The valence formula and the structure of M_k

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1 The valence formula

1.1 Preliminaries

Recall

- the complex upper half-plane $\mathbb{H} := \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$,
- the modular group $\Gamma := \mathrm{SL}_2(\mathbb{Z}) := \{A \in M_{2\times 2}(\mathbb{Z}) : \det A = 1\}$,
- the Möbius transformations defined for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\tau \in \mathbb{H}$ as

$$M\tau = \frac{a\tau + b}{c\tau + d}.$$

With this, a *modular form of weight* $k \in \mathbb{Z}$ is defined as a function $f : \mathbb{H} \to \mathbb{C}$ such that

1. f is holomorphic on $\mathbb{H}$,
2. $f(M\tau) := f(\frac{a\tau+b}{c\tau+d}) = (c\tau + d)^k f(\tau)$,
3. f has a Fourier expansion of the form

$$f(\tau) = \sum_{n=0}^{\infty} a_f(n) q^n,$$

where $q = e^{2\pi i \tau}$

Now, let $f \in M_k$ be a modular form of weight k. As a holomorphic function on $\mathbb{H}$, it has a Taylor expansion at each point $a \in \mathbb{H}$ of the form

$$f(\tau) = \sum_{n=0}^{\infty} c_{f,a}(n)(\tau - a)^n$$

with coefficients $c_{f,a}(n) \in \mathbb{C}$. We define the order of f at a by

$$\mathrm{ord}_a(f) = \min \{n \in \mathbb{N}_0 : c_{f,a}(n) \neq 0\},$$

and we define the order of f at ∞ by

$$\mathrm{ord}_\infty(f) = \min \{n \in \mathbb{N}_0 : a_f(n) \neq 0\}.$$

Finally, note that f is 1-periodic, i.e.

$$f(\tau + 1) = f(\tau) \quad \forall \tau \in \mathbb{H}.$$

This follows directly from condition 2 in the definition of a modular form when taking $M = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

1.2 The valence formula

With these definitions, we can state and prove a formula for the sum of the orders of a modular form:

Theorem 1.1 (Valence formula)**.** *Let i be the imaginary unit and $\rho = e^{i\frac{\pi}{3}}$. For $f \in M_k$ with $f \neq 0$,*

$$\operatorname{ord}_\infty(f) + \frac{1}{2}\operatorname{ord}_i(f) + \frac{1}{3}\operatorname{ord}_\rho(f) + \sum_{\substack{\tau \in \Gamma\backslash\mathbb{H} \\ \tau \neq i,\rho \bmod \Gamma}} \operatorname{ord}_\tau(f) = \frac{k}{12}$$

Remark 1.1. 1. The formula is also called the $k/12$-formula. If we define $\operatorname{ord}(\infty) = 1$ and the order of a point $\tau \in \mathbb{H}$ as

$$\operatorname{ord}(\tau) = \frac{1}{2}|\Gamma_\tau| = \begin{cases} 3, & \text{if } \tau = \rho \pmod{\Gamma} \\ 2, & \text{if } \tau = i \pmod{\Gamma} \\ 1, & \text{otherwise} \end{cases}$$

we can write the valence formula more compactly as

$$\sum_{\tau \in \Gamma\backslash\mathbb{H}\cup\{\infty\}} \frac{\operatorname{ord}_\tau(f)}{\operatorname{ord}(\tau)} = \frac{k}{12}.$$

2. The sum on the left-hand side of the valence formula is finite, as the zeros of $0 \neq f \in M_k$ do not accumulate in $\Gamma\backslash\mathbb{H}$.

Example 1.2. From the valence formula, it follows that the Eisenstein series E_4 (or E_6) has a simple zero at $\tau = \rho$ (or $\tau = i$) and no further zeros modulo Γ. This can be obtained by comparing the coefficients and since the order is an element of $\mathbb{N}_0$.

We now proceed to prove the weight formula. First, recall the fundamental domain $\mathcal{F}$:

$$\mathcal{F} = \{\tau \in \mathbb{H} : |\tau| \geq 1, -1/2 \geq \Re(\tau) \geq 1/2\}.$$

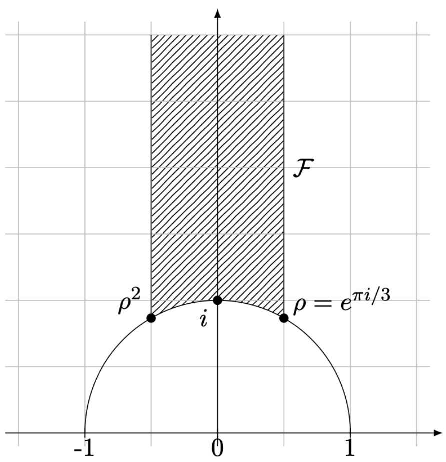

Figure 1: The fundamental domain $\mathcal{F}$, which contains the points i, ρ, and ρ^2. If we remove the left edge and the part of $\mathcal{F}$ which lies on the segment of the unit circle whose real part is < 0, this is a system of representants for the quotient $\Gamma/\mathbb{H}$. Graphic from [1].

Proof of Theorem 1.1. **The sum is well-defined:** We claim that for $f \in M_k$,

$$\mathrm{ord}_{M\tau}(f) = \mathrm{ord}_\tau(f)$$

for all $\tau \in \mathbb{H}$ and $M \in \Gamma$. Indeed, this follows directly from the second condition in the definition of a modular form by noting that

$$f(M\tau) = \underbrace{(c\tau + d)^k}_{\neq 0} f(\tau).$$

Hence, the order of f at τ depends only on the class of τ in $\Gamma\backslash\mathbb{H}$ and the sum on the left-hand side of the weight formula is well-defined.

The formula: Let

$$f = \sum_{n=n_0}^{\infty} a_f(n) q^n \in M_k$$

with $f \neq 0$ and $a_f(n_0) \neq 0$. In particular, $n_0 = \mathrm{ord}_\infty(f)$. There exists a $T > 0$ such that $f(\tau)$ has no zeros for $\mathrm{Im}(\tau) > T$. Otherwise, the zeros of the function $g(q) = \sum_{n=0}^{\infty} a_f(n) q^n$ (which is holomorphic for $|q| < 1$) would accumulate at $q = 0$, implying $g(q) = 0$ for all $|q| < 1$, which contradicts $f \neq 0$. For simplicity, let's assume f also has no zeros on the boundary of the fundamental domain $\mathcal{F}$. Later, we will see why this is allowed. We now consider the following path γ in $\mathbb{H}$:

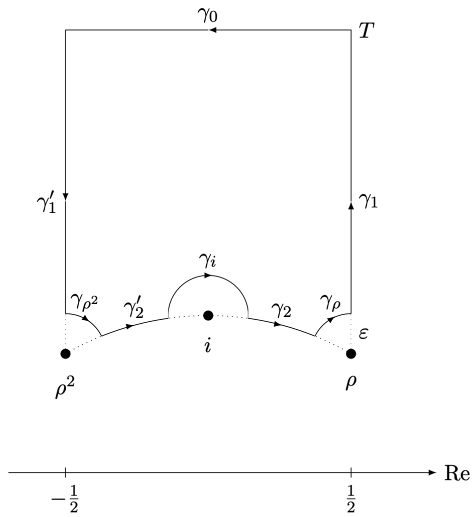

Figure 2: The path γ, taken from [1]

Here, $\varepsilon > 0$ is chosen small enough so that the circles around i, ρ, and ρ^2 do not contain any zeros inside $\mathcal{F}$. Note that γ has winding number 1.

According to the residue theorem, with $F = f'/f$, we have the formula

$$\int_\gamma F(\tau) d\tau = 2\pi i \sum_{w \in \mathcal{F}^\circ} \operatorname{ord}_w(f)$$

where $\mathcal{F}^\circ$ denotes the interior of $\mathcal{F}$. This is because F only has poles where f vanishes. Since the right-hand side does not depend on ε, we can take the limit $\varepsilon \to 0$ on the left-hand side. We compute the integrals over the individual subpaths:

- The path γ_0: The function $F = f'/f$ is 1-periodic and has a Fourier expansion of the form

$$F(\tau) = \frac{2\pi i \sum_{n=n_0}^\infty n a_f(n) q^n}{\sum_{n=n_0}^\infty a_f(n) q^n} = 2\pi i n_0 + \ldots$$

  We parameterize $-\gamma_0$ (where the minus indicates the reversal of orientation) by $x \mapsto x + iT$ with $x \in [-1/2, 1/2]$. Then,

$$\int_{\gamma_0} F(\tau) d\tau = -\int_{-1/2}^{1/2} F(x + iT) dx = -2\pi i n_0 = -2\pi i \operatorname{ord}_\infty(f)$$

- The paths γ_1 and γ_1': Since F is 1-periodic, and γ_1' is the reverse of γ_1, we have

$$\int_{\gamma_1} F(\tau) d\tau + \int_{\gamma_1'} F(\tau) d\tau = 0$$

- The paths γ_2 and γ_2': Given $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, the function $F = f'/f$ transforms as follows:

  Since $f(M\tau) = (c\tau + d)^k f(\tau)$, applying the chain rule on the left-hand side and the product rule on the right-hand side gives

  $$f'(M\tau)\frac{dM\tau}{d\tau} = k(c\tau + d)^{k-1} f(\tau) + (c\tau + d)^k f'(\tau),$$

  and hence

  $$F(M\tau)dM\tau = \left(\frac{kc}{c\tau + d} + F(\tau)\right) d\tau$$

  The matrix $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ maps γ_2 to $-\gamma_2'$. Hence,

  $$\int_{\gamma_2'} F(\tau)d\tau = -\int_{S\gamma_2} F(\tau)d\tau = -\int_{\gamma_2} F(S\tau)dS\tau = -\int_{\gamma_2} \left(\frac{k}{\tau} + F(\tau)\right) d\tau$$

  and thus

  $$\int_{\gamma_2} F(\tau)d\tau + \int_{\gamma_2'} F(\tau) = -k \int_{\gamma_2} \frac{d\tau}{\tau}$$

  Using a substitution $\tau \mapsto e^{it}$, further computation yields

  $$\lim_{\varepsilon \to 0} \left(\int_{\gamma_2} \frac{d\tau}{\tau}\right) = \int_i^\rho \frac{d\tau}{\tau} = -\frac{2\pi i}{12}.$$

- The paths $\gamma_\rho, \gamma_{\rho^2}$, and γ_i: The function $F(\tau)$ has a Laurent expansion around ρ of the form

  $$F(\tau) = \sum_{n=-1}^{\infty} c_F(n)(\tau - \rho)^n$$

  Here, $c_F(-1) = \mathrm{ord}_\rho(f)$, which can be seen by substituting the Taylor expansion of f into $F = f'/f$.

  We write

  $$\lim_{\varepsilon \to 0} \int_{\gamma_\rho} F(\tau)d\tau = \lim_{\varepsilon \to 0} \left(c_F(-1) \int_{\gamma_\rho} \frac{d\tau}{\tau - \rho} + \int_{\gamma_\rho} \left(F(\tau) - \frac{c_F(-1)}{\tau - \rho}\right) d\tau\right).$$

  The second integral vanishes as $\varepsilon \to 0$ since the integrand is holomorphic at $\tau = \rho$ and the length of the integration path approaches 0. With $c_F(-1) = \mathrm{ord}_\rho(f)$, we obtain

  $$\lim_{\varepsilon \to 0} \int_{\gamma_\rho} F(\tau)d\tau = \mathrm{ord}_\rho(f) \lim_{\varepsilon \to 0} \int_{\gamma_\rho} \frac{d\tau}{\tau - \rho}$$

  To compute the integral, we choose $\alpha = \alpha(\varepsilon)$ such that $e^{i\alpha}$ lies on the unit circle to the left of ρ and $\left|e^{i\alpha} - \rho\right| = \varepsilon$. Let $\varphi = \varphi(\varepsilon) < 0$ be the angle between $\rho + i\varepsilon$ and $e^{i\alpha}$.

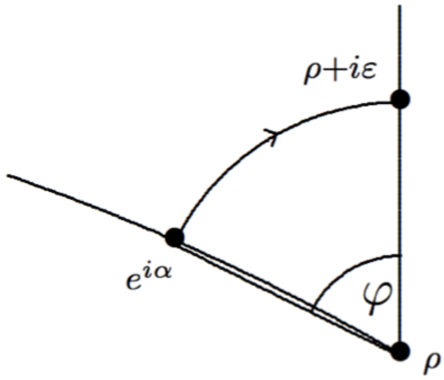

Figure 3: The path γ_ρ, taken from [1]

Then, we can parameterize the path γ_ρ as

$$\rho + \varepsilon e^{it}, \quad \frac{\pi}{2} + \varphi \leq t \leq \frac{\pi}{2}$$

Hence,

$$\lim_{\varepsilon \to 0} \int_{\gamma_\rho} \frac{d\tau}{\tau - \rho} = \lim_{\varepsilon \to 0} \int_{\pi/2+\varphi}^{\pi/2} \frac{1}{\varepsilon e^{it}} i\varepsilon e^{it} dt = -i \lim_{\varepsilon \to 0} \varphi = \frac{\pi i}{3}$$

Overall, we have

$$\lim_{\varepsilon \to 0} \int_{\gamma_\rho} F(\tau) d\tau = -\frac{2\pi i}{6} \operatorname{ord}_\rho(f)$$

Analogously, we show

$$\lim_{\varepsilon \to 0} \int_{\gamma_{\rho^2}} F(\tau) d\tau = -\frac{2\pi i}{6} \operatorname{ord}_\rho(f)$$

and

$$\lim_{\varepsilon \to 0} \int_{\gamma_i} F(\tau) d\tau = -\frac{2\pi i}{2} \operatorname{ord}_i(f).$$

Combining all paths, we obtain the valence formula from 1.1. If zeros lie on the boundary of $\mathcal{F}$, as with i, ρ, and ρ^2, we excise small circles around the zeros. The additional boundary integrals are treated similarly to above. $\square$

Lemma 1.2. *The Δ-function does not have any zeros on $\mathbb{H}$. In particular, it induces an Isomorphism*

$$M_k \to S_{k+12}, \quad f \mapsto \Delta \cdot f$$

Proof. Recall that if $f \in M_k$ and $g \in M_l$, then $fg \in M_{k+l}$. Moreover, if f or g is a cusp form, then so is fg. Since $\Delta \in S_{12}$, the map is well-defined. To see that Δ does not vanish on $\mathbb{H}$ we make use of the valence formula. For $k = 12$, the RHS of the formula equals 1. Moreover $\mathrm{ord}_\infty(\Delta) = 1$. Since $\mathrm{ord}_\tau(\Delta) \geq 0$ for all $\tau \in \mathbb{H}$, the valence formula implies $\mathrm{ord}_\tau(\Delta) = 0$ for all $\tau \in \mathbb{H}$, i.e. $\Delta(\tau) \neq 0$ for all $\tau \in \mathbb{H}$. Thus the map

$$S_{k+12} \to M_k, \quad g \mapsto g/\Delta$$

is well-defined and yields an inverse to $f \mapsto f \cdot \Delta$ □

Theorem 1.3. *For $k \leq 12$ the spaces M_k and S_k are given as follows:*

1. $M_k = S_k = \{0\}$ *for* $k < 0$.
2. $M_0 = \mathbb{C}$ *and* $S_0 = \{0\}$.
3. $M_2 = S_2 = \{0\}$.
4. $M_k = \mathbb{C}E_k$ *and* $S_k = \{0\}$ *for* $k = 4, 6, 8, 10$.
5. $S_{12} = \mathbb{C}\Delta$ *and* $M_{12} = \mathbb{C}E_{12} \oplus \mathbb{C}\Delta$.

Proof.
1. For $k < 0$, the RHS of the valence formula is negative, while the LHS is always positive (the valence formula holds for $f \neq 0$).
2. Assume $f \in M_0$ is not constant. Then $f - f(i)$ is not constant either and vanishes at i. Hence the LHS of the valence formula applied to $f - f(i)$ is greater or equal to $\frac{1}{2}$, and the RHS is equal to 0.
3. For $k = 2$, the RHS of the valence formula is equal to $\frac{1}{6}$. The LHS cannot attain the value $\frac{1}{6}$, since $\mathrm{ord}_\infty(f)$ and $\mathrm{ord}_\tau(f)$, $\tau \in \mathbb{H}$, are whole numbers.
4. For $k = 4, 6, 8, 10$ and $f \in S_k$ we have $f/\Delta \in M_{k-12}$ is a modular form of negative weight, which by the first point vanishes identically. Since the map $f \mapsto f/\Delta$ is injective, it follows that $S_k = \{0\}$. Hence also $M_k = \mathbb{C}E_k \oplus S_k = \mathbb{C}E_k$.
5. Let $f \in S_{12}$ be a cusp form. Then $f/\Delta \in M_0$ is a modular form of weight 0, hence constant.

□

Theorem 1.4. *For even $k > 2$ we have the following dimension formula*

$$\dim(M_k) = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor, & \text{if } k \equiv 2 \pmod{12} \\ \left\lfloor \frac{k}{12} \right\rfloor + 1, & \text{if } k \not\equiv 2 \pmod{12}. \end{cases}$$

and $\dim(S_k) = \dim(M_k) - 1$

Proof. We do induction on k. By the previous theorem the formula is valid for $k = 4, 6, 8, 10, 12$. Let now $k > 12$ and assume that the dimension formula holds for all weights less than k. Since the map $S_k \to M_{k-12}, f \mapsto f/\Delta$ is an isomorphism, we get

$$\begin{aligned}\dim(M_k) &= 1 + \dim(S_k) = 1 + \dim(M_{k-12}) \\ &= 1 + \begin{cases} \left\lfloor \frac{k-12}{12} \right\rfloor, & \text{if } k-12 \equiv 2 \pmod{12} \\ \left\lfloor \frac{k-12}{12} \right\rfloor + 1, & \text{if } k-12 \not\equiv 2 \pmod{12}, \end{cases} \\ &= \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor, & \text{if } k \equiv 2 \pmod{12} \\ \left\lfloor \frac{k}{12} \right\rfloor + 1, & \text{if } k \not\equiv 2 \pmod{12}, \end{cases}\end{aligned}$$

which is the desired dimension formula. □

Theorem 1.5. *For even $k \geq 4$, a basis of M_k is given by the functions*

$$E_4^\alpha E_6^\beta, \quad \alpha, \beta \in \mathbb{N}_0, 4\alpha + 6\beta = k.$$

In particular, every modular form $f \in M_k$ can be written uniquely as a polynomial in E_4 and E_6, i.e. we have an isomorphism of rings

$$M_* = \bigoplus_{\substack{k=0 \\ k \text{ even}}}^{\infty} M_k \cong \mathbb{C}[E_4, E_6] \cong \mathbb{C}[X, Y]$$

Proof. We prove by induction on $k \geq 0$ that we can write every $f \in M_k$ as a polynomial in E_4 and E_6. The monomials will then be of the form $CE_4^\alpha E_6^\beta$ with $4\alpha + 6\beta = k$, so that f can be written as a linear combination of the $E_4^\alpha E_6^\beta$.

For $k = 0, 2, 4, 6$ the statement is true due to theorem 1.3. For $k = 8$ and $k = 10$, from a dimension argument and by comparing the constant Fourier coefficients, we have $E_8 = E_4^2$ and $E_{10} = E_4 E_6$.

Let now $k \geq 12$ and assume that every modular form of weight less than k can be written as a polynomial in E_4 and E_6. Choose $\alpha, \beta \in \mathbb{N}_0$ sucht that $4\alpha + 6\beta = k$. Then $E_4^\alpha E_6^\beta$ has weight k and constant Fourier coefficient 1. By decomposing each $f \in M_k$ adequately, it follows that $M_k = \mathbb{C}E_4^\alpha E_6^\beta \bigoplus S_k$. Now, for $f \in S_k$ we have $f = g \cdot \Delta$ for some $g \in M_{k-12}$, which by induction hypothesis can be written as a polynomial in E_4 and E_6. Since we also have $\Delta = \frac{E_4^3 - E_6^2}{1728}$, it follows that f can be written as a polynomial in E_4 and E_6. We have thus showed that every modular form $f \in M_k$ can be written as a linear combination of the functions

$$E_4^\alpha E_6^\beta, \quad \alpha, \beta \in \mathbb{N}_0, 4\alpha + 6\beta = k.$$

Since the set of all $\alpha, \beta \in \mathbb{N}_0$ such that $4\alpha + 6\beta = k$ has cardinality equal to $\dim(M_k)$, these functions form a basis of the space M_k.

□

Remark 1.3. In particular, note the astonishing fact that for each k the basis of M_k consists of functions with *entire* Fourier coefficients.

References

[1] Markus Schwagenscheidt, *Modulformen*, lecture notes, available online.