WEIERSTRASS FUNCTIONS

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Abstract. In this paper, we will discuss the Weierstrass ζ and σ-functions and their connection with the $\wp$-function in order to prove 'Abel's Theorem on the existence of elliptic functions with prescribed zeros and poles'.
As an alternative construction method for elliptic functions we will present the Jacobi theta function, which also has numerous other applications in mathematics and physics.

For the whole paper $\Omega \subseteq \mathbb{C}$ is a lattice:

Definition 0.1. Let $V = \mathbb{R}^n$, with $n \geqslant 1$. A subset $\Omega \subseteq V$ is called a *lattice in* V if there exists an $\mathbb{R}-$basis $(w_1, \ldots, w_n)$ of V such that

$$\Omega = \mathbb{Z}w_1 + \ldots + \mathbb{Z}w_n.$$

We also call $(w_1, \ldots, w_n)$ a basis of Ω.

1. Convergence of Infinite Products

In this short chapter, we will introduce some important facts about *infinite products*, which we will further need to define the σ-function.
Let $a_1, a_2, a_3, \ldots$ be a sequence of complex numbers that converges to 0. Then by definition of convergence, there exists $N \in \mathbb{N}$ such that $|a_n| < 1$ for every $n \geqslant N$. We define the infinite product of the sequence $(1 + a_n)_{n\in\mathbb{N}}$ as

$$\prod_{n=1}^{\infty}(1 + a_n) := (1 + a_1) \cdots (1 + a_N) \exp\left(\sum_{n=N+1}^{\infty} \log(1 + a_n)\right)$$

Since we work with complex numbers, it is important to specify that the above definition uses the principal branch of the logarithm.
We say that the infinite product *converges absolutely* if $\sum_{n=N+1}^{\infty} |a_n|$ converges. In this case, the remaining sum $\sum_{n=N+1}^{\infty} \log(1 + a_n)$ converges absolutely. Moreover, an infinite product that converges is equal 0 if and only if there is an $n \in \mathbb{N}$ with $(1 + a_n) = 0$.

If $f_1, f_2, \ldots$ is a sequence of holomorphic functions on some domain $D \subseteq \mathbb{C}$, we say that $\prod_{n=1}^{\infty}(1 + f_n)$ *converges absolutely* and *locally uniformly* if the series $\sum_{n=1}^{\infty} f_n$ converges absolutely and locally uniformly. In this case, the infinite product $\prod_{n=1}^{\infty}(1 + f_n)$ defines a holomorphic function on D.

2. The three Weierstrass functions: σ, ζ and η.

Proposition 2.1 (*Weierstrass σ-function*). *For $z \in \mathbb{C}$ the Weierstrass σ-function*

$$\sigma(z) := \sigma(z; \Omega) := z \prod_{0\neq\omega\in\Omega} \left(1 - \frac{z}{w}\right) e^{\frac{z}{w} + \frac{1}{2}(\frac{z}{w})^2}$$

converges absolutely and uniformly on every compact subset of $\mathbb{C}$ and hence defines an entire function. It has zeros of order one precisely at the points in Ω. Moreover, $\sigma(z)$ is an odd funciton.

To prove this statement we recall a known Lemma.

Lemma 2.2. *Let $\alpha \in \mathbb{R}$ The series*

$$\sum_{0\neq w\in\Omega} |w|^{-\alpha}$$

converges if and only if $\alpha > 2$.

Now, let us begin the proof of the Proposition 2.1.

Proof of Proposition 2.1. We divide the proof into three parts:

- *Convergence*: Let $K \subseteq \mathbb{C}$ be a compact set. We will do a computation using the Series representation of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, which we use in the first equality.

$$\left|1-\left(1-\frac{z}{w}\right)e^{\frac{z}{w}+\frac{1}{2}(\frac{z}{w})^2}\right| = \left|\frac{1}{8}\left(\frac{z}{w}\right)^3\left(\left(\frac{z}{w}\right)^2+3\left(\frac{z}{w}\right)+4\right)+\sum_{n=3}^{\infty}\frac{(\frac{z}{w}+\frac{1}{2}(\frac{z}{w})^2)^n}{n!}\right|$$
$$\leqslant C_K|w|^{-3}$$

  where C_K is a constant that depends only on the compact set, which exists by the boundedness of K. Moreover, $\sum_{w\neq 0}|w|^{-3}$ is a convergent series by Lemma 2.2, so by the above introduction of infinite products we immediately get that it converges absolutely and locally uniformly.

- *Zeros of σ*: The σ-function vanishes if and only if $z = 0$ or one of the factors $(1-\frac{z}{w})e^{\frac{z}{w}+\frac{1}{2}(\frac{z}{w})^2}$ vanishes for some $0 \neq w \in \Omega$. This happens only when $z = w$. From this reasoning, it directly follows by its definition that σ has zeros of order 1 exactly at the lattice points in Ω.

- *σ is odd*: By replacing z with $-z$ and w with $-w$ in the infinite product we get:

$$\sigma(-z) = -z\prod_{0\neq w\in\Omega}\left(1-\frac{-z}{-w}\right)e^{\frac{-z}{-w}+\frac{1}{2}(\frac{-z}{-w})^2} = -z\prod_{0\neq\omega\in\Omega}\left(1-\frac{z}{w}\right)e^{\frac{z}{w}+\frac{1}{2}(\frac{z}{w})^2} =$$
$$= -\sigma(z)$$

  From which we can conclude that σ is odd.

□

Proposition 2.3. *For $z \in \mathbb{C}\setminus\Omega$ the Weierstrass ζ-function is given by:*

$$\zeta(z) := \zeta(z;\Omega) := \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{0\neq w\in\Omega}\left(\frac{1}{z-w}+\frac{1}{w}+\frac{z}{w^2}\right).$$

This function converges absolutely and uniformly on every compact subset of $\mathbb{C}\setminus\Omega$. It has poles of first order and residue 1 precisely at the points in Ω and it is an odd function.

Proof. Three aspects need to be shown.

- *Covergence*: Let $K \subseteq \mathbb{C}\setminus\Omega$. We estimate for $z \in K$:

$$\left|\frac{1}{z-w}+\frac{1}{w}+\frac{z}{w^2}\right| = \left|\frac{z^2}{w^2(z-w)}\right| \leqslant C_K|w|^{-3},$$

the constant C_K only depends on K and exists because of boundedness of K. Then again by Lemma 2.2 the sum $\sum_{0\neq w\in\Omega}|w|^{-3}$ converges absolutely and uniformly on K.

- *Poles of* ζ: It is clear from the definition of ζ, that the poles are exactly at the lattice points and of order and residue one.
- ζ *is odd*: From Proposition 2.1 it is known that the σ-function is an odd function, from which follows that $\sigma'(z)$ is even. By using the definition $\zeta(z) := \frac{\sigma'(z)}{\sigma(z)}$ we have
$$\zeta(-z) = \frac{\sigma'(-z)}{\sigma(-z)} = \frac{\sigma'(z)}{-\sigma(z)} = \sigma(z),$$
which ends the proof.

□

It is now beautiful to discover the connection between the ζ- function and the $\wp$-function. First, we need to recall the $\wp$-function and some of its properties. Secondly, it is important to recall the following definition of the Eisenstein series:

Definition 2.4 (Eisentein Series)**.** Let $k \in \mathbb{N}$ with $k \geqslant 3$, then
$$G_k := G_k(\Omega) := \sum_{0\neq w\in\Omega} w^{-k} \tag{1}$$
defines the *Eisenstein Series*.

Theorem 2.5. *The Weierstrass* $\wp$*- function*
$$\wp(z) := \wp_\Omega(z) := \frac{1}{z^2} + \sum_{0\neq w\in\Omega}\left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right), \quad z \in \mathbb{C}\setminus\Omega \tag{2}$$
converges absolutely and uniformly in every compact subset of $\mathbb{C}\setminus\Omega$*. Moreover, it is an even elliptic function with respect to* Ω *and has poles of second order with residue* 0 *in every lattice point of* Ω*. The Laurent expansion at* 0 *has the form:*
$$\wp(z) = z^{-2} + a_2 z^2, \dots.$$

Corollary 2.6. *For* $z \in \mathbb{C}\setminus\Omega$ *we have*
$$\zeta'(z) = -\wp(z)$$

Proof. To do the comparison let us consider
$$\zeta'(z) = -\frac{1}{z^{-2}} + \sum_{0\neq w\in\Omega}\left(-\frac{1}{(z-w)^2} + \frac{1}{w^2}\right).$$
Then by direct comparison with $\wp(z)$ (2), it can be seen that $\zeta(z) = -\wp(z)$. □

Corollary 2.7. *We have the Laurent expansion*
$$\zeta(z;\Omega) = \frac{1}{z} + \sum_{k=2}^{\infty} G_{2k}(\Omega) z^{2k-1}$$
around $z = 0$.

Proof. First note that we have
$$\frac{1}{1-t} = \sum_{m=0}^{\infty} t^m, \quad (|t| < 1) \tag{3}$$

Hence for $0 \neq w \in \Omega$ we may write

$$\frac{1}{z-w}+\frac{1}{w}+\frac{z}{w^2}=\frac{1}{w}\left(1-\frac{1}{1-\frac{z}{w}}+\frac{z}{w}\right)\overset{(3)}{=}\frac{1}{w}\left(1-\sum_{m=0}^{\infty}\left(\frac{z}{w}\right)^m+\frac{z}{w}\right)=$$
$$=\frac{1}{w}\left(-\sum_{m=2}^{\infty}\left(\frac{z}{w}\right)^m\right)=-\sum_{m=2}^{\infty}\left(\frac{z^m}{w^{m+1}}\right),\quad (|z|<\gamma)$$

and thus we get

$$\zeta(z)=\frac{1}{z}+\sum_{0\neq w\in\Omega}\left(-\sum_{m=2}^{\infty}\left(\frac{z^m}{w^{m+1}}\right)\right),\quad (0<|z|<\gamma). \tag{4}$$

Since

$$\left|\left(\frac{z^m}{w^{m+1}}\right)\right|\leqslant |w|^{-3}\left(\frac{|z|}{w}\right)^m\gamma^2 \tag{5}$$

we see from the estimation in (5) that the double series in (4) has to converge absolutely by Lemma 2.2. Hence, we change the order of the summation and obtain

$$\zeta(z)=\frac{1}{z}-\sum_{m=2}^{\infty}\left(\sum_{0\neq w\in\Omega}\left(\frac{1}{w^{m+1}}\right)z^m\right)=\frac{1}{z}-\sum_{m=2}^{\infty}\left(\sum_{0\neq w\in\Omega}G_{m+1}(\Omega)z^m\right)=$$
$$=\frac{1}{z}-\sum_{m=2}^{\infty}G_{m+1}(\Omega)z^m=\frac{1}{z}-\sum_{k=2}^{\infty}G_{2k}(\Omega)z^{2k-1}$$

for $0<|z|<\gamma$. Recall that $G_k=0$ for odd k, which gives the stated Laurent expansion. □

The ζ-function is not elliptic. However, we have the following result.

Lemma 2.8. *For $w\in\Omega$ the Weierstrass η-function*

$$\eta(w):=\eta(w;\Omega):=\zeta(z+w)-\zeta(z)$$

is independent of the choice of $z\in\mathbb{C}\setminus\Omega$. In particular, we have

$$\eta(w+w')=\eta(w)+\eta(w'),\quad w,w'\in\Omega$$

that is, $\eta:\Omega\to\mathbb{C}$ is a group homomorphism.

Proof. We know by Theorem 2.5 that $\wp$ is an elliptic function, thus we find

$$(\zeta(z+w)-\zeta(z))'=-\wp(z+w)+\wp(z)=0$$

by this equality we then see that $\zeta(z+w)-\zeta(z)$ is independent of z. We can now compute:

$$\eta(w+w')=\zeta(z+w+w')-\zeta(z)=\zeta(z+w)-\zeta(z+w)+\zeta(z+w+w')-\zeta(z)=$$
$$=(\zeta(z+w)-\zeta(z))+(\zeta((z+w)+w')-\zeta(z+w))=\eta(w)+\eta(w').$$

By this last equality, we get that $\eta:\Omega\to\mathbb{C}$ is a group homomorphism. □

We conclude this chapter by showing a nice property of the η-function, which is called the *Legendre Relation*. But before proving it, we would like to recall the definition of a *fundamental parallelogram*.

Definition 2.9. Let $\Omega \subseteq \mathbb{C}$ be a lattice and let (w_1, w_2) be a basis pf Ω. For $u \in \mathbb{C}$ we define the *fundamental parallelogram* w.r.t (w_1, w_2) and base point u by

$$P(u; w_1, w_2) = \{u + \alpha_1 w_1 + \alpha_2 w_2 : \alpha_1, \alpha_2 \in [0, 1)\}$$

For $u = 0$ we also write $P(w_1, w_2) = P(0; w_1, w_2)$

Proposition 2.10 (*Legendre Relation*)**.** *Let $\Omega = \mathbb{Z}w_1 + \mathbb{Z}w_2$ with $Im(\frac{w_1}{w_2}) > 0$. Then we have:*

$$\eta(w_2)w_1 - \eta(w_1)w_2 = 2\pi i$$

In particular, for $w, w' \in \Omega$ we have

$$\eta(w)w' - \eta(w')w \in 2\pi i\mathbb{Z}.$$

Proof. Let $P = P(u; w_1, w_2)$ be a fundamental parallelogram, where the base point $u \in \mathbb{C}$ is chosen in a way that 0 is still contained in the interior of $P(u; w_1, w_2)$. We now analyze the integral of $\zeta(z)$ over the positively oriented boundary of P and using the residue theorem we get that

$$\int_{\partial P} \zeta(z)dz = 2\pi i,$$

since the function $\zeta(z)$ has only one pole of first order and residue 1 in P, which is exactly at $z = 0$. At the same time, we can split our integral as follows:

$$\begin{aligned}\int_{\partial P} \zeta(z)dz &= \int_{u}^{u+w_2} \zeta(z)dz + \int_{u+w_2}^{u+w_1+w_2} \zeta(z)dz + \int_{u+w_1+w_2}^{u+w_1} \zeta(z)dz + \int_{u+w_1}^{u} \zeta(z)dz = \\ &= \int_{u}^{u+w_2} (\zeta(z) - \zeta(z + w_1))dz + \int_{u+w_1}^{u} (\zeta(z) - \zeta(z + w_2))dz = \\ &= \eta(w_1)w_2 - \eta(w_2)w_1,\end{aligned}$$

where the last equality follows directly by the definition of the η-function. Moreover, we used that the parallelogram has positive orientation.
To conclude we know that $\eta : \Omega \to \mathbb{C}$ is a group homomorphism, from which we get that for each $w, w' \in \Omega$

$$\eta(w)w' - \eta(w')w \in 2\pi i\mathbb{Z}$$

□

Remark 1. It follows as a consequence of the above-discussed propositions, that for $0 \neq \lambda \in \mathbb{C}$ it holds:

$$\begin{aligned}\sigma(\lambda z; \lambda\Omega) &= \lambda\sigma(z; \Omega), \\ \zeta(\lambda w; \lambda\Omega) &= \frac{1}{\lambda}\zeta(z; \Omega), \\ \eta(\lambda w; \lambda\Omega) &= \frac{1}{\lambda}\eta(w; \Omega).\end{aligned}$$

3. The Transformation Law for σ

Recall that $\sigma(z;\Omega)$ is *not* an elliptic function since it is entire and non-constant. Nonetheless, it satisfies an interesting transformation law which associates $\sigma(z+w)$ to $\sigma(z)$ for $w \in \Omega$ and $z \in \mathbb{C}$.

Theorem 3.1. *For $w \in \Omega$ and $z \in \mathbb{C}$ we have*

$$\sigma(z+w) = \chi(w) e^{\eta(w)(z+\frac{w}{2})} \sigma(z), \tag{6}$$

where

$$\chi(w) = \begin{cases} 1 & \text{if } \frac{w}{2} \in \Omega, \\ -1 & \text{if } \frac{w}{2} \notin \Omega. \end{cases}$$

Proof. Since σ vanishes at every point in Ω, the above theorem trivially holds in that case. Let us now assume that $z \notin \Omega$. By the definition of the ζ-function we have $\sigma'(z) = \sigma(z)\zeta(z)$. We thus get:

$$\begin{aligned} \frac{d}{dz}\left(\frac{\sigma(z+w)}{\sigma(z)}\right) &= \frac{\sigma'(z+w)\sigma(z) - \sigma(z+w)\sigma'(z)}{\sigma(z)^2} \\ &= \frac{\sigma(z+w)\zeta(z+w)\sigma(z) - \sigma(z+w)\sigma(z)\zeta(z)}{\sigma(z)^2} \\ &= \frac{\sigma(z+w)}{\sigma(z)}\eta(w) \end{aligned}$$

where we used the definition $\eta(w) \coloneqq \zeta(z+w) - \zeta(z)$ in the last equality.
Now let us define

$$\psi(w) \coloneqq \frac{\sigma(z+w)}{\sigma(z)} e^{-\eta(w)(z+\frac{w}{2})},$$

which is independent of z. This can be verified easily by computing its derivative w.r.t. z and checking that it is equal to zero.
If we can show that $\psi(w) = \chi(w)$, then we have concluded the proof. Indeed, if $\frac{w}{2} \notin \Omega$ choose $z = -\frac{w}{2}$ and use the fact that σ is odd. This yields

$$\psi(w) = \frac{\sigma(\frac{w}{2})}{\sigma(-\frac{w}{2})} = -1 = \chi(w)$$

Now consider the case where $0 \neq \frac{w}{2} \notin \Omega$. We have for any $\tilde{w} \in \Omega$ that

$$\psi(2\tilde{w}) = \frac{\sigma(z+2\tilde{w})\sigma(z+\tilde{w})}{\sigma(z+\tilde{w})\sigma(z)} e^{-2\eta(\tilde{w})(z+\tilde{w})} = \psi(\tilde{w})^2, \tag{7}$$

where we used the fact that η is a homomorphism.
By the discreteness of Ω, $\exists n \geqslant 1$ s.t. $w' \coloneqq 2^{-n}\tilde{w} \in \Omega$ and $w'/2 = 2^{-n-1}\tilde{w} \notin \Omega$. In our case $w' = w$ where $w'/2 \notin \Omega$. We have seen in the previous case, that we then have $\psi(w') = -1$.
We obtain from equation (7)

$$\psi(w) = \psi(2^n w') = \psi(w')^{2^n} = (-1)^{2^n} = 1$$

since $n \geqslant 1$. This shows that $\psi = \chi$ and thus concludes the proof. □

Corollary 3.2. *Let $a, b \in \mathbb{C}$, for $z \in \mathbb{C}$ such that $z \notin b + \Omega$ we define $f(z) := \frac{\sigma(z-a)}{\sigma(z-b)}$. For $w \in \Omega$ we then obtain with above transformation law (6)*

$$f(z+w) = e^{\eta(w)(b-a)} f(z).$$

Proof. Indeed,

$$\begin{aligned} f(z+w) &\overset{\text{def.}}{=} \frac{\sigma(z+w-a)}{\sigma(z+w-b)} \\ &= \frac{\sigma((z-a)+w)}{\sigma((z-b)+w)} \\ &\overset{(3.1)}{=} \frac{\chi(w)e^{\eta(w)(z-a+\frac{w}{2})}\sigma(z-a)}{\chi(w)e^{\eta(w)(z-b+\frac{w}{2})}\sigma(z-b)} \\ &= e^{\eta(w)(b-a)} f(z). \end{aligned}$$

□

4. Abel's Theorem on the Existence of Elliptic Functions

Recall Abel's relation which states the following: Let f be a (non-constant) elliptic function for a given lattice Ω with fundamental parallelogram P. If we list the zeros $a_1, \ldots, a_r$ and poles $b_1, \ldots, b_r$ of f in P (with repetitions for the respective order), then we have

$$a_1 + \ldots + a_r \equiv b_1 + \ldots + b_r \pmod{\Omega}.$$

The following existence theorem can be considered as the converse of Abel's relation and is the culmination of this chapter. Its proof makes use of nearly all tools developed up until now in this paper.

Theorem 4.1. *Let $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$ be two finite sequences in $\mathbb{C}$, such that $\{a_1 + \Omega, \ldots, a_r + \Omega\}$ and $\{b_1 + \Omega, \ldots, b_r + \Omega\}$ are disjoint and such that*

$$w_0 := (b_1 + \ldots + b_r) - (a_1 + \ldots + a_r) \in \Omega.$$

Then

$$f(z) := e^{-\eta(w_0)z} \frac{\sigma(z-a_1)\cdots\sigma(z-a_r)}{\sigma(z-b_1)\cdots\sigma(z-b_r)}$$

is an elliptic function that has zeros precisely at the points $a_1 + \Omega, \ldots, a_r + \Omega$ and poles precisely at the points $b_1 + \Omega, \ldots, b_r + \Omega$ (with order at such a point given by its number of repetitions in the respective sequence). Moreover, every elliptic function with zeros $a_1, \ldots, a_r$ and poles at $b_1, \ldots, b_r$ is a constant multiple of f.

This means that we know how to construct any possible elliptic function up to a constant factor given its zeros and poles (which have to satisfy Abel's relation!). Now let us turn our attention to the proof of this existence theorem.

Proof. Let us define

$$f_{a,b}(z) := \frac{\sigma(z-a)}{\sigma(z-b)},$$

then we can rewrite $f(z)$ as

$$f(z) = e^{-\eta(w_0)z} \prod_{j=1}^{r} f_{a_j,b_j}(z).$$

From above corollary (3.2), it follows that

$$\begin{aligned} f(z+w) &= e^{-\eta(w_0)(z+w)} \prod_{j=1}^{r} f_{a_j,b_j}(z+w) \\ &= e^{-\eta(w_0)(z+w)} \prod_{j=1}^{r} e^{\eta(w)(b_j-a_j)} f_{a_j,b_j}(z) \\ &= e^{-\eta(w_0)(z+w)} e^{\eta(w)\sum_{j=1}^{r}(b_j-a_j)} \prod_{j=1}^{r} f_{a_j,b_j}(z) \\ &= e^{\eta(w)w_0-\eta(w_0)w} e^{-\eta(w_0)z} \prod_{j=1}^{r} f_{a_j,b_j}(z) \\ &= f(z). \end{aligned}$$

We have used that $\eta(w)w_0 - \eta(w_0)w \in 2\pi i\mathbb{Z}$ by the Legendre relation.
Since $\sigma(z)$ has zeros of order 1 at the lattice points it follows immediately that $\sigma(z-b_i)$ has zeros of order 1 at b_i for all $i \in \{1, \ldots, r\}$. From this is follows that $f(z)$ has poles of order exactly one at the points $\{b_1 + \Omega, \ldots, b_r + \Omega\}$.
The last statement of the theorem follows from a fact seen during the course. Any two elliptic functions with the same zeros and poles only differ by a constant factor. This concludes the proof. □

5. Jacobi's Theta Function

Definition 5.1. The Jacobi theta function for the lattice $\Omega = \mathbb{Z}\tau + \mathbb{Z}$ for $\tau \in \mathbb{H}$ is defined by

$$\begin{aligned} \theta\colon \mathbb{C} \times \mathbb{H} &\to \mathbb{C} \\ (z,\tau) &\mapsto \theta(z|\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z} \end{aligned}$$

Lemma 5.2. *The Jacobi theta function converges absolutely and locally uniformly on $\mathbb{C} \times \mathbb{H}$. For every fixed $\tau \in \mathbb{H}$ it defines an entire function $\theta(\cdot|\tau)$ in z. It has zeros (at least) at the points $\frac{\tau+1}{2} + \Omega$.*
Moreover, it satisfies the transformation laws

$$\theta(z+1|\tau) = \theta(z|\tau) \text{ and } \theta(z+\tau|\tau) = e^{-\pi i\tau - 2\pi i z}\theta(z|\tau)$$

Remark 2. Since $\theta(z|\tau)$ is entire and non-constant in z it cannot be doubly periodic in z. Nonetheless we get to relate $\theta(z+\tau|\tau)$ to $\theta(z|\tau)$. Therefore and because $\theta(z|\tau)$ also has period 1 in z, the θ-function is sometimes called a *quasiperiodic* function.

Proof. We prove the three points separately:

- *Convergence*: Let $K \subseteq \mathbb{C} \times \mathbb{H}$ be compact. Then there exists $\varepsilon > 0$ such that $|Im(z)| \leqslant \frac{1}{\varepsilon}$ and $Im(\tau) > \varepsilon$ for all $(z,\tau) \in K$. Let now $(z,\tau) \in K$ be arbitrary, then
$$\sum_{n\in\mathbb{Z}} |e^{\pi i n^2\tau + 2\pi i n z}| = \sum_{n=1}^{\infty} e^{-\pi n^2 Im(\tau) - 2\pi n Im(z)} \leqslant 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2\varepsilon + \frac{2\pi n}{\varepsilon}}.$$
We see that there must exist a constant $C > 0$ such that the right hand side of the calculation can be estimated by $\sum_{n=1}^{\infty} e^{-Cn^2}$, which is a convergent series as a subseries of a convergent series. Moreover, $\theta(z;\tau)$ defines an entire function in z and a holomorphic in τ.

- *Transformation laws*:
  1. It holds that $\theta(z+1;\tau) = \theta(z;\tau)$, since

$$\theta(z+1;\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n(z+1)} = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z + 2\pi i n} =$$
$$= \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z} e^{2\pi i n} \overset{(*)}{=} \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z} = \theta(z;\tau).$$

  In (*) we used the fact that for any $n \in \mathbb{Z}$ it holds $e^{2\pi i n} = 1$.

  2. We now prove that $\theta(z+\tau;\tau) = \theta(z;\tau)$.

$$\theta(z+\tau;\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n(z+\tau)} = e^{-\pi i \tau - 2\pi i z} \sum_{n\in\mathbb{Z}} e^{\pi i (n+1)^2 \tau + 2\pi i (n+1) z} =$$
$$= e^{-\pi i \tau - 2\pi i z} \theta(z;\tau).$$

- Zeros at $\frac{\tau+1}{2} + \Omega$: We directly compute

$$\theta\left(\frac{\tau+1}{2};\tau\right) = \sum_{m\in\mathbb{Z}} (-1)^n e^{\pi i n(n+1)\tau} = \sum_{n\in\mathbb{Z}} (-1)^{-m-1} e^{\pi i(-m-1)(-m)\tau} =$$
$$= -\theta\left(\frac{\tau+1}{2};\tau\right),$$

  which holds only in the case where $\theta\left(\frac{\tau+1}{2};\tau\right) = 0$. This shows that the zeros of $\theta(z;\tau)$ are at least at the points $\frac{\tau+1}{2} + \Omega$.

□

The main purpose of defining the θ-function in this paper is to show that there exists an alternative way of constructing elliptic functions.

Theorem 5.3. *For $a_1, \ldots, a_r, b_1, \ldots, b_r \in \mathbb{C}$ with $(a_1 + \ldots + a_r) - (b_1 + \ldots + b_r) \in \Omega$ the function*

$$f(z) := \frac{\theta(z-a_1|\tau)\cdots\theta(z-a_r|\tau)}{\theta(z-b_1|\tau)\cdots\theta(z-b_r|\tau)}$$

is an elliptic function for the lattice $\Omega = \mathbb{Z}\tau + \mathbb{Z}$. If the sets $\{a_1 + \Omega, \ldots, a_r + \Omega\}$ and $\{b_1 + \Omega, \ldots, b_r + \Omega\}$ are disjoint, then $f(z)$ has zeros at the points $z \in \frac{\tau+1}{2} + a_j + \Omega$ and poles at the points $z \in \frac{\tau+1}{2} + b_j + \Omega$ for $1 \leqslant j \leqslant r$, where the order of the poles is given by the number of repetitions of a_j and b_j.

Proof. Uses the Jacobi triple product identity. Can be found in [1]. □

Remark 3. In comparison to Abel's Existence Theorem the poles and zeros seem to be shifted by $\frac{\tau+1}{2}$, which is due to the fact that the zeros of the θ-function are located at $\frac{\tau+1}{2} + \Omega$ instead of at $0 + \Omega$ for the σ-function

References

[1] M. Schwagenscheidt, *Elliptic Functions and Elliptic Curves*, ETH Zurich (2022) https://people.math.ethz.ch/~mschwagen/ellipticfunctions_script.pdf