Eisenstein series and the Delta function

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Introduction

Before the Easter break, we introduced modular forms, which are holomorphic functions on the upper half plane which are invariant under Moebius transformations and satisfy a certain growth condition. So far, we have not encountered any non-trivial modular forms and by the definition, it's not immediately clear whether it's even possible to find non-trivial modular forms. In this talk, we give a first example of a non-trivial modular form (the Eisenstein series) and use it to construct a non-trivial modular cusp form (the Ramanujan delta function). As usual, we closely follow the lecture notes [12].

1 Preliminaries

In the following, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ will denote the modular group, as introduced in the last talk. Recall that Γ acts on the set of holomorphic functions on the upper half plane via $f|_k M(\tau) = j(M,\tau)^{-k} f(M\tau)$, where $j(M,\tau) = c\tau + d$ is the factor of automorphy (with $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$).

Definition 1. Let $k \in \mathbb{Z}$ and let $f : \mathbb{H} \to \mathbb{C}$ be a complex valued function. We call f a *modular form of weight* k for Γ if f fullfills the following three conditions:

1. f is homolomorphic on $\mathbb{H}$,
2. $f|_k M = f$ for all $M \in \Gamma$,
3. There are no negative terms in the fourier expansion of f, i.e. it's fourier expansion is of the form
$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n,$$
where $q = e^{2\pi i \tau}$. If also $a_0 = 0$, then f is called a *cusp form.*

To show holomorphy, we will need the following statement:

Theorem 1. *Let $(f_n)_n$ be a sequence of complex-valued functions on a set S. Assume there are non-negative numbers $(M_n)_n$ s.t.*

1. *$|f_n(x)| \leq M_n$ for all $n \geq 1$ and all $x \in S$.*
2. *$\sum_{n=1}^{\infty} M_n$ converges*

Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely and uniformly on S.

2 Eisenstein Series

Definition 2. Let $k \in \mathbb{Z}$, $k \geq 4$ be even. We define *the Eisenstein Series of weight k* as

$$G_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m\tau + n)^{-k}$$

for all $\tau \in \mathbb{H}$.

In week 1, we already introduced the Eisenstein Series for a given lattice Ω. Recall that we defined

$$G_k(\Omega) = \sum_{0\neq\omega\in\Omega} \omega^{-k}$$

for $k \in \mathbb{Z}, k \geq 4$ and even. In the special case of $\Omega = \mathbb{Z}\tau + \mathbb{Z}$, this is simply the Eisenstein Series as in Definition 2. We will now show the following proposition, which shows the relationship between modular forms and the Eisenstein Series.

Proposition 1. *Let $k \in \mathbb{Z}$, $k \geq 4$ be even. Then G_k is a non-zero modular form of weight k with respect to Γ.*

Recall that there are no modular forms of odd weight on Γ. For k odd, we have that $-(m\tau + n)^{-k} = (-m\tau - n)^{-k}$, so $G_k = 0$. This means that the Eisenstein series does not give a new non-trivial example of a modular form in this case, as we expected.

Lemma 1. *Let $K \subset \mathbb{H}$ be compact. Then there are constants $\alpha, \beta > 0$ such that*

$$\forall \tau \in K\ \forall m, n \in \mathbb{R} \colon \alpha|mi + n| \leq |m\tau + n| \leq \beta|mi + n|. \tag{1}$$

Proof. We can assume WLOG that $m^2 + n^2 = 1$. Otherwise, replace (m, n) by $\frac{1}{\sqrt{m^2+n^2}}(m, n)$ and use the fact that (1) is homogeneous in m, n. Now consider the continuous function $(\tau, (m, n)) \mapsto |m\tau + n|$ on the compact set $K \times S^1$ (products of compact sets are compact, by Tychonoff's theorem). This function admits a minimum and a maximum, hence there are constants α, β such that $\alpha \leq |m\tau + n| \leq \beta$. It only remains to show that $\alpha > 0$. To this end, notice that $z \mapsto \mathrm{Im}(z)$ is continuous, so it admits a minimum on the compact set K. Hence, there exists a constant C s.t. $\forall z \in K \colon \mathrm{Im}(z) \geq C > 0$. Now $|m\tau + n| = \left((mx + n)^2 + iym\right)^{1/2}$. So if $m = 0$, then $|m\tau + n| = |n| = 1 > 0$ (because of the relation $m^2 + n^2 = 1$) while if $m \neq 0$, then $|m\tau + n| \geq |my| \geq |m|C > 0$. In either case, the value of the function is positive and hence it's minimum (which is α) is positive too. This completes the proof. $\square$

Lemma 2. *Let $k \in \mathbb{Z}$ be even, $k \geq 4$. Then G_k defines a holomorphic function on $\mathbb{H}$.*

Proof. By the Weierstrass convergence theorem, it suffices to show G_k converges locally uniformly. To this end, we want to apply the Weierstrass M-test. Using the lemma above, we can bound

$$\begin{aligned} |G_k| \leq \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} |m\tau + n|^{-k} &\leq \alpha^{-k} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} |mi + n|^{-k} \\ &= \alpha^{-k} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m^2 + n^2)^{-k/2} \end{aligned}$$

Now we only need to show that $\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}(m^2+n^2)^{-k/2}$ converges. Indeed,

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}(m^2+n^2)^{-k/2} = \sum_{n\in\mathbb{Z}\setminus\{0\}} n^{-k} + \sum_{m\in\mathbb{Z}\setminus\{0\}} m^{-k} + \sum_{m,n\in\mathbb{Z}\setminus\{0\}}(m^2+n^2)^{-k/2}.$$

Now we use that fact that $\sum_{n\in\mathbb{Z}\setminus\{0\}} n^{-k} = 2\zeta(k)$ and that $m^2+n^2 \geq 2|mn|$ (this is simply $\big(|m|-|n|\big)^2 \geq 0$) to obtain the bound

$$\begin{aligned}\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}(m^2+n^2)^{-k/2} &\leq 4\zeta(k) + \sum_{m,n\in\mathbb{Z}\setminus\{0\}} 2^{-k/2}|mn|^{-k/2} \\ &= 4\zeta(k) + 4\cdot 2^{-k/2}\zeta(k/2)^2 \\ &< \infty.\end{aligned}$$

By the Weierstrass M-test, we obtain that G_k converges absolutely and uniformly on every compact set, which proves the claim. □

Lemma 3. *Let $k\in\mathbb{Z}$ be even, $k\geq 4$. Then $G_k|_kM = G_k$ for every $M\in\Gamma$, i.e. G_k is Γ-invariant.*

Proof. The proof is a simple computation: let $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in\Gamma$. Then

$$\begin{aligned}(G_k|_kM)(\tau) &= (c\tau+d)^{-k}\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\left(m\frac{a\tau+b}{c\tau+d}+n\right)^{-k} \\ &= \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\big(m(a\tau+b)+n(c\tau+d)\big)^{-k} \\ &= \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\big((ma+nc)\tau+(mb+nd)\big)^{-k}.\end{aligned}$$

Now the map $(m,n)\mapsto(ma+nc,mb+nd)$ from $\mathbb{Z}^2\setminus\{(0,0)\}$ to itself is a bijection, because it is given by right multiplication of the matrix M, which has determinant 1 and is thus invertible over $\mathbb{Z}$. Also, notice that we can reorder the series because it converges absolutely. □

To conclude that G_k is indeed a modular form, we need to calculate it's Fourier expansion. We want to show the following result:

Proposition 2. *Let $k\in\mathbb{Z}$ be even, $k\geq 4$. Then the Eisenstein Series G_k has the Fourier expansion*

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n,$$

where $q = e^{2\pi i\tau}$, $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$ is the Riemann-Zeta function and $\sigma_k(n) = \sum_{d|n} d^k$ is the generalized divisor sum function.

Lemma 4. *Let $k\in\mathbb{Z}$ be even, $k\geq 4$. Then for every $\tau\in\mathbb{H}$ we have that*

$$\sum_{n\in\mathbb{Z}}(\tau+n)^{-k} = \frac{(2\pi i)^k}{(k-1)!}\sum_{d=1}^{\infty} d^{k-1}e^{2\pi i d\tau}.$$

With this lemma we can already calculate the Fourier expansion of G_k. Write

$$G_k(\tau) = \sum_{n\neq 0} n^{-k} + \sum_{m\in\mathbb{Z}\setminus\{0\}} \sum_{n\in\mathbb{Z}} (m\tau+n)^{-k} = 2\zeta(k) + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}(m\tau+n)^{-k}.$$

The second equality comes from the fact that k is even and thus

$$\sum_{n\in\mathbb{Z}}(-m\tau+n)^{-k} = \sum_{n\in\mathbb{Z}}(-m\tau-n)^{-k} = \sum_{n\in\mathbb{Z}}(m\tau+n)^{-k}.$$

Now we apply Lemma 4 (but with τ replaced by $m\tau$ to obtain that

$$\begin{aligned}
G_k(\tau) &= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sum_{d=1}^{\infty} d^{k-1}e^{2\pi i d m\tau} \\
&= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sum_{d\mid n} d^{k-1}e^{2\pi i n\tau} \\
&= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi i n\tau},
\end{aligned}$$

as desired. In the second equality, we made the subsitution $n = md$. So it remains to proof the lemma.

Proof. Consider the function $f(\tau) = \sum_{n\in\mathbb{Z}}(\tau+n)^{-k}$ on the upper-half plane. Then $f(\tau+1) = f(\tau)$, because increasing τ by an integer corresponds to a shift of $\mathbb{Z}$, which does not change the sum. f is also holomorphic (we can essentially copy the argument that G_k is holomorphic). Therefore, f has a Fourier expansion

$$\sum_{n\in\mathbb{Z}}(\tau+n)^{-k} = \sum_{d\in\mathbb{Z}}\left(\int_0^1 f(\tau)e^{-2\pi i d\tau}\right)e^{2\pi i d\tau},$$

where the inner integral does not depend on $y = \mathrm{Im}(\tau)$. The same proof as in Lemma 3 shows that $\sum_{n\in\mathbb{Z}}(\tau+n)^{-k}$ converges locally uniformly and thus uniformly on the compact set $[0,1]\times\{y\}$. Hence, we can exchange limit and integral to obtain

$$\begin{aligned}
\int_0^1 \sum_{n\in\mathbb{Z}}(\tau+n)^{-k}e^{-2\pi i d\tau}dx &= \sum_{n\in\mathbb{Z}}\int_0^1(\tau+n)^{-k}e^{-2\pi i d\tau}dx \\
&= \sum_{n\in\mathbb{Z}}\int_n^{n+1}\tau^{-k}e^{-2\pi i d\tau}dx \qquad (2)\\
&= \int_{-\infty}^{\infty}\tau^{-k}e^{-2\pi i d\tau}dx
\end{aligned}$$

So it only remains to calculate this last integral. We distinguish three cases:

1. $d < 0$. In this case, we can bound

$$\begin{aligned}
\left|\int_{-\infty}^{\infty}\tau^{-k}e^{-2\pi i d\tau}dx\right| &\leq \int_{-\infty}^{\infty}|\tau^{-k}e^{-2\pi i d\tau}|dx \\
&= e^{2\pi i d y}\int_{-\infty}^{\infty}|\tau^{-k}|dx
\end{aligned}$$

Now we can take the limit $y \to \infty$ on both sides (recall that the integral in (2) is independent of y). Then the right hand side will be zero, given that the integral $\int_{-\infty}^{\infty} |\tau^{-k}| dx$ is finite. To show this, bound $|\tau|^{-k}$ by y^{-k} in the unit ball around 0 and by x^{-k} outside the unit ball. Then:

$$\begin{aligned}
\int_{-\infty}^{\infty} |\tau^{-k}| dx &\leq 2 \int_{0}^{1} y^{-k} dx + 2 \int_{1}^{\infty} x^{-k} dx \\
&= 2y^{-k} + 2\frac{1}{1-k} x^{1-k} \Big|_{1}^{\infty} \\
&= 2y^{-k} + 2\frac{1}{1-k} \\
&< \infty
\end{aligned}$$

This calculation shows that $\int_{-\infty}^{\infty} \tau^{-k} e^{-2\pi i d\tau} dx = 0$ for $d < 0$.

2. $d = 0$. In this case, the above argument won't work, because $e^{2\pi i dy} = 1$. However, now

$$\begin{aligned}
\int_{-\infty}^{\infty} \tau^{-k} e^{-2\pi i d\tau} dx &= \int_{-\infty}^{\infty} \tau^{-k} dx \\
&= \int_{-\infty}^{\infty} (x+iy)^{-k} dx \\
&= \frac{1}{-k+1} (x+iy)^{-k+1} \Big|_{x=-\infty}^{x=\infty} \\
&= 0,
\end{aligned}$$

because $\lim_{x\to\pm\infty} (x+iy)^{-k} = 0$ (simply calculate the modulus of the expression, which is $(x^2+y^2)^{-k/2}$, which goes to 0 as $x \to \pm\infty$).

3. $d > 0$ In this case we use the integral representation

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (c+it)^{-z} e^{c+it} dt$$

for every $z \in \mathbb{C}$ and any $c > 0$ as follows:

set $t := -2\pi dx$ and $c := 2\pi dy > 0$. Then we calculate

$$\begin{aligned}
\int_{-\infty}^{\infty} \tau^{-k} e^{-2\pi i d\tau} dx &= \int_{-\infty}^{\infty} (x+iy)^{-k} e^{-2\pi i dx + 2\pi dy} dx \\
&= \int_{-\infty}^{\infty} (\frac{t}{-2\pi d} + \frac{ic}{2\pi d})^{-k} e^{c+it} \frac{1}{2\pi d} dt \\
&= \frac{1}{2\pi d} \int_{-\infty}^{\infty} (\frac{1}{2\pi d})^{-k} (-t+ic)^{-k} e^{c+it} dt \\
&= \frac{(2\pi d)^k}{2\pi d} (-i)^k \int_{-\infty}^{\infty} \big((-i)(-t+ic)\big)^{-k} e^{c+it} dt \\
&= \frac{(2\pi d i)^k}{2\pi d} \int_{-\infty}^{\infty} (c+it)^{-k} e^{c+it} dt \\
&= \frac{(2\pi d i)^k}{2\pi d} \frac{2\pi}{\Gamma(k)} \\
&= \frac{(2\pi d)^k}{\Gamma(k)} d^{k-1},
\end{aligned}$$

where we used that $(-i)^k = i^k$ because k is even in the fifth equality. The result follows using the fact that $\Gamma(k) = (k-1)!$ for any positive integer k.

□

Putting all our results together we finally obtain Proposition 1:

Proof. Lemma 2 states that G_k is holomorphic and by Lemma 3 it transforms invariantly under Γ with weight k. Moreover, by Proposition 2, G_k has a Fourier expansion where all coefficients of negative index vanish. Also, the zeroth Fourier coefficient is $\zeta(k) \neq 0$, so G_k is non-zero. □

Note that if k is even, G_k is no *cusp form.*

Definition 3. For even $k \in \mathbb{Z}$ with $k \geq 4$ we call the map

$$E_k : \tau \in \mathbb{H} \mapsto \frac{1}{2\zeta(k)} G_k(\tau) = 1 + \frac{(2\pi i)^k}{\zeta(k)(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

the *normalized Eisenstein series.*

Proposition 3. *The normalized Eisenstein series has the representation*

$$E_k = \sum_{M \in \Gamma_\infty \backslash \Gamma} 1|_k M,$$

where 1 *denotes the constant function* $\equiv 1$ *on the upper half plane and* Γ_∞ *the subgroup* $\{\pm T^n | n \in \mathbb{Z}\}$.

Proof. For every $\tau \in \mathbb{H}$ it holds

$$\begin{aligned}
E_k(\tau) = \frac{1}{2\zeta(k)} G_k(\tau) &= \frac{1}{2\zeta(k)} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} (m\tau+n)^{-k} \\
&= \frac{1}{2\zeta(k)} \sum_{l=1}^{\infty} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=l}} (m\tau+n)^{-k} \\
&= \frac{1}{2\zeta(k)} \sum_{l=1}^{\infty} l^{-k} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1}} (m\tau+n)^{-k} \\
&= \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1}} (m\tau+n)^{-k}
\end{aligned}$$

Now let $M_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ be arbitrary. Note that $\gcd(c,d) = 1$ and $j(M_1,\tau)^{-k} = (c\tau+d)^{-k}$. If $M_2 = \left(\begin{smallmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{smallmatrix}\right)$ is a second arbitrary element of Γ, we claim that

$$c = \tilde{c} \text{ and } d = \tilde{d} \Leftrightarrow \exists n \in \mathbb{Z}\colon\ M_1 = T^n M_2,$$

where $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Since $T^n = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ leaves the bottom row of M_2 invariant, the implication ("$\Leftarrow$") is clear. Conversely, assume $c = \tilde{c}$, $d = \tilde{d}$ and let $m_1, m_2 \in \mathbb{Z}$ such that $a = \tilde{a} + m_1$ and $b = \tilde{b} + m_2$. Then

$$0 = \det(M_1) - \det(M_2) = (a-\tilde{a})d - (b-\tilde{b})c = dm_1 - cm_2$$

But c and d are coprime, which implies the existence of $n \in \mathbb{Z}$ such that $m_1 = cn$ and $m_2 = dn$. Therefore it holds $M_1 = T^n M_2$, which yields the claim.
This defines a natural bijection between coprime pairs of integers and the cosets of $\Gamma_\infty^+\backslash\Gamma$, where $\Gamma_\infty^+ = \{T^n | n \in \mathbb{Z}\}$. Hence we have

$$\begin{aligned}
E_k(\tau) &= \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1}} (m\tau+n)^{-k} \\
&= \frac{1}{2} \sum_{M\in\Gamma_\infty^+\backslash\Gamma} j(M,\tau)^{-k} \\
&= \sum_{M\in\Gamma_\infty\backslash\Gamma} j(M,\tau)^{-k} \\
&= \sum_{M\in\Gamma_\infty\backslash\Gamma} 1|_k M.
\end{aligned}$$

□

It is easily seen that for $k \geq 4$ the modular forms of weight k define a $\mathbb{C}$-vector space M_k. The same is true for S_k, the set of all cusp forms of weight k. The following lemma states, that M_k is the direct sum of S_k and the subspace spanned by the Eisenstein series of weight k.

Lemma 5. *For even $k \in \mathbb{Z}$ with $k \geq 4$ we have $M_k = \mathbb{C}E_k \oplus S_k$.*

Proof. We have $M_k = \mathbb{C}E_k + S_k$, since $f - a_f(0)E_k \in S_k$ where $a_f(0)$ denotes the constant coefficient of the Fourier expansion of f. As for all $\lambda \in \mathbb{C}$ we have $a_{\lambda E_K}(0) = \lambda a_{E_k}(0) = \lambda$, we also have $\mathbb{C}E_k \cap S_k = \{0\}$. $\square$

This lemma allows us to estimate the growth of the Fourier coefficients of a modular form of weight k.

Theorem 2. *Let $k \in \mathbb{Z}$ such that $k \geq 4$ and $f \in M_k$ with Fourier coefficients $a_f(n)$ for all $n \geq 0$. Then there exists a constant $C > 0$, such that $a_f(n) \leq Cn^{k-1}$ for all $n \geq 1$.*

Proof. By Lemma 5 we find $\alpha \in \mathbb{C}$ and $g \in S_k$, such that $f = \alpha E_k + g$. The Hecke Bound yields the estimation for the cusp form g. Hence it suffices to proof the estimation for E_k, i.e for the sequence $\sigma_{k-1}(n)$:

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1} = \sum_{d|n} \frac{n^{k-1}}{d^{k-1}} < n^{k-1} \sum_{d=1}^{\infty} d^{1-k} = n^{k-1}\zeta(k-1).$$

Since $k - 1 > 1$, $\zeta(k-1)$ is indeed well-defined, which proofs the claim. $\square$

Surprisingly, the Fourier coefficients of the normalized Eisenstein series E_k turn out to be rational numbers. To proof this remarkable fact, it is clear from Definition 3 that it is enough to show that the constant factor $\frac{(2\pi i)^k}{\zeta(k)}$ is rational. In order to do so, we need the following definition.

Definition 4. The *Bernoulli numbers* B_n are the coefficients of the Taylor expansion around $x_0 = 0$ of the map $b : x \in B_{2\pi}(0) \subset \mathbb{C} \mapsto \begin{cases} \frac{x}{e^x - 1} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$, i.e. for all $n \in \mathbb{N}_0$ we define $B_n = b^{(n)}(0)$. They satisfy for all $x \in B_{2\pi}(0)$

$$b(x) = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

It can be shown, that the Bernoulli numbers are all rational. Furthermore we have $B_n = 0$ for all odd $n > 1$. The link to the Fourier coefficients of the Eisenstein series is now given by the *Euler formula*:

Proposition 4. *For even $n \in \mathbb{N}$ with $n \geq 2$ it holds*

$$2\zeta(n) = -\frac{(2\pi i)^n}{n!} B_n.$$

Proof. Consider the partial fraction expansion of the cotangent: For all $x \in \mathbb{R}\backslash\mathbb{Z}$ it holds

$$\pi \cot(\pi x) = \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x+n} + \frac{1}{x-n} \right).$$

Multiplying by x removes the pole at 0 and we obtain an expression for the

Taylor expansion of $x\pi\cot(\pi x)$ around 0. For all $x \neq 0$ small enough

$$\begin{aligned}
x\pi\cot(\pi x) &= 1 + \sum_{n=1}^{\infty}\left(\frac{x}{x+n} + \frac{x}{x-n}\right) \\
&= 1 + \sum_{n=1}^{\infty}\frac{x}{n}\left(\sum_{k=0}^{\infty}\left(-\frac{x}{n}\right)^k - \left(\frac{x}{n}\right)^k\right) \\
&= 1 - 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{x}{n}\right)^{2k} \\
&\overset{(*)}{=} 1 - 2\sum_{k=1}^{\infty}x^{2k}\sum_{n=1}^{\infty}n^{-2k} \\
&= 1 - 2\sum_{k=1}^{\infty}x^{2k}\zeta(2k).
\end{aligned}$$

In $(*)$ we used that the series over the doubly-indexed sequence converges absolutely and thus, by Fubini, the order of summation can be changed. Substituting $y = \pi x$ we get the Taylor expansion of $y\cot(y)$ around 0. But now, using the formula $\cot(y) = i\frac{e^{2iy}+1}{e^{2iy}-1}$, we obtain a different expression for this Taylor expansion. For all $y \neq 0$ small enough

$$y\cot(y) = iy + \frac{2iy}{e^{2iy}-1} = \sum_{k=0}^{\infty} B_{2k}\frac{(2iy)^{2k}}{(2k)!}.$$

Here we used that $B_1 = -\frac{1}{2}$ and that all Bernoulli numbers of higher odd index vanish. Comparison of the two expressions for the Taylor series of $y\cot(y)$ yields the claim. $\square$

Using the Euler formula, we immediately obtain the following expression for the Fourier series of the normalized Eisenstein series:

Theorem 3. *For even $k \in \mathbb{Z}$ with $k \geq 4$ it holds*

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n.$$

In particular, the Fourier coefficients of E_k are rational.

Example 1. For $k = 4$ and $k = 6$ we have the following Fourier expansions:

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 \ldots$$

$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + 1575504q^5 \ldots$$

Remark 1.

(i) The Fourier coefficients of E_k have bounded denominators. Indeed, for C the enumerator of B_k, the modular form CE_k has integer-valued Fourier coefficients.

(ii) For *odd* $k \in \mathbb{N}$ with $k \geq 3$ we know very little about the values $\zeta(k)$. It was shown by Apéry in 1979 [1] that $\zeta(3)$ is irrational, and in 2001 Zudilin [13] was able to prove that at least one of the values $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.

3 The Ramanujan Delta Function

In this section we give a first example of a non-trivial cusp form.

Definition 5. The *Ramanujan delta function* is defined as

$$\Delta = \frac{E_4^3 - E_6^2}{1728}.$$

The delta function is indeed the example we were looking for:

Theorem 4. *Δ is a non-trivial cusp form of weight 12.*

The proof is a simple calculation using the Cauchy product. However, this will require the absolute convergence of the Fourier series of E_4 and E_6, which we would like to show in the general framework of arbitrary modular forms.

Lemma 6. *Let f be a modular form. Then f has an absolutely convergent Fourier expansion.*

Proof. Consider the map $g : \mathbb{H} \to \Omega, \tau \mapsto e^{2\pi i \tau}$, where Ω denotes the punctured open unit disc $B_1(0) \setminus \{0\}$. For every $\omega \in \Omega$ the preimage $g^{-1}(\omega)$ is non-empty and a translated lattice of the type $\tau_0 + \mathbb{Z}$ for some $\tau_0 \in g^{-1}(\omega)$. Since f is a modular form, it is 1-periodic and thus constant on $g^{-1}(\omega)$. Then, by the factorization theorem for holomorphic functions [11], it exists $h : \Omega \to \mathbb{C}$ holomorphic, such that $f = h \circ g$. As a holomorphic map on the punctured open unit disc, h has a normally convergent Laurant expansion [3]. Thus we find complex coefficients $(a_n)_{n \in \mathbb{Z}}$, such that for all $z \in \Omega$ it holds $h(z) = \sum_{n \in \mathbb{Z}} a_n z^n$. Hence, $f(\tau) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n \tau}$ for all $\tau \in \mathbb{H}$. This series converges normally and therefore in particular absolutely on $\mathbb{H}$. By uniqueness of the Fourier expansion, this is indeed the Fourier series of f, which completes the proof. $\square$

Now back to the proof of the theorem.

Proof. By a lemma seen in the last talk, E_4^3 and E_6^2 are both modular forms of weight 12, thus Δ is indeed a modular form of weight k. To show that it is a non-trivial cusp form, we would like to compute its first two Fourier coefficients. By respective exponentiation of the Fourier series of E_4 and E_6 according to the Cauchy product rule, we obtain the Fourier expansions of E_4^3 and E_6^2 thanks to the lemma above. They both have constant coefficient 1, which shows that Δ is a cusp form. Similarly we find that their coefficients of index 1 are 720 and -1008. Thus Δ has 1 as a Fourier coefficient and is therefore non-trivial. $\square$

Definition 6. Denoting the delta function's Fourier coefficients by $\tau(n)$, i.e. $\Delta(z) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}$, we define the *Ramanujan tau function* as the map $\tau : n \in \mathbb{Z}^{\geq 1} \mapsto \tau(n)$.

Remark 2. The delta function has the product expansion

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

We will prove this fact in two weeks from now. From the product expansion it is clear that τ maps into $\mathbb{Z}$. In addition, this formula allows us to proof, which we will do next week already, that Δ doesn't vanish anywhere.

Remark 3. The Ramanujan tau function satisfies (or at least seems to satisfy) some interesting properties. Ramanujan himself conjectured properties (i)-(iv) (cf. [9]).

(i) The tau function is multiplicative for coprime numbers, i.e.

$$\tau(nm) = \tau(n)\tau(m),$$

for all $(n, m) = 1$. For instance, we have

$$\tau(6) = -6048 = -24 \cdot 252 = \tau(2)\tau(3).$$

We will prove this in one of the talks on Hecke operators.

(ii) There is a recursive formula for the tau function on powers of even primes, namely

$$\tau(p^n) = \tau(p)\tau(p^{n-1}) - p^{11}\tau(p^{n-2})$$

for all primes p and integers $n \geq 2$. [6]

(iii) τ satisfies many congruences. A famous one, conjectured by Ramanujan, is the following:

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}$$

for all $n \in \mathbb{N}$. A proof can be found in [7].

(iv) For every prime p it holds

$$|\tau(p)| \leq 2p^{11/2}.$$

This property is known as the *Ramanujan conjecture* and was proven in 1979 by Deligne [4],[5] as a consequence of his work on the Weil conjectures. Note that it is only a tiny improvement of the Hecke bound for this special case - nevertheless this is a significantly deeper result, as evidenced by Deligne's proof.

(v) Lehmer famously conjectured that $\tau(n) \neq 0$ for all $n \in \mathbb{N}$. The conjecture is still open to date.
However, Lehmer was able to prove [8] that if the tau function ever vanishes, then the smallest zero of τ is a prime.

(vi) There exists an effectively computable constant $C > 0$, such that for all $n \geq 1$ for which $\tau(n)$ is an *odd* value, it holds

$$|\tau(n)| \geq \log(n)^C.$$

This result is due to Murty, Murty and Shorey [10] and in particular it implies that for odd integers a, the equation

$$\tau(n) = a$$

has only finitely many solutions.

(vii) We have that

$$\tau(n) \notin \{\pm 1, \pm 3, \pm 5, \pm 7, \pm 13, \pm 17, -19, \pm 23, \pm 37, \pm 691\}$$

for all $n > 1$. This is a very recent result from 2023 by Balakrishnan, Craig, Ono and Tsai [2].

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