}$ is sometimes directly referred to by the original function f. An elliptic function may thus be viewed as a function on the torus $\mathbb{C}/\Omega$.

3.2. **The structure of a field.**

Reminder 3.2. *Any function $0 \neq f \in \mathcal{M}(\mathbb{C})$ is (uniquely) represented by a* Laurent series *of the form*

$$f(z) = \sum_{n \geqslant m} a_n (z-c)^n,\ m \in \mathbb{Z},\ a_n \in \mathbb{C},\ a_m \neq 0, \tag{3}$$

which converges normally (in particular locally uniformly) in a punctured neighbourhood of $c \in \mathbb{C}$.

By definition, m is the order *of the meromorphic function f at c, written as* $\mathrm{ord}_c(f)$. *The order is positive if and only if f has a zero at c, and it is negative iff. c is a pole. The order vanishes otherwise. A pole of order one is called* simple. *The coefficient a_{-1} in the series* (3) *is the* residue *of f at c, denoted by* $\mathrm{res}_c(f)$. *The residue of a simple pole is always nonzero.*

For $f \in \mathcal{K}(\Omega)$, $\omega \in \Omega$, and z in an appropriate neighbourhood of $c + \omega$, one can write:

$$f(z) = f(z-\omega) = \sum_{n \geqslant m} a_n (z - (c+\omega))^n,$$

which implies the invariance of orders and residues under translation wrt. to the lattice :

$$\mathrm{ord}_{c+\omega}(f) = \mathrm{ord}_c(f) \quad \text{and} \quad \mathrm{res}_{c+\omega}(f) = \mathrm{res}_{c+\omega}(f). \tag{4}$$

The periodicity of orders and residues (equation (4)), together with the discreteness of P_f, yields to the

Proposition 3.3. *The elliptic functions $\mathcal{K}(\Omega)$ form a subfield of the field $\mathcal{M}(\mathbb{C})$, containing all meromorphic functions on $\mathbb{C}$. It contains all constant functions. Each $f \in \mathcal{K}(\Omega)$ has finitely many poles in any fundamental parallelogram of Ω.*

Remark. The natural field homomophism $\iota : \mathbb{C} \hookrightarrow \mathcal{K}(\Omega)$, $z \mapsto (u \mapsto z)$ $(\forall z, u \in \mathbb{C})$, allows for identifying the subfield $\mathrm{Im}(\iota)$ of all constant functions on $\mathbb{C}$ with the field of complex numbers itself. One accordingly writes the field tower $\mathcal{M}(\mathbb{C})/\mathcal{K}(\Omega)/\,\mathbb{C}$.

A useful lemma follows from the two defining properties (3.1) and (3.2).

Lemma 3.4. *Let $f \in \mathcal{K}(\Omega)$, Then $f'(z)$ and $g(z) := f(nz + u)$ with $0 \neq n \in \mathbb{Z}$ and $u \in \mathbb{C}$ are also elliptic functions wrt. Ω.*

Let $p(z) \in \mathbb{C}[z]$ be a polynomial. Because $\mathcal{K}(\Omega)$ is closed under addition and multiplication, and it contains all constant maps on $\mathbb{C}$, the composition $z \mapsto p(f(z))$ is also an elliptic function wrt. Ω. For $r(z) \in \mathbb{C}(z)$ a rational function, the composition $z \mapsto r(f(z))$ is also in $\mathcal{K}(\Omega)$. It is moreover independent of the fractional representation of the rational function r. The composition map $r \mapsto r(f)$ having trivial kernel, it induces an isomorphism from the field of rational functions $\mathbb{C}(z)$ onto a subfield of $\mathcal{K}(\Omega)$. Latter will be denoted by

$$\mathbb{C}(f) := \{g : g = r(f) \text{ for } r \text{ a rational function}\}\,.$$

The next step will be to determine all even elliptic functions f $(f(z) = f(-z))$. One will first restrict to those, whose poles are contained in the lattice. Such an example is the Weierstrass $\wp$-function, which has exactly second order poles at all lattice positions.

4. The four theorems of Liouville

Reminder 4.1. *A function $f \in \mathcal{H}(\mathbb{C})$ is called* entire. *Liouville's original theorem states that every entire bounded function $f : \mathbb{C} \to \mathbb{C}$ is constant. There are at least three standard proofs for this fundamental result. They may use Cauchy's inequality (for Taylor coefficients), or more directly, Cauchy's integral formula.*

In 1847, J. Liouville derived four theorems, which are specific to elliptic functions. Like entire functions, they satisfy indeed strong conditions.

4.1. Liouville's first theorem (for elliptic functions).

The double periodicity of elliptic functions allows for dropping the assumption about the existence of a global upper bound in Liouville's original theorem. An elliptic function, holomorphic on the entire complex plane, is necessarily bounded.

Theorem 4.2 (Liouville, I)**.** *Every entire elliptic function (i.e. $f \in \mathcal{K}(\Omega) \cap \mathcal{H}(\mathbb{C})$) is constant.*

Proof. Let Q be a fundamental parallelogram. Its closure $\overline{Q}$ is compact, and the function f is continuous on this domain. There consequently exists a real constant $C > 0$, so that $|f(z)| \leqslant C$ for all $z \in Q$. For an arbitrary $z \in \mathbb{C}$, theorem 2.8 provides an $\omega \in \Omega$, so that $z + \omega \in Q$ holds. The periodicity condition (3.1) then ensures that f is bounded on $\mathbb{C}$: $|f(z)| = |f(z + \omega)| \leqslant C$. The statement finally follows by applying Liouville's classical theorem (4.1) to the entire function f. $\square$

4.2. Liouville's second theorem.

On the basis of theorem 4.2, it is natural to study the poles of an elliptic function.

Theorem 4.3 (Liouville, II)**.** *Let Q be a fundamental parallelogram of Ω and f and elliptic function. The sum of all residues of f, evaluated in Q, vanishes: $\sum_{c \in Q} \mathrm{res}_c(f) = 0$.*

Remark. The function f being meromorphic on $\mathbb{C}$, its set of poles is discrete. Besides, f has nonzero residues at most at $c \in P_f$, for it is holomorphic on $\mathbb{C} - P_f$. The summation accordingly reduces to the set $Q \cap P_f$, which is finite.

Proof. Equation (4) makes the sum of residues independent of the choice of a lattice basis.

Claim. After a suitable translation of Q, there is no pole on the boundary of the resulting fundamental parallelogram.

The claim is a consequence of the discreteness of P_f.

From now on, $Q := \mathrm{Par}(q; \omega_1, \omega_2)$ will denote (without loss of generality) this translated parallelogram. In particular, the function f is holomorphic on its (positively oriented) boundary ∂Q. One evaluates the integral of f along the boundary of the fundamental parallelogram by applying the residue theorem:

$$\pm 2\pi i \cdot \sum_{c \in Q} \mathrm{res}_c(f) = \pm \int_{\partial Q} f(z)dz = \int_q^{q+\omega_1} f(z) - f(z+\omega_2)dz + \int_{q+\omega_2}^{q} f(z) - f(z+\omega_1)dz. \tag{5}$$

The sign depends on the mutual orientation of the "basis vectors" ω_1 and ω_2. From the periodicity of the elliptic function f (see equation (3.1)), both integrands are zero, and the sum of residues vanishes. □

One property of elliptic functions immediately follows from theorem 4.3.

Corollary 4.4. *There is no elliptic function with a unique pole of order one. One either needs two distinct poles of order one, or a single pole of order two with vanishing residue.*

K. T. W. Weierstrass was able to provide an example for each of the two cases. In particular, his eponymous $\wp$-function has exactly one pole of order two in each fundamental parallelogram, namely at its bottom left corner.

4.3. **Liouville's third theorem.** The two next theorems of Liouville consider the zeros of the elliptic function $g_u : \mathbb{C} - P_f \to \mathbb{C}$, $z \mapsto f(z) - u$, for a fixed function $f \in \mathcal{K}(\Omega)$, and a parameter $u \in \mathbb{C}$. The zeros of g_u are exactly the pre-images of u wrt. f. They have equal orders: $\mathrm{ord}_0(g_u) = \mathrm{ord}_u(f)$. The functions f and g_u also share the same poles, with equal orders.

Theorem 4.5 (Liouville, III)**.** *Let $f \in \mathcal{K}(\Omega)$ be nonconstant, and Q a fundamental parallelogram. The sum of all orders of the elliptic function g_u, evaluated in Q, vanishes: $\sum_{c \in Q} \mathrm{ord}_c(g_u) = 0$.*

Remark. The zero set[1] and the set of poles of $0 \neq g_u \in \mathcal{M}(\mathbb{C})$ is discrete. The order of g_u at any other point in $\mathbb{C}$ vanishes. The sum of orders is consequently finite.

Proof. On the basis of:

(4.1) the non-constantness of f,
(4.2) $f' \in \mathcal{K}(\Omega)$ according to lemma (3.4),
(4.3) and the field structure of the set of all elliptic functions,

one defines the meromorphic function $h(z) := \frac{f'(z)}{f(z)-u} \in \mathcal{K}(\omega)$.

Because $\mathrm{res}_c(h) = \mathrm{ord}_c(f - u)$, the statement immediately follows from theorem 4.3. □

Remark. Last equality is justified by the generalized argument principle (see proposition 6.4). One sets $\phi(z) := 1$ and $\psi(z) := f(z)$. The definition of the residue $\mathrm{res}_c(h) = \frac{1}{2\pi i} \cdot \int_\gamma h(z)dz$ appears on the left-hand side of equation (15). The singularities of h correspond exactly to the poles of f and to the pre-images of u wrt. f. The zero set Z_{g_u} and the set of poles P_f being discrete, on can choose a simple closed path γ around c, which does not contain any singularity of h but possibly c. The right-hand side of equation (15) finally reduces to the single term $\mathrm{ord}_c(f)$.

Definition 4.6. *Let f be a meromorphic function on the set D, and $u \in \overline{\mathbb{C}}$ a complex number. Consider a subset $M \subset D$, such that M contains only finitely many pre-images of u wrt. f. The number of the corresponding points (counted with multiplicities) is*

$$\mathrm{num}_u(f, M) := \sum_{c \in f^{-1}(\{u\}) \cap M} \mathrm{ord}_c(f) \textit{ for any } u \in \mathbb{C}\,.$$

The case $u := 0$ (i.e. $f^{-1}(\{0\}) =: Z_f$) yields to the number of zeros of f. The case $u := \infty$ corresponds to poles. It is treated separately, in order to account for their negative orders:

$$\mathrm{num}_\infty(f, M) := - \sum_{c \in P_f \cap M} \mathrm{ord}_c(f). \tag{6}$$

[1]as follows from theorem 1.5

An interesting characterization of the image of elliptic functions results from theorem 4.5.

Corollary 4.7. *Every nonconstant elliptic function f has an equal number of poles, pre-images of u ($\forall u \in \mathbb{C}$), and zeros in Q, when these are counted with their multiplicities:* $\mathrm{num}_\infty(f,Q) = \mathrm{num}_u(f,Q) = \mathrm{num}_0(f,Q)$. *In particular, the function f takes every complex value in Q.*

Sketch of the proof. Because $f(z) - u \in \mathcal{K}(\Omega)$ is not constant, it must have a pole in Q according to theorem 4.2. Then, it must also have a pre-image of u in Q according to theorem 4.5. The statement then follows from the arbitrariness of the complex number u. □

Definition 4.8. *The* order *of an elliptic function f is its number $r \geqslant 0$ of zeros or poles inside a fundamental parallelogram Q, counted with multiplicities:*

$$\begin{aligned} \mathrm{ord}(f)) &:= \mathrm{num}_0(f,Q) \\ &:= \mathrm{num}_\infty(f,Q). \end{aligned}$$

Definition 4.9. *Let $f \in \mathcal{K}(\Omega)$. A points $u \in \overline{\mathbb{C}}$ is a* ramification point *(with respect to f), if there exists a $c \in \mathbb{C}$ so that* $\mathrm{ord}_c(g_u) \geqslant 2$ *holds. For $u := 0,\, \infty$, the point c respectively corresponds to a zero or pole of order at leasts two.*

Remark. For a nonconstant elliptic function of order $r > 0$, theorem 4.5 yields to $\left|f^{-1}\left(\{z\}\right)\right| < r$ iff. z is a ramification point of f. It is equal to r otherwise.

4.4. **Liouville's fourth theorem.**

Theorem 4.10 (Liouville, IV)**.** *Let $f \in \mathcal{K}(\Omega)$ be non-zero, and Q a fundamental parallelogram of Ω. The sum of all points in Q, weighted by the order of f at the given point, belongs to the lattice Ω:* $\sum_{c\in Q} \mathrm{ord}_c(f)\cdot c \in \Omega$.

Proof. The proof of Liouville's fourth theorem reuses the method from the proof of theorem 4.3. The meromorphic function $z \cdot \frac{f'(z)}{f(z)}$ must be integrated on the boundary ∂Q of a fundamental parallelogram Q. Instead of the residue theorem, one now applies the generalized argument principle[2] to evaluate the integral:

$$2\pi i \cdot \sum_{c\in Q} \mathrm{ord}_c(f) \cdot c = \int_{\partial Q} z \cdot \frac{f'(z)}{f(z)} dz. \tag{7}$$

Applying the same changes of variable as for the calculation of the integral (5), equation (7) yields to

$$\pm\left(\omega_1 \cdot \int_u^{u+\omega_2} \frac{f'(z)}{f(z)} dz - \omega_1 \cdot \int_u^{u+\omega_1} \frac{f'(z)}{f(z)} dz\right). \tag{8}$$

The double periodicity of f, together with the constant difference between any two branches of the complex logarithm ($2\pi i \cdot n, \exists n \in \mathbb{Z}$), yields to

$$\int_u^{u+\omega_i} \frac{f'(z)}{f(z)} dz \in 2\pi i \cdot \mathbb{Z} \text{ for } j = 1,2. \tag{9}$$

The statement finally follows from equations (8) and (9). □

Let $a_1, ..., a_r$ and $b_1, ..., b_r$ denote the zeros, respectively the poles of a nonconstant function $f \in \mathcal{K}(\Omega)$, on the basis of corollary 4.7. In both enumerations every point is repeated according to its multiplicity. Then, theorem 4.10 can be reformulated in the form

$$a_1 + \cdots + a_r \equiv b_1 + \cdots + b_r \mod \Omega, \tag{10}$$

which equation is sometimes called *Abel's relation*.

Theorem 4.2 consequently claims that every elliptic function of order zero is constant, while theorem 4.3 corresponds to the non-existence of an elliptic function of order one. On the other hand, the relation $r \geqslant 2$, together with equation (10), provide sufficient conditions for the existence of an elliptic function with prescribed zeros and poles.

[2]by setting $\phi(z) := f(z)$, $\psi(z) := z$, and $u := 0$ in equation (16)

5. Lattice invariants: the Eisenstein series

Reminder 5.1. *Let*

$$\sum_{g\in\mathbb{Z}^n} \alpha_g \text{ with } \alpha_g \in \mathbb{C}\,. \tag{11}$$

be an absolutely convergent n-dimensional (for $n > 1$) series. There consequently exists a constant $C > 0$, such that for every finite subset $E \subset \mathbb{Z}^n$, $\sum_{g\in E} |\alpha_g| < C$ holds. For every bijection $\phi : \mathbb{N} \to \mathbb{Z}^n$, the series $\sum_{k\in\mathbb{N}} \alpha_{\phi(k)}$ can unambiguously be written in the form (11), as it does not depend on the enumeration ϕ, according to Riemann's rearrangement theorem.

5.1. Convergence of series over a lattice.

Consider a fundamental parallelogram $Q := \operatorname{Par}(\omega_1, \omega_2)$ of the lattice $\Omega \subset \mathbb{C}$.

Definition 5.2. *The* diameter *of Q is $\delta := \delta(\omega_1, \omega_2) := \sup\{|z-u| : z, u \in Q\}$.*

Let $\overline{B_\rho(0)} := \{z \in \mathbb{C} : |z| \leqslant \rho\}$ denote the (closed) disc of radius $\rho > 0$, around zero. The number of lattice points inside this disc is written: $A_\rho(\Omega) := \left|\Omega \cap \overline{B_\rho(0)}\right|$.

Lemma 5.3. *For every radius $\rho \geqslant \delta$, the number of lattice points inside $\overline{B_\rho(0)}$ satisfies the following inequalities: $\frac{\pi}{\operatorname{vol}(\Omega)} \cdot (\rho-\delta)^2 \leqslant A_\rho(\Omega) \leqslant \frac{\pi}{\operatorname{vol}(\Omega)} \cdot (\rho+\delta)^2$.*

Proof. Let $M_\rho := \bigcup_{q\in\Omega:|q|\leqslant\rho} \operatorname{Par}(q; \omega_1, \omega_2)$ be the smallest area covering $\overline{B_\rho(0)}$, composed of period parallelograms. The two inclusions $\overline{B_{\rho-\delta}(0)} \subset M_\rho \subset \overline{B_{\rho+\delta}(0)}$ follow from the definitions of the three sets involved. The second inclusion also appeals to the definition of the diameter δ of the fundamental parallelogram Q.

The statement follows after translating the two inclusions of sets to inequalities between their respective areas: $\operatorname{area}(\overline{B_r(0)}) = \pi r^2$ (for $r := \rho - \delta,\ \rho + \delta$), and $\operatorname{area}(M_\rho) = \operatorname{vol}(\Omega) \cdot A_\rho(\Omega)$. □

Theorem 5.4 (Absolute convergence). *The series $\sum_{0\neq\omega\in\Omega} |\omega|^{-\alpha}$ converges for (and only for) $\alpha > 2$.*

Proof. The argument distinguishes three cases:

(5.1) $\alpha > 2$, which justifies the direct implication,
(5.2) $\alpha \leqslant 0$, for which the series[3] trivially diverges,
(5.3) and $0 < \alpha \leqslant 2$, which together with the previous case shows the converse.

For the case (5.1), let $\emptyset \neq E \subset \Omega - \{0\}$ be a finite set and $M := \max\{|\omega| : \omega \in E\}$. Consider the positive integer $\delta \leqslant n \in \mathbb{N}$. Lemma 5.3 provides the following estimate:

$$A_{n+1}(\Omega) - A_n(\Omega) \leqslant \frac{\pi}{\operatorname{vol}(\Omega)} \cdot \left((n+1+\delta)^2 - (n-\delta)^2\right) = \frac{\pi}{\operatorname{vol}(\Omega)} \cdot (2\delta+1) \cdot (2n+1) \leqslant c_2 \cdot n,$$

for the constant $c_2 = \frac{3\pi(2\delta+1)}{2\cdot\operatorname{vol}(\Omega)} > 0$ (for instance).

Defining the positive constant $c_1 := \sum_{\substack{0\neq\omega\in\Omega \\ |\omega|\leqslant\delta+1}} |\omega|^{-\alpha}$, the statement follows from the upper bound

$$\sum_{\omega\in E} |\omega|^{-\alpha} \leqslant c_1 + \sum_{\substack{n\in\mathbb{N} \\ \delta<n<M}} (A_{n+1}(\Omega) - A_n(\Omega)) \cdot n^{-\alpha} \leqslant c_1 + c_2 \cdot \sum_{n=1}^{\infty} n^{1-\alpha} < \infty,$$

since $1 - \alpha < -1$ has been assumed.

For the case (5.3), define $N \in \mathbb{N}$, $N > 2\delta$. From lemma 5.3, the following estimate holds for every integer $k \geqslant 2$:

$$A_{kN}(\Omega) - A_{(k-1)N}(\Omega) \geqslant \frac{\pi}{\operatorname{vol}(\Omega)} \cdot \left(2N^2 \cdot k - N(N+2\delta)\right) \geqslant c_3 \cdot k,$$

with the constant $c_3 := \frac{2\pi N^2}{\operatorname{vol}(\Omega)} > 0$ (for example).

For the finite set $E_m := \{\omega \in \Omega : 0 < |\omega| \leqslant mN\}$, one obtains the following lower bound:

$$\sum_{\omega\in E_m} |\omega|^{-\alpha} \geqslant \sum_{k=2}^{m} (A_{kN}(\Omega) - A_{(k-1)N}(\Omega)) \cdot (kN)^{-\alpha} \geqslant c_3 N^{-\alpha} \cdot \sum_{k=2}^{m} k^{1-\alpha}.$$

[3] of infinitely many positive terms

The last sum is bounded from below by the harmonic sum $\sum_{k=2}^{m} k^{-1}$, for $-1 \leqslant 1-\alpha$ holds. The statement for case (5.3) finally follows by allowing an arbitrarily large upper limit $m \to \infty$, noting that the harmonic series $\sum_{k=2}^{\infty} k^{-1}$ diverges. □

5.2. **The Eisenstein series: First properties.**

Definition 5.5. *The* Eisenstein series *(for a lattice Ω) are infinite sums of negative powers of the nonzero elements of the lattice:*

$$G_k := G_k(\Omega) := \sum_{0 \neq \omega \in \Omega} \omega^{-k} \text{ for } k \geqslant 3. \tag{12}$$

The parameter k is the weight *of the Eisenstein series.*

The convergence of the series (12) follows from theorem 5.4.

Corollary 5.6. *For any lattice Ω, the Eisenstein series $G_k(\Omega)$ (with $k \geqslant 3$) converge absolutely.*

Because $\omega \in \Omega \Leftrightarrow -\omega \in \Omega$ holds, Riemann's rearrangement theorem yields to $G_k = (-1)^k G_k$, and to the

Proposition 5.7. *For any lattice Ω, the Eisenstein series of odd weights k vanish: $G_k(\Omega) = 0$.*

Remark. On the other hand, all Eisenstein series of even weight $k \geqslant 4$ are nonzero. Besides, each such series can be expressed as a polynomial over the field of the rational numbers $\mathbb{Q}$ in the variables G_4 and G_6 (e.g. $7G_8 = 3G_4^2$ and $11G_{10} = 5G_4G_6$).

Remark. The regular part[4] of the Laurent expansion of the Weierstrass $\wp$-function (for a lattice Ω), around zero, has coefficients $a_m = m \cdot G_{m+1}(\Omega)$, for $m \geqslant 2$ even. All its terms of odd powers vanish, for the function is even.

5.3. **A natural elliptic function: First example.** For any $2 < k \in \mathbb{N}$, the series

$$f_k(z) = \sum_{\omega \in \Omega} (z-\omega)^{-k}$$

defines an of elliptic function wrt. the lattice Ω.

The absolute convergence of the Eisenstein series (see corollary 5.6) implies[5] the well-definiteness of the functions $f_k(z)$ at all $z \in \mathbb{C} - \Omega$.

The functions f_k are[6] periodic wrt. to Ω:

$$f_k(z+\widetilde{\omega}) = \sum_{\omega \in \Omega} (z+\widetilde{\omega}-\omega)^{-k} = \sum_{\omega \in \Omega - \widetilde{\omega}} (z-\omega)^{-k} = f(z), \text{ for any } \widetilde{\omega} \in \Omega, \tag{13}$$

since Ω is an additive subgroup (i.e. $\Omega - \widetilde{\omega} = \Omega$).

The doubly periodic functions f_k have poles of order k exactly at the lattice points. From equation (13), the principal part[7] of the Laurent expansion of f_k around a point $\widetilde{\omega} \in \Omega$ is indeed equal to $(z-\widetilde{\omega})^{-k}$. The sets of poles P_{f_k} are therefore discrete and periodic wrt. Ω. Because the functions f_k are holomorphic on $\mathbb{C} - \Omega$, $f_k \in \mathcal{K}(\Omega)$ finally follows for all $k \geqslant 3$.

6. Addendum: The argument principle

The argument principle is a consequence of the residue theorem. It allows for the evaluation of a generic integral in terms of two sums. There exists a generalized version of this theorem. The argument principle itself can then be viewed as a special case of the latter.

Definition 6.1 (winding number, or index)**.** *The* winding number *or* index *of a closed path γ intuitively counts the number of times the curve revolves around a given point $z \in \mathbb{C} - \mathrm{Im}(\gamma)$:*

$$\mathrm{ind}_z(\gamma) := \frac{1}{2\pi i} \cdot \int_\gamma \frac{du}{u-z}.$$

[4] i.e. the terms with non-negative powers

[5] One simply chooses the lattice $\Omega + z$ ($\forall z \in \mathbb{C}$) in the definition of the Eisenstein series: $f_k(z) = G_k(\Omega + z)$.

[6] essentially by definition

[7] i.e. the terms with negative powers

Definition 6.2 (simple closed path)**.** *A simple closed path γ has non-empty interior*[8] *($\mathrm{int}(\gamma) \neq \emptyset$), and the index of every point $z \in \mathrm{int}(\gamma)$ (i.e. inside the path) is one ($\mathrm{ind}_z(\gamma) = 1$).*

In particular, the boundary of a bounded domain corresponds to a simple closed path.

Theorem 6.3 (Residue theorem)**.** *Let γ be a null-homotopic path in $D \subset \mathbb{C}$, $A \subset D$ a finite set, which satisfies $A \cap \mathrm{Im}(\gamma) = \emptyset$, and $h \in \mathcal{H}(D - A)$. Then, it holds:*

$$\frac{1}{2\pi i} \cdot \int_\gamma h(z)dz = \sum_{c \in \mathrm{int}(\gamma)} \mathrm{ind}_c(\gamma) \cdot \mathrm{res}_c(h). \tag{14}$$

Remark. Because the residues of h in $D - A$ vanish, the sum on the right-hand side of (14) is finite.

Remark. In particular, theorem 6.3 provides an alternative definition of the residue a_{-1} of a meromorphic function h at a point $c \in \mathbb{C}$:

$$a_{-1} := \frac{1}{2\pi i} \cdot \int_\gamma h(z)dz,$$

in which γ is a positively oriented simple closed path around c, which includes no other singularity of f.

Theorem 6.4 (Generalized argument principle)**.** *Consider*

(6.1) a meromorphic function $\phi \neq 0$ in $D \subset \mathbb{C}$, with at most finitely many poles in D,
(6.2) a null-homotopic path γ, which contains no pole of ϕ,
(6.3) a complex number $u \notin \mathrm{Im}(\gamma)$, so that $\phi^{-1}(\{u\})$ is finite,
(6.4) and $\psi \in \mathcal{H}(D)$.

It then holds:

$$\frac{1}{2\pi i} \cdot \int_\gamma \psi(z) \cdot \frac{\phi'(z)}{\phi(z) - u} dz = \sum_{c \in \phi^{-1}(\{u\})} \mathrm{ind}_c(\gamma) \cdot \mathrm{ord}_c(\phi) \cdot \psi(c) + \sum_{d \in P_\phi} \mathrm{ind}_d(\gamma) \cdot \mathrm{ord}_d(\phi) \cdot \psi(d). \tag{15}$$

Remark. The first sum runs over the pre-images of u wrt. ϕ inside γ, the second one over its poles, enclosed by the path. Both sums are finite as a consequence of the identity theorem for meromorphic functions.

The proof is an application of the residue theorem to the function $\psi(z) \cdot \frac{\phi'(z)}{\phi(z)-u}$. Then, two cases must be treated separately, although in a similar fashion, whether

(6.1) ϕ is holomorphic at $c \in D$,
(6.2) or ϕ has a pole at c.

Example 6.5. *Considering a bounded domain D, γ its boundary (i.e. $\mathrm{ind}_z(\gamma) = 0$, $\forall z \in \mathbb{C} - D$), and setting $\psi(z) := z^n$ ($n \in \mathbb{N}$), equation* (15) *yields to the formula:*

$$\frac{1}{2\pi i} \cdot \int_\gamma z^n \cdot \frac{\phi'(z)}{\phi(z)} dz = \sum_{c \in (Z_\phi \cup P_\phi) \cap \mathrm{int}(\gamma)} c^n \cdot \mathrm{ord}_\phi(c). \tag{16}$$

The argument principle finally relates the number of poles and zeros (with multiplicities) of a given meromorphic function, contained in a simple closed path, with the curvilinear integral of a specific derived function along this path.

Theorem 6.6 (Argument principle)**.** *By setting $\psi(z) := 1$ and $u := 0$, the generalized argument principle reduces to:*

$$\frac{1}{2\pi i} \cdot \int_\gamma \frac{\phi'(z)}{\phi(z)} dz = \mathrm{num}_0(\phi, \mathrm{int}(\gamma)) - \mathrm{num}_\infty(\phi, \mathrm{int}(\gamma)). \tag{17}$$

Alternatively, equation (17) follows more directly from formula (16) in the previous example, by setting $n := 0$.

[8] The interior of the closed path γ is defined as $\mathrm{int}(\gamma) := \{z \in \mathbb{C} - \mathrm{Im}(\gamma) : \mathrm{ind}_z(\gamma) \neq 0\}$.

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