The Eisenstein series of weight 2 and the Dedekind-Eta function

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1 The Eisenstein series of weight 2

In the first part of this talk, we will answer a question that comes up naturally when looking at the definition of Eisenstein series: Why can we only define it for even numbers greater or equal to 4?

Reminder *For $k \geq 4$ even, we defined the Eisenstein series $G_k : \mathbb{H} \to \mathbb{C}$,*

$$G_k(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} (m\tau + n)^{-k} \tag{1}$$

and showed that it is a modular form of weight k. Furthermore, its Fourier expansion is given by

$$G_k(\tau) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi i n\tau}, \tag{2}$$

where $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$ is the Riemann zeta function and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the sum of positive divisors function.

If we try to extend the definition of G_k to other integers k, we find that the sum (1) vanishes for odd $k \geq 3$, since we then have an absolutely convergent series and $-(m\tau + n)^{-k} = (-m\tau - n)^{-k}$ for $m, n \in \mathbb{Z}$.

For $k = 1, 2$, it turns out that the series is not absolutely convergent, hence the functions G_1 and G_2 are only well-defined once we fix an order of summation, as there is no conventional order to sum over $\mathbb{Z}^2\setminus\{0\}$. For $k = 1$, we again have the property $-(m\tau + n)^{-k} = (-m\tau - n)^{-k}$ for $m, n \in \mathbb{Z}$, which makes the series vanish for the most natural summation orders over $\mathbb{Z}^2\setminus\{(0,0)\}$. In the following, we shall take a closer look at the more interesting case of $k = 2$, i.e. the Eisenstein series of weight 2.

Firstly, we need to fix our order of summation. The one we choose is

$$G_2(\tau) = \sum_{n\in\mathbb{Z}\setminus\{0\}} n^{-2} + \sum_{m\in\mathbb{Z}\setminus\{0\}} (\sum_{n\in\mathbb{Z}} (m\tau+n)^{-2}) \tag{3}$$

where we employ the usual conventions of

$$\sum_{k\in\mathbb{Z}} \equiv \lim_{K\to\infty} \sum_{k=-K}^{K}, \sum_{k\in\mathbb{Z}\setminus\{0\}} \equiv \lim_{K\to\infty} \sum_{k=-K}^{-1} + \sum_{k=1}^{K}.$$

By defining G_2 in this way, the expression (2) for the Fourier expansion still holds true. Indeed, when proving this formula a few weeks ago, we began by fixing the summation order in the exact way as in (3). We did not need absolute convergence outside of that, hence the same proof works.

We already know that we cannot find modular forms of weight 2, but let us see how close we got with G_2. G_2 would be weight 2 modular form, if it satisfied the following conditions:

1. G_2 is holomorphic on $\mathbb{H}$.
2. $G_2(\tau+1) = G_2(\tau)$ for all $\tau \in \mathbb{H}$.
3. $G_2(-\frac{1}{\tau}) = \tau^2 G_2(\tau)$ for all $\tau \in \mathbb{H}$.
4. The Fourier coefficients $c_k(G_2)$ vanish for all $k < 0$.

Recall that the second and third conditions are equivalent to the invariance under the 2-slash operator, since they correspond to the invariance under two generators of the modular group Γ.

Proposition 1.1 *The function G_2 satisfies the conditions 1,2 and 4 above.*

Proof As noted, we have the Fourier expansion

$$G_2(\tau) = 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n\tau}. \tag{4}$$

From this we immediately obtain condition 4. For condition 2, we compute

$$\begin{aligned} G_2(\tau+1) &= \sum_{n\in\mathbb{Z}\setminus\{0\}} n^{-2} + \sum_{m\in\mathbb{Z}\setminus\{0\}} (\sum_{n\in\mathbb{Z}} (m\tau+m+n)^{-2}) \\ &= \sum_{n\in\mathbb{Z}\setminus\{0\}} n^{-2} + \sum_{m\in\mathbb{Z}\setminus\{0\}} (\sum_{n'\in\mathbb{Z}} (m\tau+n')^{-2}) = G_2(\tau). \end{aligned}$$

Note that the index shifting here will not change the value of the (conditionally convergent) series, since the inner series are in fact absolutely convergent.

Finally, we again use the Fourier expansion to show that G_2 is holomorphic. Note that the series

$$\sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n\tau} = \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{e^{2\pi i n \mathrm{Im}(\tau)}} e^{2\pi i n \mathrm{Re}(\tau)} \tag{5}$$

converges absolutely for $\tau \in \mathbb{H}$, i.e. $\mathrm{Im}(\tau) > 0$, since $\sigma_1(n) \leq 1 + \cdots + n \in O(n^2)$. Moreover, we can see that it even converges uniformly on sets that satisfy $\mathrm{Im}(z) \geq c > 0$ and in particular on compact subsets of $\mathbb{H}$. Hence the partial sums converge locally uniformly to G_2, which implies that G_2 is holomorphic by a well-known result in complex analysis (see e.g. [1, Thm. 2.21]). $\square$

While G_2 fails to satisfy the last condition of weight 2 modular forms, it only misses the mark by a linear summand:

Proposition 1.2 *We have* $G_2(-\frac{1}{\tau}) = \tau^2 G_2(\tau) - 2\pi i \tau$ *for* $\tau \in \mathbb{H}$.

A proof of this statement can be found in [2, Sec. 2.7].

We can use G_2 to define a function that transforms like a weight 2 modular form while sacrificing holomorphicity.

Example 1.3 *The non-holomorphic Eisenstein series* $G_2^*(\tau) := G_2(\tau) - \frac{\pi}{\mathrm{Im}(\tau)}$ *is invariant under the 2-slash operator.*

Proof We need to check that $G_2^*(\tau + 1) = G_2^*(\tau)$ and $G_2^*(-\frac{1}{\tau}) = \tau^2 G_2^*(\tau)$ hold for all $\tau \in \mathbb{H}$. The first is clearly true since we already know that G_2 and $\mathrm{Im}(\cdot)$ are 1-periodic. Setting $a := \mathrm{Re}(\tau), b := \mathrm{Im}(\tau)$, we have $\mathrm{Im}(-\frac{1}{\tau}) = \frac{b}{a^2+b^2}$. We check that:

$$\begin{aligned}
\tau^2 G_2^* - G_2^*(-\frac{1}{\tau}) &= \tau^2 (G_2(\tau) - \frac{\pi}{\mathrm{Im}(\tau)}) - G_2(-\frac{1}{\tau}) + \frac{\pi}{\mathrm{Im}(-\frac{1}{\tau})} \\
&= \tau^2 G_2(\tau) - \frac{\pi(a+bi)^2}{b} - \tau^2 G_2(\tau) + 2\pi i(a+bi) + \frac{\pi(a^2+b^2)}{b} \\
&= -\frac{\pi(a^2 + 2iab - b^2)}{b} + 2\pi i a - 2\pi b + \frac{\pi a^2}{b} + \pi b = 0,
\end{aligned}$$

where we used the previous proposition for the second equality. $\square$

2 The Dedekind-Eta function

In the second half of the talk we will introduce the Dedekind-Eta function and use it to prove an alternative representation of the Ramanujan-Delta function Δ.

Definition 2.1 *The Dedekind-Eta function is defined as*

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

for $\tau \in \mathbb{H}$*, where we have set* $q = e^{2\pi i \tau}$

The product converges absolutely and locally uniformly on $\mathbb{H}$ and thus defines a holomorphic function on $\mathbb{H}$ with $\eta(\tau) \neq 0$ for all $\tau \in \mathbb{H}$. This convergence is proven using the following result from complex analysis (see e.g. [3, Ch. 14]).

Lemma 2.2 *Let $\{a_n\}$ be a sequence of complex numbers. Then, if the sum $\sum_{n\in\mathbb{N}} |a_n|$ converges, the infinite product $\prod_{n\in\mathbb{N}}(1-a_n)$ converges absolutely to a non-zero value. Furthermore, if $\sum_{n\in\mathbb{N}} |a_n|$ converges uniformly, so does $\prod_{n\in\mathbb{N}}(1-a_n)$.*

Using this, we can reduce the convergence of the Dedekind-Eta function to the convergence of the infinite sum $\sum_{n\in\mathbb{N}} |q^n| = \sum_{n\in\mathbb{N}} |e^{2\pi i\tau n}|$.
Before continuing with the transformation property of the Dedekind-Eta function and deducing from this the product expansion of the Ramanujan-Delta function, we will briefly take a look at one possible use of the Dedekind-Eta function: proving the asymptotic growth of the partition function. We recall the definition of the partition function p(n)

Definition 2.3 *Let $n \in \mathbb{N}$. We define the partition function at n p(n) to be the number of distinct ways of representing n as a sum of positive integers. Here the order of the terms in the sum is irrelevant and we set the convention $p(0) = 1$ the unique empty sum.*

Hardy and Ramanujan conjectured the following asymptotic for the partition function:

$$p(n) \sim \frac{1}{4n\sqrt{3}} exp\left(\pi\sqrt{\frac{2n}{3}}\right) \quad as \quad n \to \infty$$

This was improved by Rademacher,using the Dedekind-Eta function, resulting in the following convergent series expression for p(n):

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\frac{d}{dn}\left(\frac{1}{\sqrt{n-1/24}} sinh\left[\frac{\pi}{k}\sqrt{\frac{2}{3}(n-1/24)}\right]\right)$$

How did Rademacher use the Dedekind-Eta function in his proof? For this we will examine the relation between the partition function and the Dedekind-Eta function. Firstly, let us recall the definition of a generating function:

Definition 2.4 *Let $\{a_n\}$ be a sequence. Then we define the generating function of that sequence as the formal power series*

$$f(q) = \sum_{n=0}^{\infty} a_n q^n$$

We can view the values of the partition function as a series and thus write the generating function of the partition function:

$$f(q) = \sum_{n=0}^{\infty} p(n) q^n$$

By Euler we have the following expression for this generating function:

$$f(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-1}$$

This bears remarkable similarity to the Dedekind-Eta function. In particular we get the following relation:

$$f(q) = q^{1/24} \frac{1}{\eta(\tau)}$$

Now that we have established some motivation for the definition of the Dedekind-Eta function, we will proceed with its relation to the Eisenstein-series of weight 2.

Lemma 2.5 *For $\tau \in \mathbb{H}$ it holds*

$$\frac{\eta'(\tau)}{\eta(\tau)} = \frac{i}{4\pi} G_2(\tau)$$

Proof We calculate the logarithmic derivative of η as follows:

$$\begin{aligned} \frac{\eta'(\tau)}{\eta(\tau)} &= \frac{\partial}{\partial \tau} log(\eta(\tau)) = \frac{\partial}{\partial \tau} \left(\frac{1}{24} log(q) + \sum_{n=1}^{\infty} log(1 - q^n) \right) \\ &= \frac{2\pi i}{24} + \sum_{n=1}^{\infty} \frac{-2\pi i n q^n}{1 - q^n} = \frac{\pi i}{12} \left(1 - 24 \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n} \right) \\ &= \frac{\pi i}{12} \left(1 - 24 \sum_{n \geq 1} n \sum_{m \geq 1} q^{nm} \right) \end{aligned}$$

where we have used the convergence of the geometric sum $\sum_{m \geq 1} q^{nm} = \frac{q^n}{1-q^n}$. Now we rewrite this further using the sum-of-divisors function $\sigma_1(n) = \sum_{d|n} d$

$$\frac{\eta'(\tau)}{\eta(\tau)} = \frac{\pi i}{12} (1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n) = \frac{i}{4\pi} G_2(\tau)$$

where in the last step we have plugged in the definition of the Eisenstein-series of weight 2: $G_2(\tau) = \pi^2/3 + 8\pi^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n$ □

From this we now want to deduce a transformation property of η. We first arrive at an auxiliary result.

Lemma 2.6 *For $\tau \in \mathbb{H}$ it holds*

$$\eta(\tau + 1) = e^{2\pi i/24} \eta(\tau)$$

and

$$\eta(-1/\tau) = \sqrt{-i\tau}\, \eta(\tau)$$

where we mean the principal branch of the square root.

Proof The first equality follows immediately from the definition of η:

$$\eta(\tau+1) = e^{\pi i(\tau+1)/12}\prod_{n=1}^{\infty}(1-e^{2\pi i(\tau+1)}) = e^{\pi i/12}\eta(\tau)$$

For the second equality we consider again the logarithmic derivative and define the following helping function.

$$f(\tau) = \frac{\eta'(\tau)}{\eta(\tau)} = \frac{i}{4\pi}G_2(\tau)$$

Using the transformation law of the Eisenstein-series of weight 2 i.e.

$$G_2(-1/\tau) = \tau^2 G_2(\tau) - 2\pi i\tau$$

we get the following identity for $f(\tau)$:

$$f\left(\frac{-1}{\tau}\right)\frac{1}{\tau^2} - f(\tau) - \frac{1}{2\tau} = 0 \quad \forall \tau \in \mathbb{H}$$

We define a second helping function:

$$g(y) = \frac{\eta(i/y)}{\eta(iy)\sqrt{y}} \quad for \quad y > 0$$

which fulfills the following equation

$$-i\frac{g'(y)}{g(y)} = f(i/y)\frac{i}{(iy)^2} - f(iy) - \frac{1}{2iy} = 0$$

From this it follows in particular that $g'(y) = 0$ for all $y > 0$ and so g is constant there i.e.

$$\eta(i/y) = \gamma\sqrt{y}\eta(iy)$$

with γ constant. To find this constant we plug in $y = 1$ and find $\gamma = 1$ and therefore

$$\eta(i/y) = \sqrt{y}\eta(iy) \quad for \quad y > 0 \tag{6}$$

We now have two holomorphic functions that agree on $y > 0$ which has an accumulation point in $\mathbb{H}$. Therefore, by the identity theorem, they agree on all of $\mathbb{H}$ and we get the desired equality. $\square$

From this, we can deduce a transformation property of the Dedekind-Eta function under the action of the group $\Gamma = SL_2(\mathbb{Z})$, which we will state without proof.

Lemma 2.7 *For* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ *we have the following transformation-law:*

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \nu_\eta \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sqrt{c\tau+d}\,\eta(\tau)$$

Where $\nu_\eta \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is a 24th root of unity.*

It is thus natural to call η a *modular form of weight 1/2*.
Before continuing with the product expansion of the Ramanujan-Delta function, we will see an application of the transformation law to the partition-function, using the relation between the two that we found earlier. In particular, we will use the transformation-law to prove an asymptotic behaviour of the generating function of the partition function.

Lemma 2.8 *We have the following asymptotic behaviour of the generating function of the partition function:*

$$f(q) \sim \sqrt{-\tau i}\, exp\left(\frac{\pi i}{12\tau}\right) \quad for \quad \tau \to 0, \tau \in \mathbb{H}$$

Proof Recall that we can write

$$f(q) = \frac{q^{1/24}}{\eta(\tau)} := g(\tau) \tag{7}$$

We replace τ with $-1/\tau$ and get

$$g(-1/\tau) exp\left(\frac{\pi i}{12\tau}\right) = \frac{1}{\eta(-1/\tau)} = \frac{1}{\eta(\tau)\sqrt{\tau/i}}$$

Where in the last step we have used the transformation property of the Dedekind-Eta function. Combining this with the expression of the generating function Eq. (7), we get:

$$f(q) = g(\tau) = q^{1/24} g(-1/\tau)\sqrt{\tau/i}\, exp\left(\frac{\pi i}{12\tau}\right)$$

Since the first two terms approach 1 as $\tau \to 0$ we get the final desired result. $\square$

Finally, we want to use the transformation law of the Dedekind-Eta function to prove the product expansion of the Ramanujan-Delta function.

Corollary 2.9 *We have the following product expansion of the Ramanujan-Delta function*

$$\Delta = \eta^{24} = q \prod_{n=1}^{\infty} (1-q^n)^{24}$$

Proof Let T and S denote the transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$ respectively. From Lemma 2.6 we know

$$\eta^{24}|_{12}T = (1)^{-12}e^{48\pi i/24}\eta^{24}(\tau) = \eta^{24}$$
$$\eta^{24}|_{12}S = \tau^{-12}(-i\tau)^{1}2\eta^{24} = \eta^{24}$$

Since these two generate $SL_2(\mathbb{Z})$, we get $\eta^{24} \in M_{12}$. If we look at the expansion of the product, we can see that it has no constant term, i.e. $a_{\eta^{24}}(0) = 0$ and we therefore have $\eta^{24} \in S_{12}$. Due to the classification of cusp forms we saw last week, this means that $\eta^{24} \in \mathbb{C}\Delta$. The coefficient of η^{24} at q is 1 and so, comparing to the coefficients of the Ramanujan-Delta function and using the classification, we get $\eta^{24} = \Delta$ $\square$

References

[1] Q. Kuang, *Complex Analysis*, lecture notes, 2018

[2] M. Schwagenscheidt, *Vorlesung Modulformen*, lecture notes, 2021

[3] D. Romik, *Complex Analysis Lecture Notes*, lecture notes, 2020