The modular group and modular forms

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27.03.2024

Introduction

In this seminar, we aim to introduce the modular group and modular forms. We begin by examining the group structure of the modular group and its action on the upper half-plane through Möbius transformations. We then define the fundamental domain and explore its properties, followed by a brief introduction to elliptic points. After that, we introduce Fourier expansions and define the factor of automorphy and modular forms. Finally, we prove a few general statements about modular forms, such as the Hecke bound, and that there are no modular forms of negative weight.

1 Möbius transformations and the upper half-plane

We begin with a brief review of Möbius transformations, focusing particularly on those operating on the upper half-plane.

Definition 1.1. The upper half-plane is the set of complex numbers with positive imaginary part: $\mathbb{H} = \{\tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) > 0\}$.

Definition 1.2. Let $a, b, c, d \in \mathbb{R}$ satisfy the condition $ad - bc = 1$. We define a Möbius transformation on $\mathbb{H}$ as a map of the form

$$f(\tau) = \frac{a\tau + b}{c\tau + d}$$

The condition $ad - bc = 1$ in the above definition is sometimes written as $ad - bc > 0$, but since multiplying a, b, c and d by the same constant does not change f, both conditions are equivalent. We verify that the image of f is indeed in $\mathbb{H}$ in the following lemma:

Lemma 1.3. $\mathrm{Im}(f(\tau)) = \dfrac{ad - bc}{|c\tau + d|^2}\,\mathrm{Im}(\tau)$

Proof. $f(\tau) = \dfrac{a\tau + b}{c\tau + d} = \dfrac{(a\tau + b)(c\overline{\tau} + d)}{|c\tau + d|^2} = \dfrac{ac|\tau|^2 + bd + ad\tau + bc\overline{\tau}}{|c\tau + d|^2}$;

$ac|\tau|^2 + bd,\ |c\tau + d|^2 \in \mathbb{R} \Rightarrow \mathrm{Im}(f(\tau)) = \dfrac{\mathrm{Im}(ad\tau + bc\overline{\tau})}{|c\tau + d|^2} = \dfrac{(ad - bc)\mathrm{Im}(\tau)}{|c\tau + d|^2}$ □

Since $ad - bc = 1$, $\mathrm{Im}(\tau) > 0$ and $|c\tau + d|^2 > 0$ (because $\tau \neq -\dfrac{d}{c}$), we have $\mathrm{Im}(f(\tau)) > 0$, and therefore $f(\tau) \in \mathbb{H}$. It can also be shown that f is a bijection on $\mathbb{H}$ with inverse function $f^{-1}(\tau) = \dfrac{d\tau - b}{-c\tau + a}$. Note that the coefficients a, b, c, d of f correspond to the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, since $ab - cd = 1$ by definition.

2 The modular group and its action on the upper half-plane

We define the modular group, denoted by Γ, as the subgroup of $\mathrm{SL}_2(\mathbb{R})$ consisting of the matrices with integer coefficients:

Definition 2.1. $\Gamma := \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$

We determine the structure of Γ by proving the claim:

Theorem 2.2. Γ is generated (as a multiplicative group) by the matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Proof. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an arbitrary element of Γ and let G be the group generated by S and T, that is, the group consisting of all possible combinations (products and inverses) of S and T. It is easy to verify that S and T are in Γ, therefore $G \subset \Gamma$. We use strong induction on $|c|$ to show the remaining direction, namely $\Gamma \subset G$. Note that $S^2 = -I$, so $-I \in G$. In the base case $c = 0$, $ad = ad - bc = 1$, so $a = d = \pm 1$, and since $-I \in G$, we can assume wlog that $a = d = 1$. $T^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ for any $m \in \mathbb{Z} \Rightarrow M = T^b$ is indeed in G. For $|c| > 0$, we consider the matrix $ST^mM = \begin{pmatrix} -c & -d \\ a + mc & b + md \end{pmatrix}$. We choose $m = m_0 \in \mathbb{Z}$ such that $|a + m_0 c| < |c|$, and by the induction hypothesis, $ST^{m_0}M \in G \Rightarrow M \in G$. This finishes the proof. $\square$

Before we proceed, let us mention another identity ($S^2 = -I$ was already established in the above proof) related to the group structure of the modular group, namely $(TS)^3 = (ST)^3 = -I$. The proof follows directly by simple calculations using the definitions of S and T.

In order to examine how the modular group Γ, and more generally $\mathrm{SL}_2(\mathbb{R})$, act on the upper half-plane $\mathbb{H}$, we recall what a group action is:

Definition 2.3. A group action σ of a group G on a set X is a function $\sigma : G \times X \to X$ with the following two properties:

1. $\sigma(e, x) = x$, where e is the identity of G and $x \in X$ - arbitrary
2. $\sigma(g, \sigma(h, x)) = \sigma(gh, x)$, where $g, h \in G$ and $x \in X$ - arbitrary

Such a group action is called a left action; a right action is defined similarly as $\sigma : X \times G \to X$ that satisfies $\sigma(x, e) = x$ and $\sigma(\sigma(x, g), h) = \sigma(x, gh)$

To simplify notation, we write gx or $g \cdot x$ instead of $\sigma(g, x)$ from now on.

Theorem 2.4. The group Γ acts on $\mathbb{H}$ through Möbius transformations.

$$\Gamma \times \mathbb{H} \to \mathbb{H}; \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau \to \frac{a\tau + b}{c\tau + d}$$

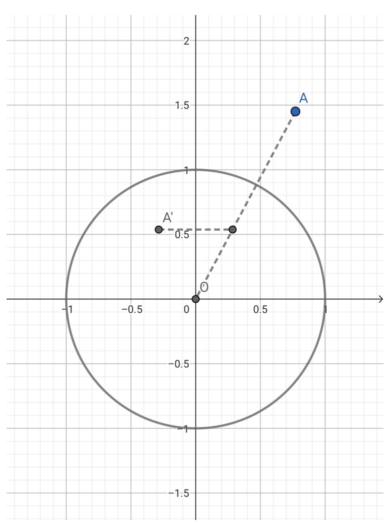
Proof. We show the above function is indeed an action. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity of $\mathrm{SL}_2(\mathbb{R})$, and we have $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \tau = \dfrac{\tau + 0}{0\tau + 1} = \tau \Rightarrow$ the first property holds. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ be arbitrary elements of $\mathrm{SL}_2(\mathbb{R})$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \cdot \tau \right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{a'\tau + b'}{c'\tau + d'} =$$

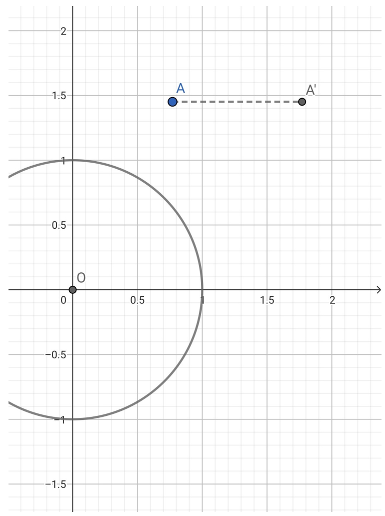
$$= \frac{a\dfrac{a'\tau + b'}{c'\tau + d'} + b}{c\dfrac{a'\tau + b'}{c'\tau + d'} + d} = \frac{(aa' + bc')\tau + ab' + bd'}{(a'c + dc')\tau + b'c + dd'} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) \cdot \tau$$

The last equality holds because $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + bc' & ab' + bd' \\ a'c + dc' & b'c + dd' \end{pmatrix}$ $\square$

Therefore, the second condition for σ to be an action is also satisfied. Next, we illustrate the action of S and T on $\mathbb{H}$. S acts on $\mathbb{H}$ through the Möbius transformation $f(\tau) = -\frac{1}{\tau}$. If we write τ as $x + yi$, then $f(\tau) = \frac{-x+yi}{x^2+y^2}$. Therefore, $f(\tau)$ first maps τ to its inversion image with respect to the unit circle (we define inversion in the next page), and then reflects it with respect to the y-axis, as shown in the picture below (A' is the image of A):

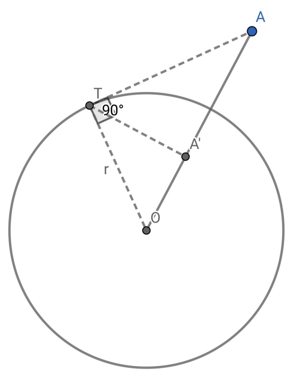


T acts on $\mathbb{H}$ through the Möbius transformation $f(\tau) = \tau + 1$, which simply shifts τ with 1 to the right, as shown in the picture below (A' is the image of A):



To better understand how S acts on $\mathbb{H}$, we define inversion:

Definition 2.5. Inversion with center O and radius $r > 0$ is the map of the plane that sends each point A to a point A' on the ray $\overrightarrow{OA}$ such that $|OA||OA'| = r^2$



Inversion has various important geometric properties. It maps the set of lines through O to itself, the set of lines not containing O to the set of circles containing O (and vice versa), and the set of circles not containing O to itself.

Let us recall the definition of an orbit of a group action:

Definition 2.6. The orbit of an element $x \in X$ is the set $O_G(x) := \{gx \mid g \in G\}$.

One can easily verify that orbits corresponding to different elements of X are either equal or disjoint.

Definition 2.7. We call a group operation transitive if there is only one orbit, i.e. for every $x, x' \in X$ there is an element $g \in G$ such that $gx = x'$.

In the following theorem we show an important property of the group action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$:

Theorem 2.8. The group action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ is transitive.

Proof. Let τ_0, τ be arbitrary elements of $\mathbb{H}$. We want to find an element g of $\mathrm{SL}_2(\mathbb{R})$ such that $g \cdot \tau_0 = \tau$. Let $\tau_0 = x_0 + y_0 i$ and $\tau = x + yi$ (note that $y_0, y > 0$). Let $g_1 = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}$, $g_2 = \begin{pmatrix} 1 & \frac{x}{y} - \frac{x_0}{y_0} \\ 0 & 1 \end{pmatrix}$, $g_3 = \begin{pmatrix} \frac{1}{\sqrt{y_0}} & 0 \\ 0 & \sqrt{y_0} \end{pmatrix}$ and let $g = g_1 g_2 g_3$. Since $g_1, g_2, g_3 \in \mathrm{SL}_2(\mathbb{R})$ it follows that $g \in \mathrm{SL}_2(\mathbb{R})$.

$$g \cdot \tau_0 = (g_1 g_2 g_3) \cdot \tau_0 = (g_1 g_2) \cdot (g_3 \cdot \tau_0) = (g_1 g_2) \cdot \left(\frac{x_0}{y_0} + i \right) =$$

$$= g_1 \cdot \left(g_2 \cdot \left(\frac{x_0}{y_0} + i \right) \right) = g_1 \cdot \left(\frac{x}{y} + 1 \right) = x + yi = \tau \qquad \square$$

However, the action of Γ on $\mathbb{H}$ is not transitive because we restrict a, b, c and d to integers. To gain a better understanding of this action, we introduce the fundamental domain - a closed subset of $\mathbb{H}$ that contains a representative of each orbit.

3 The fundamental domain and elliptic points

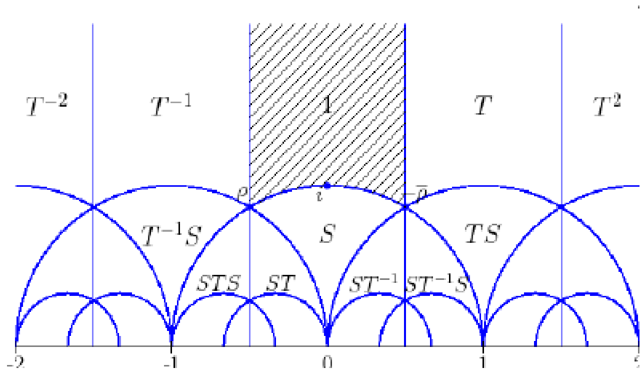
Definition 3.1. We define the fundamental domain as

$$\mathcal{F} := \left\{ \tau \in \mathbb{H} : |\mathrm{Re}(\tau)| \leq \frac{1}{2}, |\tau| \geq 1 \right\}$$

First, we prove that every orbit has a representative in $\mathcal{F}$:

Theorem 3.2. For every $\tau \in \mathbb{H}$ there is an element $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that $M\tau = \dfrac{a\tau + b}{c\tau + d} \in \mathcal{F}$.

Proof. Let $\tau \in \mathbb{H}$ be arbitrary. We already proved in Lemma 1.3. that for a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have $\mathrm{Im}(M\tau) = \mathrm{Im}\left(\dfrac{a\tau + b}{c\tau + d} \right) = \dfrac{\mathrm{Im}(\tau)}{|c\tau + d|^2}$. We can choose c and d in M in such a way that $|c\tau + d|^2$ is minimal, i.e. there is a matrix $M_0 \in \Gamma$ such that $\mathrm{Im}(M_0\tau)$ is maximal. By multiplying M_0 with a power of T, we shift the real part of $M_0\tau$ by a whole number without changing the the imaginary part. Therefore, wlog we may assume that $|\mathrm{Re}(M_0\tau)| \leq \dfrac{1}{2}$. Assume now that $|M_0\tau| < 1$. Then $\mathrm{Im}(SM_0\tau) = \dfrac{\mathrm{Im}(M_0\tau)}{|M_0\tau|^2} > \mathrm{Im}(M_0\tau)$, and since SM_0 is also in Γ, we reach a contradiction to the maximality of $\mathrm{Im}(M_0\tau)$. Therefore, our assumption is wrong and $|M_0\tau| \geq 1 \Rightarrow M_0\tau \in \mathcal{F}$, which finishes the proof. $\square$



The shaded area represents the fundamental domain. Each element of Γ in the diagram corresponds to the image resulting from the action of the respective element on the fundamental domain.

In the next theorem, we demonstrate that each representative within the set $\mathcal{F}$ is unique, except for those on the boundary of $\mathcal{F}$. We say that two elements τ and τ' are equivalent modulo Γ if there is an element $M \in \Gamma$ such that $M\tau = \tau'$. In particular, as we proved in Theorem 2.8, every two elements of $\mathbb{H}$ are equivalent modulo $\mathrm{SL}_2(\mathbb{R})$.

Theorem 3.3. If $\tau \neq \tau' \in \mathcal{F}$ are equivalent modulo Γ, then τ, τ' must be on the boundary of $\mathcal{F}$. Moreover, we either have $|\tau| = 1$ and $\tau' = \dfrac{-1}{\tau}$, or $\mathrm{Re}(\tau) = \pm\dfrac{1}{2}$ and $\tau' = \tau \mp 1$.

Proof. Let $\tau \neq \tau' \in \mathcal{F}$ be equivalent modulo Γ, i.e. there is a matrix $M \in \Gamma$ with $M\tau = \tau'$. Wlog we may assume that $\mathrm{Im}(\tau') = \mathrm{Im}(M\tau) \geq \mathrm{Im}(\tau)$, otherwise we can swap τ and τ'. This yields $|c\tau + d|^2 \leq 1$, using again the fact that $\mathrm{Im}(M\tau) = \dfrac{\mathrm{Im}(\tau)}{|c\tau + d|^2}$ (Lemma 1.3). Let us write τ as $x + iy$, and observe that by definition of $\mathcal{F}$ it holds $-\dfrac{1}{2} \leq x \leq \dfrac{1}{2}$ and $y \geq \dfrac{\sqrt{3}}{2} \Rightarrow 1 \geq |c\tau + d|^2 = (cx + d)^2 + c^2y^2 \geq \dfrac{3}{4}c^2$. Because c is an integer, we have $c \in \{-1, 0, 1\}$. Note that M and $-M$ operate in the same way on $\mathbb{H}$, as multiplying a, b, c, d by -1 does not change $\dfrac{a\tau + b}{c\tau + d}$, so we can assume wlog $c \geq 0$. We now examine both cases:

- **c = 0:** In this case, as we showed in the proof of Theorem 2.2, it holds that $M = T^m$ for some $m \in \mathbb{Z}$. Hence, M maps τ to $\tau' = \tau + m$, so τ and τ' must be on the vertical parts of the boundary of $\mathcal{F}$, and $m = \pm 1$. As a result, we get $\mathrm{Re}(\tau) = \pm\dfrac{1}{2}$ and $\tau' = \tau \mp 1$.

- **c = 1:** In this case, $1 \geq |c\tau + d|^2 = |\tau + d|^2 = (x+d)^2 + y^2 \geq (x+d)^2 + \dfrac{3}{4} \Rightarrow |x + d| \leq \dfrac{1}{2}$. Since $|x| \leq \dfrac{1}{2}$, this inequality yields $|d| \leq 1$, so $d \in \{-1, 0, 1\}$. We now examine the cases for d:

  - $d = 0$: M must be of the form $\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$, so $M = T^aS$ and $\tau' = M\tau = a - \dfrac{1}{\tau}$. If we assume $|\tau| > 1$, we get $\left|\mathrm{Re}\left(\dfrac{-1}{\tau}\right)\right| = \dfrac{|x|}{|\tau|^2} < \dfrac{1}{2}$. Thus, a bust be 0 in order to satisfy $\mathrm{Re}\left(a - \frac{1}{r}\right) = \mathrm{Re}(\tau') \leq \frac{1}{2}$, and hence $|\tau'| = |M\tau| = \left|\frac{-1}{\tau}\right| < 1$. This contradicts $\tau' \in \mathcal{F}$. Therefore, $|\tau| = 1 \Rightarrow \frac{1}{\tau} = \overline{\tau}$. As a result, $\tau' = M\tau = a - \overline{\tau} = (a - x) + iy$. $\mathrm{Re}(\tau') \leq \frac{1}{2}$ implies $a \in \{-1, 0, 1\}$. It remains to examine the three possible cases for a:

For $a = -1$ we have $M = T^{-1}S$.
For $a = 0$ we have $M = S$.
For $a = 1$ we have $M = TS$.

In the case $a = 0$, we get $\tau' = \frac{-1}{\tau}$ as we claimed. $a = 1$ yields $\tau' = \tau = \rho := e^{\frac{\pi i}{3}}$ and $a = -1$ yields $\tau' = \tau = \rho^2 = e^{\frac{2\pi i}{3}}$. We required, however, that $\tau \neq \tau'$, so we exclude the the cases $a = \pm 1$.

– $d = -1$: From the inequalities $|x + d| \leq \frac{1}{2}$ and $|x| \leq \frac{1}{2}$, we observe that $x = \frac{1}{2}$ and y $= \frac{\sqrt{3}}{2}$, so $\tau = \rho$. $1 = det(M) = ad - bc = a \cdot (-1) - b \Rightarrow a + b = -1$. Therefore, $M\rho = \frac{a\rho + b}{\rho - 1} = a - \frac{1}{\rho - 1}$, which yields $\mathrm{Im}(M\rho) = \frac{\sqrt{3}}{2}$ and $\mathrm{Re}(M\rho)$ $= a + \frac{1}{2}$. This is only possible if $a \in \{-1, 0\}$.
If $a = 0$, we have $b = -1$ and $M = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. However, this leads to $M\rho = \rho$, contradicting $\tau \neq \tau'$.
If $a = -1$, we have $b = 0$ and $M = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$. This yields $M\rho = \rho^2 = \frac{-1}{\rho}$ and since $|\rho| = 1$, the theorem is satisfied.

– $d = 1$: This case is solved similarly to $d = -1$ □

In order to define elliptic points, we recall the definitions of a stabilizer and a fixed point of a group action:

Definition 3.4. The stabilizer of an element $x \in X$ is the subgroup $G_x := \{g \in G \mid gx = x\}$ of G.

Definition 3.5. A fix point of G is an element $x \in X$ such that the orbit of x consists only of x, i.e. $O_G(x) = \{x\}$ (or equivalently the stabilizer of x is $G_x = G$).

We now examine the fixed points $\tau \in \mathbb{H}$ of the action defined by Γ. Let $\Gamma_\tau = \{M \in \Gamma : M\tau = \tau\}$ be the stabilizer of τ. For all $\tau \in \mathbb{H}$ and $M \in \Gamma$, we have $\pm I \in \Gamma_\tau$ and $|\Gamma_\tau| = |\Gamma_{M\tau}|$.

Definition 3.6. A point $\tau \in \mathbb{H}$ is called an elliptic point of order $\frac{1}{2}\,|\Gamma_\tau|$ if $\Gamma_\tau \neq \{\pm I\}$, that is, τ has a nontrivial stabilizer.

The following theorem shows that there are only 2 elliptic points modulo the operation of Γ, which have orders 2 and 3.

Theorem 3.7. The elliptic points of Γ in $\mathcal{F}$ are the following:

1. $\tau = \mathrm{i}$ with stabilizer $\{\pm I, \pm S\}$,
2. $\tau = \rho = e^{\frac{\pi i}{3}}$ with stabilizer $\{\pm I, \pm TS, \pm (TS)^2\}$,
3. $\tau = \rho^2 = e^{\frac{2\pi i}{3}}$ with stabilizer $\{\pm I, \pm ST, \pm (ST)^2\}$

The proof follows easily from the proof of Theorem 3.3.

4 Fourier series

Theorem 4.1. Let f be a 1-periodic (i.e. $f(\tau+1) = f(\tau)$) holomorphic function on the region $D = \{\tau \in \mathbb{C} : a < \mathrm{Im}(\tau) < b\}$, where $-\infty \leq a < b \leq \infty$. Then f can be represented as a convergent complex Fourier series $f(\tau) = \sum\limits_{n=-\infty}^{\infty} a_n e^{2\pi i n \tau}$, where $a_n = \displaystyle\int_0^1 f(x+iy)e^{-2\pi i n(x+iy)}dx$, and y can be chosen arbitrarily in the interval (a, b).

Proof. For each y, the function $f(\tau) = f(x+iy)$ is two times continuously differentiable in the variable x, and by a well-known result in real analysis it admits a Fourier expansion:

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n(y) e^{2\pi i n x}$$

with

$$a_n(y) = \int_0^1 f(\tau)e^{-2\pi i n x}\,dx \quad \text{or equivalently} \quad a_n(y)e^{2\pi n y} = \int_0^1 f(\tau)e^{-2\pi i n \tau}\,dx$$

We are done if we show that $a_n := a_n(y)e^{2\pi n y}$ does not depend on y, because this would imply $f(\tau) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n \tau}$. We must therefore verify that the derivative $\dfrac{d}{dy}(a_n(y)e^{2\pi i n y})$ is 0 for all y or, equivalently, $a_n'(y) = -2\pi n a_n(y)$. To prove this, we use the Cauchy-Riemann differential equations

$$\frac{\partial f(\tau)}{\partial x} = -i\frac{\partial f(\tau)}{\partial y}$$

The integral formula for $a_n(y)$ together with the Leibniz rule yield

$$a_n' = \int_0^1 \frac{\partial f(\tau)}{\partial y} e^{-2\pi i n x} dx = \int_0^1 i\frac{\partial f(\tau)}{\partial x} e^{-2\pi i n x}$$

and after partial integration we obtain the desired differential equation for $a_n(y)$. □

5 Modular forms

Let us introduce two definitions necessary for defining modular forms:

Definition 5.1. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $\tau \in \mathbb{H}$ we define the factor of automorphy as $j(M, \tau) = c\tau + d$.

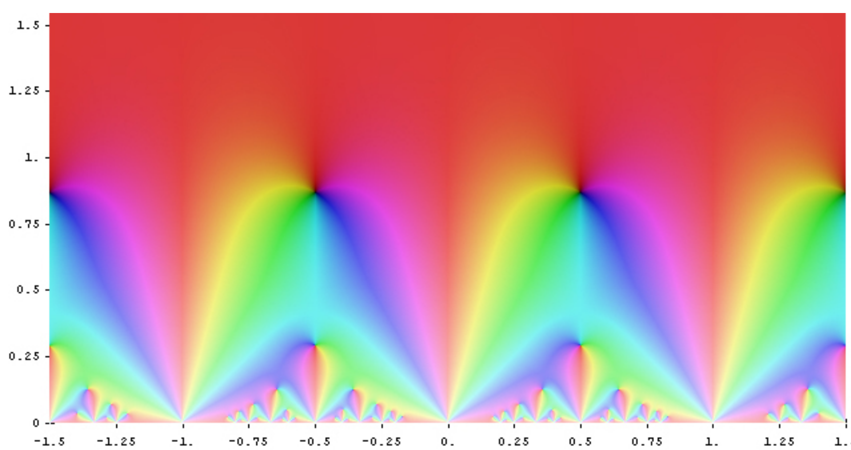
Definition 5.2. For $k \in \mathbb{Z}$, $M \in \Gamma$ and $f : \mathbb{H} \to \mathbb{C}$ we define the weight k slash-operator as $(f|_k M)(\tau) := j(M, \tau)^{-k} f(M\tau)$.

With the relation $j(MN, \tau) = j(M, N\tau)j(N, \tau)$, one can verify that the weight k slash-operator defines a right group action of $\mathrm{SL}_2(\mathbb{R})$ on the set of holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ i.e. $f|_k MN = (f|_k M)|_k N$. We can now define modular forms:

Definition 5.3. Let $k \in \mathbb{Z}$. A function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form of weight k for Γ if the following conditions hold:

1. f is holomorphic on $\mathbb{H}$
2. $f|_k M = f$ for all $M \in \Gamma$
3. f can be written as a Fourier series of the form $f(\tau) = \sum_{n=0}^{\infty} a_f(n) q^n$ with $q = e^{2\pi i \tau}$

If $a_f(0) = 0$, f is called a cusp form.



The diagram displays a colour plot of a modular form, specifically the holomorphic Eisenstein series of weight 4.

Remarks:

1. The second condition is equivalent to $f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$. Since T and S generate Γ, this condition is equivalent to the two inequalities $f(\tau + 1) = f(\tau)$ (for $M = T$) and $f\left(\frac{-1}{\tau}\right) = \tau^k f(\tau)$ (for $M = S$). The first inequality means f must be 1-periodic.

2. As shown in Theorem 4.1, if f is 1 - periodic, f must have a Fourier expansion of the form $f(\tau) = \sum\limits_{n=-\infty}^{\infty} a_f(n)q^n$ with $a_f(n) \in \mathbb{C}$ and $q = e^{2\pi i\tau}$. The third condition therefore requires that this Fourier expansion has no negative terms, which is equivalent to $f(\tau) = f(x+iy)$ remaining bounded as $y \to \infty$. In particular, if f is a cusp form, then $f(\tau)$ approaches 0 exponentially as y tends to ∞.

3. $a_f(0) = \lim\limits_{y\to\infty} f(iy)$

4. The set of modular forms of weight k for Γ forms a $\mathbb{C}$-vector space M_k; we denote by S_k the subspace of M_k consisting of cusp forms.

Finally, we establish some important properties of modular and cusp forms.

Lemma 5.4. Given $f \in M_k$ and $g \in M_l$, we have $fg \in M_{k+l}$. Moreover, if f and g are cusp forms, then so is fg.

Proof. For M $\in \Gamma$, we calculate that

$$(fg)|_{k+l}M = j(M,\tau)^{-k-l}(fg)(M\tau) = j(M,\tau)^{-k}f(M\tau)j(M,\tau)^{-l}g(M\tau) =$$

$$= f|_k M \cdot g|_l M = fg$$

This implies the first part of the Lemma. The second part follows directly from multiplying the Fourier expansions of f and g. □

Lemma 5.5. Let $k \in \mathbb{Z}$ be odd. Then $M_k = \{0\}$.

Proof. Let $f \in M_k$. The transformation of f under the matrix $-I$ yields $f(\tau) = f((-I)\tau) = (-1)^k f(\tau)$. Since k is odd, we therefore have $f(\tau) = 0$ for all $\tau \in \mathbb{H}$. □

Theorem 5.6 (Hecke bound)**.** Let $f = \sum\limits_{n=1}^{\infty} a_f(n)q^n$ be a cusp form. There exists a constant $C > 0$ such that $|a_f(n)| \leq Cn^{k/2}$ for all $n \geq 1$.

Proof. Consider the function $h(\tau) = \text{Im}(\tau)^{k/2}|f(\tau)|$. It follows from the second property of modular forms and the identity $\text{Im}(M\tau) = \dfrac{\text{Im}(\tau)}{|c\tau+d|^2}$ (Lemma 1.3.) that h is Γ-invariant (this means $h(M\tau) = h(\tau)$ for all $M \in \Gamma$ and $\tau \in \mathbb{H}$). Since $f(\tau) \to 0$ exponentially as $\text{Im}(\tau) \to \infty$ (remark 2.), $h(\tau)$ is bounded on the entire fundamental domain $\mathcal{F}$. Combining this property with $h-\Gamma$-invariant, we get that $h(\tau)$ is bounded on the entire $\mathbb{H}$ by a constant C', i.e. $h(\tau) \leq C'$ for all $\tau \in \mathbb{H}$. We can now also derive a bound for the $n-$th Fourier coefficient $a_f(n)$:

$$|a_f(n)| \leq \int_0^1 |f(x+iy)|e^{2\pi ny}dx = \int_0^1 y^{-k/2}h(x+iy)e^{2\pi ny}dx \leq C'y^{-k/2}e^{2\pi ny}$$

As pointed out in Theorem 4.1, we can choose y freely. In particular, $y = 1/n$ gives us $|a_f(n)| \leq C'e^{2\pi}n^{k/2}$, which finishes the proof. □

Theorem 5.7. For $k < 0$ it holds that $M_k = \{0\}$.

Proof. We consider again the Γ-invariant function $h(\tau) = \mathrm{Im}(\tau)^{k/2}|f(\tau)|$. $k < 0$ and remark 2. imply that h is bounded on the fundamental domain $\mathcal{F}$, and therefore also on $\mathbb{H} \Rightarrow h(\tau) \leq C'$ for some constant C' and all $\tau \in \mathbb{H}$. As in the proof the Hecke bound, we have:

$$|a_f(n)| \leq \int_0^1 |f(x+iy)|e^{2\pi ny}dx = \int_0^1 y^{-k/2}h(x+iy)e^{2\pi ny}dx \leq C'y^{-k/2}e^{2\pi ny}$$

If $y \to 0$, the right side of the above inequality tends to 0, because $-k > 0$. Therefore, $|a_f(n)| = 0$ for all $n \Rightarrow f \equiv 0$. $\square$

References

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[3] Pink, *Algebra I und II*, lecture notes