Skript Weierstrass Function

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Outline

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1 Definition and motivation

1.1 Motivation

We are given a lattice Ω and our goal is to find a meromorphic function such that :

- It is elliptic with respect to Ω.
- It has poles of order 2 at every point in the lattice.

Very intuitively one could think that we could just define it as :

$$\sum_{\omega \in \Omega} (z-w)^{-2}$$

However this serie is actually not absolutely convergent.

Remark 1. Although the above serie does not converge, for every $k \geqslant 3,\ k \in \mathbb{N}$, the similar series :

$$\sum_{\omega \in \Omega} (z-w)^{-k}$$

do converge absolutely and uniformly on every compact subset of $\mathbb{C} \setminus \Omega$

Remark 2. *(Eisenstein Series)* We define the Eisenstein Series as

$$G_k = G_k(\Omega) = \sum_{0 \neq \omega \in \Omega} \omega^{-k}$$

It is absolutely convergent for every whole $k \geqslant 3$.

1.2 Weierstrass function

To circumvent the problem of convergence the remedy is simple, we renormalise our series:

Definition 1.1. *We define our function for $z \in \mathbb{C} \setminus \Omega$ as :*

$$\wp(z) := \wp_\Omega(z) := z^{-2} + \sum_{0 \neq \omega \in \Omega} (z-w)^{-2} - w^{-2}$$

This is the so-called **Weierstrass $\wp$ function**.

We will now show and demonstrate multiple important properties of our function :

Theorem 1.2. *For any grid Ω, our function*

$$\wp(z) = z^{-2} + \sum_{0 \neq \omega \in \Omega} (z-w)^{-2} - w^{-2},\ z \in \mathbb{C} \setminus \Omega$$

converges absolutely and uniformly in every compact subset of $\mathbb{C} \setminus \Omega$ and is thus well-defined.

Proof. Let $K \subseteq \mathbb{C} \setminus \Omega$ be compact and choose $\rho > 0$ such that $K \subseteq B_\rho = \{x \in \mathbb{C} \mid |x| \leqslant \rho\}$. We know that

$$\sum_{\substack{0 \neq \omega \in \Omega: \\ |\omega| < \rho+1}} (z-w)^{-2} - w^{-2}$$

converges absolutely and locally uniformly as it is a finite sum. Thus we may now assume $|w| \geqslant \rho + 1$. Take $z \in K_\rho \backslash \Omega$:

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| = \left| \frac{z^2 - 2\omega z}{(z-\omega)^2 \omega^2} \right| = \left| \frac{2 - \frac{z}{\omega}}{(1 - \frac{z}{\omega})^2} \right| \cdot \frac{|z|}{|\omega^3|} \leqslant \frac{2 + \frac{\rho}{\rho+1}}{(1 - \frac{\rho}{\rho+1})^2} \cdot \frac{\rho}{|\omega|^3}$$

From the convergence of the Eisenstein serie G_3 we conclude the proof. ☐

Theorem 1.3. *$\wp$ is meromorphic on $\mathbb{C}$ and has poles precisely at the lattice points Ω, which are of order 2 and have residue 0.*

Proof. Let $\rho > 0$ and write $\forall z \in K_\rho \backslash \Omega$:

$$\wp(z) = z^{-2} + \sum_{\substack{0 \neq \omega \in \Omega: \\ |\omega| < \rho+1}} \left[(z-w)^{-2} - w^{-2}\right] + \sum_{\substack{0 \neq \omega \in \Omega: \\ |\omega| \geqslant \rho+1}} \left[(z-w)^{-2} - w^{-2}\right]$$

The first part is meromorphic on K_ρ with poles of second order and residue 0 at lattice points in K_ρ. The second part is holomorphic on K_ρ. ☐

Theorem 1.4. *$\wp$ is even and has Laurent expansion of the form :*

$$\wp(z) = z^{-2} + \sum_{j=1}^{\infty} a_{2j} \ z^{2j}$$

Proof. We replace ω by $-\omega$ in the sum and use the absolute convergence of the series to see that $\wp(z) = \wp(-z)$. Above we have shown that $\wp$ has a pole of second order with residue 0 at $z = 0$, so it has the Laurent expansion :

$$\wp(z) = z^{-2} + a_0 + a_1 z^2 + ...$$

but we have that $(z-w)^{-2} - w^{-2} = 0$ for $z = 0, \ w \neq 0$. Thus we get as wanted $a_0 = 0$. ☐

Theorem 1.5. *$\wp$ is elliptic*

Proof. By the absolute and locally uniform convergence of the series defining $\wp$ we can differentiate $\wp$ termwise and get:

$$\forall z \in \mathbb{C} \backslash \Omega : \ \wp'(z) = -2 \cdot \sum_{\omega \in \Omega} (z-\omega)^{-3}$$

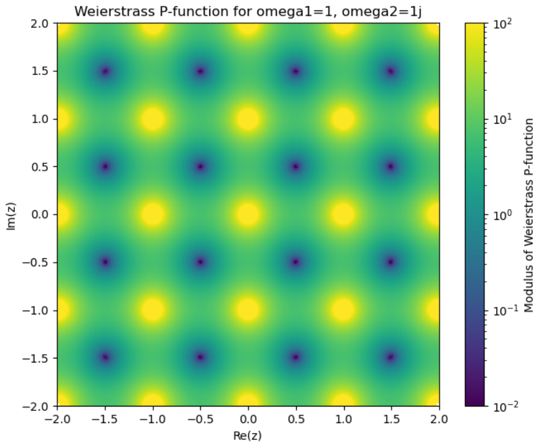
Thus $\forall \omega \in \Omega : \ \wp'(z+\omega) = \wp'(z)$. This implies:

$$\forall \omega \in \Omega \ \exists C_\omega : \wp(z+\omega) = \wp(z) + C_\omega$$

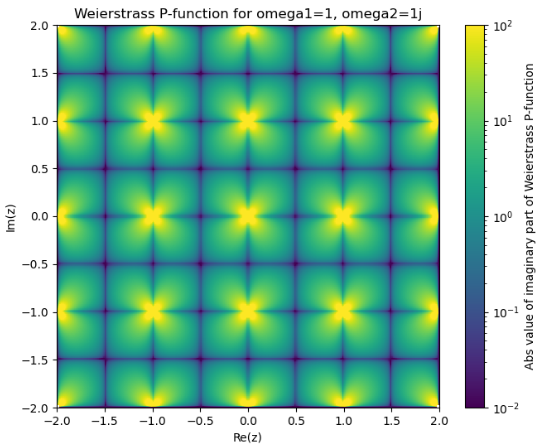
But setting $z = -\frac{\omega}{2}$ yields: $\wp(-\frac{\omega}{2}) = \wp(\frac{\omega}{2}) = \wp(-\frac{\omega}{2}) + C_w$, so $\forall \omega \in \Omega : \ C_\omega = 0$. Hence $\forall \omega \in \Omega$: $\wp(z+\omega) = \wp(z)$. ☐

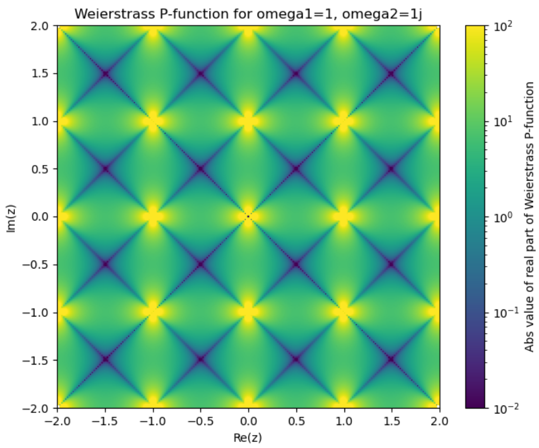
1.3 Illustrations

Here are some illustrations of our $\wp$ function using a heat map:



We have here used a lattice $\Omega = \mathbb{Z} + i\mathbb{Z}$. We can clearly see the poles in yellow right where we wanted them and the elliptic behavior of the function.
To have some more insight into the symmetry involved in this function here are the heat map of both the imaginary and real part of our function :





1.4 The derivative of the Weierstrass function

We will know inspect the properties of its derivative.

Proposition 1.6. *$\wp'$ is an odd elliptic function with poles of order 3 at lattice points and is holomorphic everywhere else.*

Proof. We already have seen in the last proof that we can have the expression :

$$\wp(z)' = -2 \cdot \sum_{\omega \in \Omega} (z - \omega)^{-3}$$

The proposition follows. □

Proposition 1.7. *The roots of $\wp'$ are of order 1 and lie precisely at the points $\frac{\omega}{2}$ for which $\omega \in \Omega, \frac{\omega}{2} \notin \Omega$*

Proof. Since $\wp'$ is an odd elliptic function:

$$\wp'(z + \omega) = \wp'(z) = -\wp'(-z) \text{ for } \omega \in \Omega$$

If $\frac{\omega}{2} \notin \Omega$, then $\frac{\omega}{2}$ is not a pole of $\wp'$ and we can take $z = -\frac{\omega}{2}$ to get:

$$\wp'(\frac{\omega}{2}) = -\wp'(\frac{\omega}{2})$$

which implies $\wp'(\frac{\omega}{2}) = 0$. Let ω_1, ω_2 be a basis of Ω and let $P = P(\omega_1, \omega_2)$. Then $\wp'$ has at least the roots $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ in P. From the Laurent expansion $\wp(z) = z^{-2} + a_2 z^2 + ...$ of $\wp$ around 0, we see that $\wp'$ has precisely one pole in P and it is of order 3.

We need the following theorem :

For $f \in \mathcal{K}(\Omega)$ non constant, any fundamental parallelogram P for Ω and any $x \in \mathbb{C}$, we have

$$\sum_{c \in P} \text{ord}_c(f - x) = 0$$

By the theorem above, $\wp'$ has precisely 3 roots in P and they must have order 1.For an arbitrary $z \in \mathbb{C}$, we find $\omega' \in \Omega$, s.t. $z - \omega' \in P$. If z is a root of $\wp'$, then $z - \omega'$ is one of the roots in P, so z is of the form $\frac{\omega}{2}$ with $\omega \in \Omega$ but $\frac{\omega}{2} \notin \Omega$. □ □

We will now just go through some further properties :

Let $P = P(\omega_1, \omega_2)$ be a fundamental parallelogram for Ω and set

$$e_1 := \wp(\frac{\omega_1}{2}), e_2 := \wp(\frac{\omega_2}{2}), e_3 := \wp(\frac{\omega_3}{2}), \text{ where } \omega_3 = \omega_1 + \omega_2$$

.

Lemma 1.8. *We have:*

- *$\wp(z) - e_k$ has precisely one double root in P at $z = \frac{\omega_k}{2}$*
- *$\wp(z) - x$ has precisely two simple roots in P if $x \notin \{e_1, e_2, e_3\}$*

2 The field of elliptic functions

We will now discuss the important role that the Weierstrass $\wp$ function and its derivative $\wp'$ play in the field of elliptic functions $\mathcal{K}(\Omega)$. More precisely, we will show how $\wp$ and $\wp'$ can be used to generate the whole field of elliptic functions. We have a proposition :

Proposition 2.1. *We have a for a given lattice Ω :*

(1) The even elliptic functions in $\mathcal{K}(\Omega)$ are precisely the rational functions in $\wp$.

(2) $\mathcal{K}(\Omega) = \mathbb{C}(\wp)[\wp']$

(3) $dim_{\mathbb{C}(\wp)}(\mathcal{K}(\Omega)) = 2$

So in particular every $f \in \mathcal{K}(\Omega)$ can be written in a unique way as $f = R(\wp) + Q(\wp) \cdot \wp'$ with rational functions R, Q over $\mathbb{C}$.

We will need this little lemma for the proof :

Lemma 2.2. *$\forall m \in \mathbb{N}_0\ \exists! \wp_m \in \mathcal{K}(\Omega)$ which has poles of order $2m$ at the lattice points Ω, is holomorphic elsewhere and has Laurent expansion at 0 of the shape $\wp_m(z) = z^{-2m} + O(z^2)$. Moreover, $\wp_m$ is a polynomial in $\wp$.*

Proof.

Existence:
Set $\varphi_0 = 1, \varphi_1 = \wp$ For $m = 2$ consider the Laurent expansion of $\wp^2$ at $z = 0$:

$$\wp^2(z) = (z^{-2} + a_2 z^2 + O(z^4))^2 = z^{-4} + 2a_2 + O(z^2)$$

So we set $\wp_2 := \wp^2 - 2a_2$.
We can do this $\forall m$ and define $\wp_m$ recursively.

Uniqueness :
Assume $\wp_m, \tilde{\wp}_m$ both satisfy all of the conditions above. Then $f := \varphi_m - \tilde{\varphi}_m \in \mathcal{K}(\Omega)$ and is holomorphic on $\mathbb{C}$ and thus by Liouville constant.
But f is also vanishing at $z = 0$ and hence equal to 0. □

Proof. (of proposition 2.1)
(1) Let $f \in \mathcal{K}(\Omega)$ be an even function and $c_1, ..., c_k$ be the poles of f in the fundamental parallelogram P which do not already lie in Ω.
Then $g(z) := f(z) \cdot \prod_{j=1}^{k} (\wp(z) - \wp(c_j))^{-\mathrm{ord}_{c_j}(f)}$ is an even elliptic function with poles only at the lattice points in Ω.
At $z = 0$ we get the Laurent expansion:

$$g(z) = a_{-2d} z^{-2d} + a_{-2d+2} z^{-2d+2} + ... + a_{-2} z^{-2} + a_0 + O(z^2)$$

with $a_j \in \mathbb{C}$ and $-2d = \mathrm{ord}_0(g)$ using g is even.
But $g_j(z) := a_{-2j} z^{-2j} - a_{-2j} \wp_j(z)$ is an entire elliptic function and $g_j(0) = 0$, so by Liouvilles theorem $g_j = 0\ \forall j$ Hence

$$g(z) = \sum_{j=0}^{d} a_{-2j} \wp_j(z)$$

Hence, we find:

$$f(z) = \left(\sum_{j=0}^{d} a_{-2j} \varphi_j(z) \right) \cdot \left(\prod_{j=1}^{k} (\wp(z) - \wp(c_j))^{-\mathrm{ord}_{c_j}(f)} \right)$$

Since, all φ_j's are polynomial in $\wp$, we find f is a rational function in $\wp$. ✓

(2) For $f \in \mathcal{K}(\Omega)$ we may write: $f = g + h\wp'$, where $g(z) = \frac{1}{2}(f(z) + f(-z))$ and $h(z) = \frac{1}{2\wp'(z)}(f(z) - f(-z))$.
Since h and g are even elliptic functions, they are rational functions in $\wp$ by (1) ✓

(3) Since $\wp'$ is odd, we have $\wp' \notin \mathbb{C}(\wp)$. But we have $\wp'^2 \in \mathbb{C}(\wp)$, so the degree is 2. ✓ □

3 Eisenstein series and Laurent Expansion

We also define $\gamma := \gamma(\Omega) := \min\{|w| : 0 \neq w \in \Omega\}$, which represent the magnitude of the lattice point which is the closest to the origin.

Proposition 3.1. *For $z \in \mathbb{C}$ with $0 < |z| < \gamma(\Omega)$ and G_k the Eisenstein serie we have :*

$$\wp(z) = z^{-2} + \sum_{n=2}^{\infty} (2n-1)G_{2n} \cdot z^{2n-2} = z^{-2} + 3G_4 \cdot z^2 + 5G_6 \cdot z^4 + ...$$

Proof. We use the little trick :

$$\frac{1}{(1-t)^2} = \frac{d}{dt}(\frac{1}{1-t}) = \frac{d}{dt}\sum_{m=0}^{\infty} t^m = \sum_{m=0}^{\infty} mt^{m-1}, \ |t| < 1.$$

Which gives us after some calculations :

$$\wp(z) = z^{-2} + \sum_{0 \neq w \in \Omega} (\sum_{m=2}^{\infty} m\frac{z^{m-1}}{w^{m+1}}), \ (|z| < \gamma(\Omega))$$

Since this double series converge absolutely we can switch the summation signs and get :

$$\wp(z) = z^{-2} + \sum_{m=2}^{\infty} mG_{m+1} \cdot z^{m-1}$$

We remember that $G_k = 0$ for odd k and we are finished. □

4 Differential Equations

We will now explore some differential equations involving our Weierstrass function :

Proposition 4.1. *(First Differential Equation) For $z \in \mathbb{C} \setminus \Omega$ we have :*

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

.

Proof. Consider $f(z) := 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$

- It has double roots precisely at the points $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}$ and by one of the previous propositions the same is true for $\wp'(z)^2$.
- Moreover, the only pole of f in P is at 0 and is of order 6. From the Laurent expansions $\wp(z) = z^{-2} + ...$, $\wp'(z)^2 = 4z^{-6} + ...$, we see that the same is true for $\wp'(z)^2$.
- Thus, $\frac{\wp'(z)^2}{f(z)}$ is an elliptic function without poles and hence constant by Thm. 3.2.1. (Liouville) So $\wp'(z)^2 = c \cdot f(z)$ and using the Laurent expansions, we see $c = 1$.

□

Proposition 4.2. *(Second Differential Equation) For $z \in \mathbb{C} \setminus \Omega$ we have :*

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

where $g_2 := g_2(\Omega) := 60G_4(\Omega)$ and $g_3 := g_3(\Omega) := 140G_6(\Omega)$

Proof. We use the Laurent expansion found earlier :

$$\wp(z) = z^{-2} + 3G_4 \cdot z^2 + 5G_6 \cdot z^4 + O(z^6)$$

and compute $\wp^2$, $\wp^3$, $\wp', \wp'^{\,2}$ according to the expansion.
We get from the previous computations : $\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_3 = O(z)$. The left side is elliptic and can only have poles on lattice. But the full equation shows that the left side is also holomorphic at 0, hence everywhere. This implies that it is actually constant by Liouville's Theorem and the constant is 0. □

5 Eisenstein Series Identities

Using the result :

Lemma 5.1. *For $n \geqslant 4$, we have the recursion :*

$$(n-3)(2n+1)(2n-1)G_{2n} = 3 \sum_{\substack{p \geqslant 2, q \geqslant 2 \\ p+q=n}} (2p-1)(2q-1)G_{2p}G_{2q}$$

We get the identities :

- $7G_8 = 3G_4^2$
- $11G_{10} = 5G_4G_6$
- $143G_{12} = 42G_4G_8 + 25G_6^2 = 13G_4^3 + 25G_6^2$

We then see from the above recursions that every G_k can be written as a polynomial over $\mathbb{Q}$ in G_4 and G_6

6 Application for physists

Recall the pendulum:

- $\theta(t)'' = 4c \sin\theta(t)$, where $c = -\frac{g}{4 \cdot l}$
- The conserved energy: $E = \frac{(\theta')^2}{2} + 4c\cos\theta$

Using $u = e^{i\theta}$ one finds:

$$u'' = 4c(u - \frac{E}{8c})^2 - \frac{E^2}{16c} + 4c$$

Then we substitute the time $z = \sqrt{\frac{3}{2c}} \cdot t$ and $p = \sqrt{\frac{2c}{3}}(u - \frac{E}{8c})$ to find:

$$p'' = 6p^2 - \frac{g_2}{2},$$

where $g_2 := \frac{E^2}{8c} - 8c$. $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$, where $g_2 := g_2(\Omega) := 60G_4(\Omega)$ and $g_3 := g_3(\Omega) := 140G_6(\Omega)$ Differentiating the differential equation above and cancelling out the $\wp'$s yields:

$$\wp''(z) = 6\wp^2 - \frac{g_2}{2}$$

This equation is solved by $p(z) = \wp(z + g_1, g_2, g_3)$. We can use the initial conditions $(\theta(0), \theta'(0))$ to find g_1 and g_3.
Then we find the form of our solution:

$$\theta(t) = \arg(a + b\wp(qt + g_1, g_2, g_3))$$

where a,b,q are constants depending on c and g_1, g_2, g_3 depend on $\theta(0), \theta'(0)$ and c.