The Petersson Inner Product and Poincaré Series; Hecke Operators I

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In this talk, we will begin by introducing the Petersson inner product of two modular formsâone of them being a cusp form. We'll explore its basic properties and show that the Eisenstein series is orthogonal to cusp forms under this inner product. Following this, we will define the Poincaré series and prove that they form a basis for the space of cusp forms. Finally, we'll conclude with an overview of Hecke operators.

Notation and Reminders

Here are some essential definitions and results from the previous talks that'll be relevant for this talk:

Definition 0.1. The upper half-plane is, denoted $\mathbb{H}$, is the set $\{z \in \mathbb{C} | \Im(z) > 0\}$.

Definition 0.2. Let $k \in \mathbb{Z}$. M_k denotes the vector space of all modular forms of weight k and S_k denotes the vector space of all modular forms of weight k.

Definition 0.3. The group $\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \{M \in \mathbb{Z}^{2\times 2} : det(M) = 1\}$ is the full modular group.

Definition 0.4. The standard fundamental domain, denoted $\mathcal{F}$, for Γ is the set

$$\mathcal{F} = \left\{ z \in \mathbb{H} \;\middle|\; |z| \geq 1,\ -\frac{1}{2} \leq \Re(z) \leq \frac{1}{2} \right\}.$$

Lemma 0.1. If $f \in M_k$ and $g \in M_l$, then $fg \in M_{k+l}$. If f or g is a cusp form, then fg is also a cusp form.

Definition 0.5. Let $k \geq 4$ be even. The *Eisenstein series* is defined as

$$G_k(z) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} (mz+n)^{-k}$$

for any $z \in \mathbb{H}$.

Definition 0.6. Let $k \in \mathbb{Z}$. For $M = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \in \Gamma$ and $f : \mathbb{H} \to \mathbb{C}$, the *weight-k slash operator* is defined as

$$(f|_k M)(z) = j(M,z)^{-k} f(Mz)$$

for any $z \in \mathbb{H}$. Here, $j(M,z) = cz + d$ denotes the *automorphic factor*.

Definition 0.7. Let $k \in \mathbb{Z}$. A function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k for Γ if

1. f is holomorphic on $\mathbb{H}$.
2. For any $M \in \Gamma$: $(f|_k M) = f$.
3. f has a Fourier expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_f(n) q^n$$

with $q = e^{2\pi i z}$.

Remark. Writing $z = x + iy \in \mathbb{H}$, the coefficients $a_n(f)$ are defined as

$$a_n(f) = \int_0^1 f(x+iy) e^{-2\pi i n(x+iy)} dx$$

where $y > 0$ can be arbitrarily chosen.

Definition 0.8. Let $k \geq 4$ be even. The *normalized Eisenstein series* is defined as

$$E_k(z) = \frac{G_k(z)}{2\zeta(k)}$$

for any $z \in \mathbb{H}$. Here, ζ denotes the Riemann zeta function.

Lemma 0.2. Let $k \geq 4$ be even. The normalized Eisenstein series can be written as

$$E_k = \sum_{M \in \Gamma_\infty \backslash \Gamma} 1|_k M$$

Here, Γ_∞ denotes the subgroup $\{\pm T^n | n \in \mathbb{Z}\}$, where $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$.

The Petersson Inner Product

To define the Petersson inner product, we will employ a measure known as the hyperbolic measure. Below is its definition:

Definition 0.9. On the upper half plane $\mathbb{H}$, we define the so-called *hyperbolic measure*. For $z = x + iy \in \mathbb{H}$:

$$d\mu(z) = \frac{dxdy}{y^2}.$$

where $dxdy$ denotes the two-dimensional Lebesgue measure on $\mathbb{C}$.

Lemma 0.3. $d\mu$ is invariant under $\mathrm{SL}_2(\mathbb{R})$, i.e. for $M \in \mathrm{SL}_2(\mathbb{R})$: $d\mu(Mz) = d\mu(z)$.

Proof. Let $z = x + iy \in \mathbb{H}$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ arbitrary. Then, we compute the following:

$$Mz = \frac{az+b}{cz+d} = \frac{a(x+iy)+b}{c(x+iy)+d} = \cdots = \frac{(ad+bc)x+bd+ac(x^2+y^2)+iy}{c^2(x^2+y^2)+2cdx+d^2} \tag{0.1}$$
$$=: u(x,y) + iv(x,y)$$

We remark that since $x, y, u, v \in \mathbb{R}$, we may use the well-known change of variables formula in $\mathbb{R}^2$ and write: $dudv = |det(J)|dxdy$ where J is defined to be the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

Using the above equation (0.1), we compute $det(J) = \frac{1}{y^2}$. Since $z = x + iy \in \mathbb{H}$, $y > 0$ and, consequently, $|det(J)| = |\frac{1}{y^2}| = \frac{1}{y^2}$. Hence, we conclude: $dudv = \frac{dxdy}{y^2}$. □

Having defined the hyperbolic measure, we now proceed to define the hyperbolic volume.

Definition 0.10. The *hyperbolic volume* of a subset $A \subseteq \mathbb{H}$ is

$$vol(A) = \int_A d\mu(z).$$

Theorem 0.4. The standard fundamental domain $\mathcal{F}$ for Γ has the hyperbolic volume $vol(\mathcal{F}) = \frac{\pi}{3}$. It also holds that $vol(\overline{\mathcal{F}}) = vol(\mathcal{F}^\circ) = \frac{\pi}{3}$.

Proof. A direct computation shows:

$$vol(\mathcal{F}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\, dy\, dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\, dx = 2 \arcsin \frac{1}{2} = \frac{\pi}{3}.$$

□

After defining the hyperbolic measure and volume, we can proceed with the definiton of the *Petersson inner product.*

Definition 0.11. Let $f, g \in M_k$ be such that $f \in S_k$ or $g \in S_k$.
The *Petersson inner product* of f and g is defined as

$$\langle f, g \rangle := \int_{\mathcal{F}} f(z)\overline{g(z)}\Im(z)^k d\mu(z)$$

Next step is showing that this definiton is well-defined, i.e. the Petersson inner product is indeed an inner product.

Theorem 0.5. Let $f, g \in \mathcal{M}_k$ be such that $f \in \mathcal{S}_k$ or $g \in \mathcal{S}_k$. Then the Petersson inner product $\langle f, g \rangle$ converges absolutely and has the following properties:

1. $\langle f, g \rangle$ is linear in f and conjugate linear in g;
2. $\langle f, g \rangle = \overline{\langle g, f \rangle}$;

3. $\langle f, f\rangle \geq 0$ for $f \in \mathcal{S}_k$, and $\langle f, f\rangle = 0$ if and only if $f = 0$.

In particular, $\langle f, g\rangle$ defines an inner product, i.e. a positive definite Hermitian sesquilinear form.

Before we begin proving the theorem, let's recall an important result discussed in a previous talk: As we've seen in the proof of the Hecke-Bound theorem, if $f \in S_k$, then the function h defined as $h(z) = Im(z)^{k/2}\,|f(z)|$ for $z \in \mathbb{H}$ is bounded in the upper half-plane $\mathbb{H}$. This will be useful below.

Proof. Since f or $g \in \mathcal{S}_k$, $fg \in S_{2k}$ by *Lemma* 0.1. Since $fg \in S_{2k}$, the function h defined as $h(z) = \Im(z)^k\,|f(z)g(z)|$ for $z \in \mathbb{H}$ is bounded in $\mathbb{H}$. We've shown in *Theorem* 0.3 that $vol(\mathcal{F}) = \frac{\pi}{3}$. In particular, $\mathcal{F}$ has finite volume. Then, it follows that $\langle f, g\rangle$ is absolutely convergent since

$$\begin{aligned}\int_{\mathcal{F}} |f(z)\overline{g(z)}\Im(z)^k|\,d\mu(z) &= \int_{\mathcal{F}} \Im(z)^k|f(z)\overline{g(z)}|\,d\mu(z)\\ &= \int_{\mathcal{F}} \Im(z)^k|f(z)g(z)|\,d\mu(z)\\ &= \int_{\mathcal{F}} h(z)\,d\mu(z)\end{aligned}$$

converges.

Property 1 follows easily from the definition: Indeed, for any $f_1, f_2 \in M_k$ and $a \in \mathbb{R}$, we have:

$$\begin{aligned}\langle f_1 + af_2, g\rangle &:= \int_{\mathcal{F}} (f_1(z) + af_2(z))\overline{g(z)}\Im(z)^k\,d\mu(z)\\ &= \int_{\mathcal{F}} f_1(z)\overline{g(z)}\Im(z)^k\,d\mu(z) + a\int_{\mathcal{F}} f_2(z)\overline{g(z)}\Im(z)^k\,d\mu(z)\\ &= \langle f_1.g\rangle + a\langle f_2, g\rangle\end{aligned}$$

Likewise, for any $g_1, g_2 \in M_k$ and $a \in \mathbb{R}$:

$$\begin{aligned}\langle f, g_1 + ag_2\rangle &= \int_{\mathcal{F}} f(z)\overline{(g_1(z) + ag_2(z))}\Im(z)^k\,d\mu(z)\\ &= \int_{\mathcal{F}} f(z)\overline{g_1(z)}\Im(z)^k\,d\mu(z) + \overline{a}\int_{\mathcal{F}} f(z)\overline{g_2(z)}\Im(z)^k\,d\mu(z)\\ &= \langle f, g_1\rangle + \overline{a}\langle f, g_2\rangle.\end{aligned}$$

To show property 2, we compute:

$$\langle f, g\rangle = \int_{\mathcal{F}} f(z)\overline{g(z)}\Im(z)^k d\mu(z) = \overline{\int_{\mathcal{F}} \overline{f(z)}g(z)\Im(z)^k d\mu(z)} = \overline{\langle g, f\rangle}.$$

Similarly for property 3,

$$\langle f, f\rangle = \int_{\mathcal{F}} f(z)\overline{f(z)}\Im(z)^k d\mu(z) = \int_{\mathcal{F}} |f(z)|^2\,\Im(z)^k d\mu(z) \geq 0$$

since both $|f(z)|^2$ and $\Im(z)^k$ are non-negative.

If $f = 0$, $\langle f, f\rangle = 0$ clearly holds. If $\langle f, f\rangle = 0$, it must necessarily hold that $|f(z)|^2\,\Im(z)^k = 0$ which is the case if and only if $|f(z)|^2 = 0$, because $\Im(z) > 0$ for $\forall z \in \mathbb{H}$. Hence, $f = 0$. This concludes the proof of property 3.

$\square$

Remark. If both f and g are required to be cusp forms of weight k, the vector space S_k can be equipped with an inner product and be turned into an inner product space. It is essential that either $f \in S_k$ or $g \in S_k$. In general, $f \in S_k$ if and only if the function $h(z) = \Im(z)^{k/2}|f(z)|$ for $z \in \mathbb{H}$ is bounded on the upper half-plane $\mathbb{H}$. A proof of this can be found in [4].

In the following, we wish to show that the Petersson inner product is independent from the choice of a fundamental domain. To do so, we'll first begin by formally defining what a fundamental domain is for any subgroup of $\mathrm{SL}_2(\mathbb{R})$. We denote by $\mathcal{G}^\circ$ the interior of a set $\mathcal{G}$.

Definition 0.12. Let $\Lambda \subseteq \mathrm{SL}_2(\mathbb{R})$ be a subgroup. A *fundamental domain* $\mathcal{G}$ for Λ is a set with the following properties:

1. $\mathcal{G}$ is closed in $\mathbb{H}$;
2. $\forall z \in \mathbb{H}\ \exists M \in \Lambda$: $Mz \in \mathcal{G}$;
3. If $z, Mz \in \mathcal{G}^\circ$ for some $M \in \Lambda$, then $M = \pm I_2$.

Theorem 0.6. Let $\varphi : \mathbb{H} \to \mathbb{C}$ be a continuous and bounded function. Let $\Gamma(\varphi) := \{\lambda \in \Gamma : \varphi(\lambda z) = \varphi(z) \text{ for } \forall z \in \mathbb{H}\}$ be the invariant subgroup of φ. Let $\Lambda \subseteq \Gamma(\varphi)$ be some subgroup. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are fundamental domains for Λ, then we have:

$$\int_{\mathcal{G}_1} \varphi(z)\, d\mu(z) = \int_{\mathcal{G}_2} \varphi(z)\, d\mu(z).$$

Proof. Since $\mathcal{G}_1$ and $\mathcal{G}_2$ are fundamental domains for Λ, we have:

$$\mathbb{H} = \bigcup_{M \in \Lambda} M^{-1}\mathcal{G}_1 = \bigcup_{M \in \Lambda} M\mathcal{G}_2$$

Although the unions are not disjoint, the points that occur multiple times form a null set. Hence, we can compute:

$$\begin{aligned}
\int_{\mathcal{G}_1} \varphi(z) d\mu(z) &= \sum_{M \in \Lambda} \int_{M\mathcal{G}_2 \cap \mathcal{G}_1} \varphi(z) d\mu(z) \\
&= \sum_{M \in \Lambda} \int_{\mathcal{G}_2 \cap M^{-1}\mathcal{G}_1} \varphi(Mz) d\mu(Mz) \\
&= \int_{\mathcal{G}_2} \varphi(z) d\mu(z).
\end{aligned}$$

At the second step, we do a change of variables. Furthermore, notice that in both steps, we use the following two facts:

1. $\Lambda \subseteq \Gamma(\varphi)$, so in particular, $\varphi(Mz) = \varphi(z)$ for $\forall M \in \Lambda$.
2. By *Lemma* 0.2, $d\mu$ is invariant under all $\Lambda \subseteq \mathrm{SL}_2(\mathbb{R})$, i.e. $\forall M \in \Lambda$: $d\mu(Mz) = d\mu(z)$.

This concludes the proof. $\square$

Let's explore why *Theorem* 0.6 implies that the Petersson inner product is independent of the choice of a fundamental domain. Consider the function:

$$\begin{aligned} \varphi_{f,g} : \mathbb{H} &\to \mathbb{C} \\ z &\mapsto f(z)\overline{g(z)}\Im(z)^k \end{aligned}$$

for $f, g \in M_k$ such that $f \in S_k$ or $g \in S_k$.

First step is showing that $\varphi_{f,g}$ is continuous and bounded on $\mathbb{H}$: Since f and g are modular forms, they are holomorphic on $\mathbb{H}$. This implies, in particular, that f and g are continuous on $\mathbb{H}$. It's a well-known fact that the imaginary function $\Im$ is continuous. Hence, the function $\varphi_{f,g}$ is continuous on $\mathbb{H}$. Moreover, we assume that f or g is a cusp form. This implies that $f(z)$ or $g(z)$ vanishes for $\Im(z) \to \infty$. Hence, $\varphi_{f,g}(z) = f(z)\overline{g(z)}\Im^k(z) \to 0$ as $\Im(z) \to \infty$. This implies that $\varphi_{f,g}$ is bounded on $\mathbb{H}$.

Next step is showing that $\varphi_{f,g}$ is invariant under Γ. For any $z \in \mathbb{H}$ and any $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we compute the following:

$$\begin{aligned} \varphi_{f,g}(Mz) = f(Mz)\overline{g(Mz)}\Im(Mz)^k &= (cz+d)^k f(z)\overline{(cz+d)^k g(z)}\frac{\Im(z)^k}{|cz+d|^{2k}} \\ &= f(z)\overline{g(z)}\Im(z)^k \\ &= \varphi_{f,g}(z) \end{aligned}$$

In this computation, we use the following two facts:

1. The modular forms f and g are weakly modular of weight k, i.e. $f(Mz) = (cz+d)^k f(z)$ and $g(Mz) = (cz+d)^k g(z)$.
2. It holds:

   $$\Im(Mz) = \frac{\Im(z)}{|cz+d|^2}$$

   This is an important property of the Möbius transformation.

Hence, *Theorem* 0.6 is applicable and it follows that the Petersson inner product is independent from the choice of a fundamental domain. Subsequently, we'll show that Eisenstein series are orthogonal to cusp forms.

Theorem 0.7. Let $k \geq 4$ be even. For any $f \in S_k$, we have $\langle E_k, f \rangle = 0$.

Proof. The proof involves a direct computation of $\langle E_k, f \rangle$, using the definition of E_k as specified in *Lemma* 0.2. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \setminus \Gamma$ and $z \in \mathbb{H}$ be arbitrary.

By Definition 0.6, we first compute the following:

$$(1|_k M)(z) = j(M, z)^{-k}$$

Hence, the following holds:

$$\begin{aligned}
\langle E_k, f\rangle &= \int_{\mathcal{F}} \left(\sum_{M\in\Gamma_\infty\backslash\Gamma} (1|_k M)(z) \right) \overline{f(z)}\Im(z)^k d\mu(z) \\
&= \int_{\mathcal{F}} \left(\sum_{M\in\Gamma_\infty\backslash\Gamma} j(M,z)^{-k} \right) \overline{f(z)}\Im(z)^k d\mu(z) \\
&= \sum_{M\in\Gamma_\infty\backslash\Gamma} \int_{\mathcal{F}} j(M,z)^{-k}\overline{f(z)}\Im(z)^k d\mu(z) \\
&= \sum_{M\in\Gamma_\infty\backslash\Gamma} \int_{\mathcal{F}} \frac{\overline{f(Mz)}}{(cz+d)^{2k}}\Im(z)^k d\mu(z)
\end{aligned}$$

Note that here we used the fact that f is a modular form. Hence, $f(Mz) = j(M,z)^k f(z)$. This implies in particular:

$$j(M,z)^{-k}\overline{f(z)} = \frac{\overline{f(Mz)}}{j(M,z)^{2k}} = \frac{\overline{f(Mz)}}{(cz+d)^{2k}}$$

In the following step, we'll use the following property of the Möbius transformation:

$$\Im(Mz) = \frac{\Im(z)}{|cz+d|^2}$$

Since k is even, this implies that

$$\Im(z)^k = \Im(Mz)^k |cz+d|^{2k} = \Im(Mz)^k (cz+d)^{2k}$$

Hence, we proceed as the following:

$$= \sum_{M\in\Gamma_\infty\backslash\Gamma} \int_{\mathcal{F}} \overline{f(Mz)}\Im(Mz)^k d\mu(z)$$

Using the invariance of $d\mu$ under $\mathrm{SL}_2(\mathbb{R})$

$$= \sum_{M\in\Gamma_\infty\backslash\Gamma} \int_{\mathcal{F}} \overline{f(Mz)}\Im(Mz)^k d\mu(Mz)$$

Under a change of variables

$$\begin{aligned}
&= \sum_{M\in\Gamma_\infty\backslash\Gamma} \int_{M\mathcal{F}} \overline{f(z)}\Im(z)^k d\mu(z) \\
&= \int_{\bigcup_{M\in\Gamma_\infty\backslash\Gamma} M\mathcal{F}} \overline{f(z)}\Im(z)^k d\mu(z)
\end{aligned}$$

Notice that $\bigcup_{M\in\Gamma_\infty\backslash\Gamma} M\mathcal{F}$ is a fundamental domain for Γ_∞. It's also true that $\{z \in \mathbb{H} | 0 \le \Re(z) \le 1\}$ is a fundamental domain for Γ_∞. Since we've shown that the Petersson inner product is independent of the choice of a fundamental domain, we're allowed to compute $\langle E_k, f\rangle$ over $\{z \in \mathbb{H} | 0 \le \Re(z) \le 1\}$. Hence, writing $z = x + iy$, we get the following result:

$$\begin{aligned}
\langle E_k, f\rangle &= \int_{\{z\in\mathbb{H}|0\le\Re(z)\le1\}} \overline{f(z)}\Im(z)^k d\mu(z) \\
&= \int_0^\infty \int_0^1 \overline{f(x+iy)} y^k d\mu(x+iy) \\
&= \int_0^\infty \int_0^1 \overline{f(x+iy)} y^{k-2} dxdy \\
&= \int_0^\infty \left(\int_0^1 \overline{f(x+iy)} dx\right) y^{k-2} dy
\end{aligned}$$

By the remark below Definiton 0.7, $\int_0^1 \overline{f(x+iy)}dx = \overline{a_f(0)}$. Since f is a cusp form, $a_f(0) = 0$. Hence, we get: $\langle E_k, f\rangle = 0$. □

Remark. As a result of the Valence formula, we were able to show that for $k \ge 4$: $M_k = \mathbb{C}E_k \oplus S_k$. Hence, *Theorem* 0.7 shows that the decomposition is orthogonal with respect to the Petersson inner product.

Poincaré Series

Definition 0.13. For $k \ge 4$ even and $m \in \mathbb{N}$ we define the m-th Poincaré series as follows

$$P_{m,k}(\tau) = \sum_{M\in\Gamma_\infty\backslash\Gamma} exp(2\pi i m\tau)|_k M$$

The goal of this chapter is to show that $P_{m,k}$ for $m \in \mathbb{N}$ and $k \ge 4$ is a basis of the vector space S_k.

Remark. A representation system for $\Gamma_\infty\backslash\Gamma$ is given by $\{(c,d) \in \mathbb{Z}^2 | gcd(c,d) = 1\}$. To see this, note that Γ_∞ has a single generator, the matrix $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and for a arbitrary matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $n \in \mathbb{Z}$ we can write

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = T^n M \tag{1}$$

So we see directly $c', d' = c, d$. Further note that since it has to hold $detM = 1$ that for given c, d there are unique a, b which fulfills the equation i.e. a, b are determined through given c, d. We can also see that $a' = a + nc, b' = b + nd$. So given c, d with $gcd(c, d) = 1$ (so the determinant can be 1), the matrix M' is defined.

Remark. This definition makes also sense for $m = 0$, but then we have $P_{0,k} = E_k$ the normed Eisenstein series. Since this acts differently then the Poincaré series we exclude it from the definition.

Lemma 0.8. It holds $P_{m,k} \in S_k$.

Proof. We can show that $P_{m,k}$ is absolute and uniformly convergent the same way as for the Eisenstein-series. With the Weierstrasse-convergence theorem it follows again that $P_{m,k}$ is holomorphic on $\mathbb{H}$. The invariance under the k-slash Operator holds, since for any functions f, any $L \in \Gamma$ and Λ subgroup of Γ it holds

$$\begin{aligned} g|_k L(\tau) &= \left(\sum_{M \in \Lambda \backslash \Gamma} f(\tau)|_k M \right) |_k L \\ &= \sum_{M \in \Lambda \backslash \Gamma} f(\tau)|_k M|_k L \\ &= \sum_{M \in \Lambda \backslash \Gamma} f(\tau)|_k ML \\ &= \sum_{M \in \Lambda \backslash \Gamma} f(\tau)|_k M \\ &= g(\tau) \end{aligned}$$

since ML again goes thorugh a representation system $\Lambda\backslash\Gamma$. Replacing g with $P_{m,k}$ (and Λ with Γ_∞ the statement follows. Notice that there is also an intuition for this: Γ_∞ is generated by the Matrix representing a Translation of 1. Since $exp(2\pi i\tau)$ is 1-periodic it is clear that it is invariant under the k-slash operator. We skip the Fourier Expansion of $P_{m,k}$ because it is too complicated. The coefficients itself are infinite series again. For further information consult the book of Iwanic. Lastly we have to show that $\lim_{y\to\infty} |P_{m,k}|$ remains bounded.

$$\begin{aligned} \lim_{y\to\infty} |P_{m,k}(x+iy)| &= \left| \sum_{M \in \Gamma_\infty \backslash \Gamma} \frac{exp(2\pi imM(x+iy))}{j(M, x+iy)^k} \right| \\ &\leq \lim_{y\to\infty} \sum_{M \in \Gamma_\infty \backslash \Gamma} \left| \frac{exp(2\pi mM(x+iy)}{j(M, x+iy)^k} \right| \\ &= \sum_{M \in \Gamma_\infty \backslash \Gamma} \lim_{y\to\infty} \frac{|exp(2\pi imM(x+iy))|}{|j(M, x+iy)|^k} \\ &= \sum_{M \in \Gamma_\infty \backslash \Gamma} \lim_{y\to\infty} \frac{exp(2\pi im(2i\ Im(M(x+iy)))^{1/2}}{|j(M, x+iy)|^k} \\ &= \sum_{M \in \Gamma_\infty \backslash \Gamma} \lim_{y\to\infty} \frac{exp(-2\pi m\frac{y}{|j(M,x+iy)|^2})}{|j(M, x+iy)|^k} \longrightarrow 0 \end{aligned}$$

The limit is clear in the case $c = 0$. In the case $c \neq 0$ notice that $|j(M, x+iy)|^2 \sim y^2$ and by inserting this it becomes clear that the expression tends to 0 in this case aswell. Since all coefficients of the sum tend to 0, the sum converges to 0 and the lemma is proven. □

Lemma 0.9. Let $n \geq 4$ even and $m \in \mathbb{N}$. For $f = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k$ it holds

$$\langle f, P_{m,k} \rangle = \frac{(k-2)!}{(4\pi m)^{k-1}} a_f(m).$$

Proof. Let $f \in S_k$ be arbitrary. We first make the weight-k slash operator of the series explicit

$$P_{m,k} = \sum_{M \in \Gamma_\infty \backslash \Gamma} j(M, \tau)^{-k} exp(2\pi i m M\tau)$$

where $j(M, \tau) = (c\tau + d)$ denotes the well known automorphiefactor, for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. We now plug this together with f into our definition of the Petersson inner product and use the "Entfaltungstrick" from last chapter to get

$$\begin{aligned}
\langle f, P_{m,k} \rangle &= \int_{\mathcal{F}} f(z) \overline{\sum_{M \in \Gamma_\infty \backslash \Gamma} j(M, \tau)^{-k} exp(2\pi i m M\tau)} Im(z)^k d\mu(z) \\
&= \sum_{M \in \Gamma_\infty \backslash \Gamma} \int_{\mathcal{F}} f(z) \overline{j(M, \tau)^{-k} exp(2\pi i m M\tau)} Im(z)^k d\mu(z) \\
&= \sum_{M \in \Gamma_\infty \backslash \Gamma} \int_{M\mathcal{F}} f(M^{-1}z) \overline{j(M, M^{-1}\tau)^{-k} exp(2\pi i m M M^{-1}\tau)} Im(M^{-1}z)^k d\mu(M^{-1}z) \\
&= \sum_{M \in \Gamma_\infty \backslash \Gamma} \int_{M\mathcal{F}} j(M^{-1}, z)^k f(z) \overline{j(M^{-1}, z)^k exp(2\pi i m z)} \frac{Im(z)^k}{|j(M^{-1}, z)|^{2k}} d\mu(z) \\
&= \sum_{M \in \Gamma_\infty \backslash \Gamma} \int_{M\mathcal{F}} f(z) exp(-2\pi i m \bar{z}) Im(z)^k d\mu(z) \\
&= \int_{\cup_{M \in \Gamma_\infty \backslash \Gamma} M\mathcal{F}} f(z) exp(-2\pi i m \bar{z}) Im(z)^k d\mu(z) \\
&= \int_0^\infty \int_0^1 f(x + iy) exp(-2\pi i m(x - iy)) y^k \frac{dxdy}{y^2} \\
&= \int_0^\infty \int_0^1 (\sum_{n=0}^{\infty} a_f(n) exp(2\pi i n(x + iy))) exp(-2\pi i m(x - iy)) y^{k-2} dxdy \\
&= \sum_{n=0}^{\infty} a_f(n) \int_0^\infty \int_0^1 exp(2\pi i(n - m)x) exp(-2\pi(n + m)y) y^{k-2} dxdy \\
&= \sum_{n=0}^{\infty} a_f(n) \int_0^1 exp(2\pi i(n - m)x) dx \int_0^\infty exp(-2\pi(n + m)y) y^{k-2} dy
\end{aligned}$$

We substitute with matrix M^{-1}, notice that there should also be a term with $|det(DM^{-1})|$ but this is 1 because $det(M) = 1$. To get to equality 4 we use

$$f(M^{-1}z) = j(M^{-1}, z)^k f(z)$$

as a consequence of invariance of modular forms under slash-operation,

$$\overline{j(M, M^{-1}z)} = \overline{j(M^{-1}, z)}$$

which can easily be checked,

$$Im(M^{-1}z) = \frac{y}{j(M^{-1}, z)}$$

from chapter 2.2 in [3],

$$d\mu(M^1 z) = d\mu(z)$$

from the previous chapter.
We seperatly calculate the two Integrals in the last equality

$$\int_0^1 exp(2\pi i(n-m)x)dx = \begin{cases} \int_0^1 1dx = 1 & \text{m = n} \\ [\frac{-i(1-exp(2\pi im)exp(2\pi in)}{2\pi(m-n)}]_0^1 \;\; = 0 & \text{m} \neq \text{n} \end{cases}$$

so the sum vanishes except the term with $n = m$ and with

$$\int_0^\infty exp(-4\pi my)y^{k-2} = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}$$

the proof concludes.

□

It remains to show that the Poincaré series are a basis of S_k.

Lemma 0.10. Let $k \geq 4$ be even and $m \in \mathbb{N}$ arbitrary. Then the subspace of Poincaré series P in S_k is the whole of S_k, the the vectorspace of cusp forms.

Proof. Let $\mathcal{P}$ be the span of all Poincaré series. For $f \in \mathcal{P}^\perp$ it holds by our previous lemma and the definition of orthogonality that

$$\frac{(k-2)!}{(4\pi m)^{k-1}} a_f(m) = \langle f, P_{m,k} \rangle = 0$$

for all $m \in \mathbb{N}$. Thus $a_f(m) = 0 \; \forall m \in \mathbb{N}$ and thus $f = 0$, which implies $\mathcal{P}^\perp = 0$ and thus $\mathcal{P} = S_k$. □

Hecke operator

The purpose of this chapter is to define the **Hecke operator**, which we show to be an Endomorphisms on the $\mathbb{C}$ -Vector space of all modular forms.

Definition 0.14. For $n \in \mathbb{N}$ we define the set

$$\mathcal{M}_n = \{M \in \mathbb{Z}^{2\times 2} : det(M) = n\}$$

of the integer 2×2 matrices with determinant n.

The group Γ acts on $\mathcal{M}_n$ from left and right through (matrix-)multiplication.

- This is well defined: It is well known that $det(AB) = det(A)det(B)$ for appropriate Matrices, especially for 2×2 matrices. Thus the result of the above defined group action is again in $\mathcal{M}_n$.
- Identity axiom group action: The identity matrix I is the neutral element of Γ and multiplication with it leaves any matrix invariant.

- Compatibility axiom group action: For any two matrices $A, B \in \Gamma$ and $C \in \mathcal{M}_m$ its clear it holds $(AB)C = A(BC)$.

The defining properties of a group guarantee that the set of orbits of (points x in) $\mathcal{M}_n$ under the action of Γ form a partition of $\mathcal{M}_n$, associated to the equivalence relation that two elements are equal if they have the same orbit. We denote the quotient space as $\Gamma \backslash \mathcal{M}_n := \mathcal{M}_n / \sim$.

The next Lemma gives us now a representation system for $\Gamma \backslash \mathcal{M}_n$.

Lemma 0.11. A representation system of $\Gamma \backslash \mathcal{M}_n$ is given through the set

$$\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z},\ ad = n,\ d > 0,\ b\ (mod\ d) \right\}$$

It has cardinality $\sigma_1(n)$, especially its finite.

Proof. We first show every matrix in $\mathcal{M}_n$ is equivalent to a matrix from our representation-system. For this let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_n$ be arbitrary. If a, c are relative prime choose $\gamma, \delta \in \mathbb{Z}$ as $\delta = a,\ \gamma = -c$, then clearly γ, δ are also relativ prime and $a\gamma + c\delta = 0$. If a, c are not relativ prime, first divide with their greatest common divisor, befor choosing γ, δ the same way. In this case γ, δ are also relativ prime and $a\gamma + c\delta = 0$ also holds. Now choose $\alpha, \beta \in \mathbb{Z}$ such that $\alpha\gamma - \delta\beta = 1$, i.e. $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$. The existence of such α, β follows BÃ©zout's identity and the fact that γ, δ are relativ prime. It now holds

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}$$

for some $a', b', d' \in \mathbb{Z}$. Since the determinant of the left side equals n, it follows the determinant of the right side also has to be equal n. Thus $a'd' = n$. Further we can assume $d' > 0$ since otherwise we just multiply with $-I$ and lastly by multiplication with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n$ for any $n \in \mathbb{N}$ can we change b' modulo d'.

Now we show that any two different matrices of our representation system are unequal with respect to the equivalence relation. Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$ such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}$$

Multiplication of the left hand side of the equality yields $\gamma a = 0$ and thus $\gamma = 0$ ($a \neq 0$ because $ad = n$). Since $\gamma = 0$ and the determinant of matrices in Γ has to be 1, it follows that $\alpha\delta = 1$ and thus $\alpha = \delta = \pm 1$. But it has to hold $d, d' > 0$ which implies $\delta > 0$ and thus $\alpha = \delta = 1$. Inserting the values gives us

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^\beta$$

Especially it follows $a = a'$ and $d = d'$. Lastly it holds $b' = b + \beta d = b\ (mod\ d)$. It remains to proof the cardinality is finite.

$\square$

Let $k \in \mathbb{Z}$ and $f : \mathbb{H} \to \mathbb{C}$. We now want to extend the definition of the weight-k slash operator to matrices $M \in \mathcal{M}_n$ by defining

$$(f|_k M)(\tau) = (c\tau + d)^{-k} f(M\tau)$$

where c, d and M are related by the representation system defined in Lemma 0.11.

Definition 0.15. For $n \in \mathbb{N}$ the Hecke-Operator T_n on $f \in M_k$ (the $\mathbb{C}$-Vectorspace of all Modular forms) is defined as

$$T_n f = n^{k-1} \sum_{M \in \Gamma \backslash \mathcal{M}_n} f|_k M$$

Since by definition elements of M_k are invariant under the slash operation of Γ, it follows that the definition does not relay on the representation system of $\Gamma \backslash \mathcal{M}_n$. By using the representation system of lemma 0.11 we can thus write the Hecke operator as

$$(T_n f)(\tau) = n^{k-1} \sum_{\substack{ad=n \\ d>0}} d^{-k} \sum_{b \ (mod \ d)} f\left(\frac{a\tau + b}{d}\right)$$

We can thus describe how the Hecke operator acts on the Fourier-expansion of $f \in M_k$.

Lemma 0.12. Let $f(\tau) = \sum_{m=0}^{\infty} a_f(m) q^m \in M_k$ with $q = exp(2\pi i\tau)$. Then

$$T_n f = \sum_{m=0}^{\infty} a_{T_n f}(m) q^m$$

$$a_{T_n f}(m) = \sum_{d|(m,n)} d^{k-1} a_f\left(\frac{mn}{d^2}\right)$$

Also it holds $a_{T_n f}(0) = \sigma_{k-1}(n) a_f(0)$ and $a_{T_n f}(1) = a_f(n)$, where $\sigma_{k-1}(n)$ is the generalized divisor function.

Proof. Start by inserting the Fourier expansion of f into the explicit description of the Hecke operator above (Definition 0.15) to get

$$\begin{aligned}
(T_n f)(\tau) &= n^{k-1} \sum_{\substack{ad=n \\ d>0}} d^{-k} \sum_{b \ (mod \ d)} \sum_{m=0}^{\infty} a_f(m) q^m \\
&= n^{k-1} \sum_{\substack{ad=n \\ d>0}} d^{-k} \sum_{b \ (mod \ d)} \sum_{m=0}^{\infty} a_f(m) exp\left(2\pi i \left(\frac{a\tau + b}{d}\right)\right)^m \\
&= n^{k-1} \sum_{\substack{ad=n \\ d>0}} d^{-k} \sum_{b \ (mod \ d)} \sum_{m=0}^{\infty} a_f(m) exp\left(\frac{2\pi i m a\tau}{d}\right) exp\left(\frac{2\pi i m b}{d}\right)
\end{aligned}$$

Remember that by definition it holds $f(\tau) = \sum_{m=0}^{\infty} a_f(m) exp(2\pi i\tau)^m$ for every $\tau \in \mathbb{H}$, i.e. the sum is at least conditional convergent. We want to use this to exchange the sums with the following claim:

Claim 1. Let I be a non-empty finite Set and $\sum_{l=0}^{\infty} b_l$ some conditional convergent sum. Then we can exchange the sum:

$$\begin{aligned}\sum_{i\in I}\sum_{l=0}^{\infty} b_l &= \sum_{i\in I}\lim_{k\to\infty}\sum_{l=0}^{k} b_l\\ &= \lim_{k\to\infty}\sum_{i\in I}\sum_{l=0}^{k} b_l\\ &= \lim_{k\to\infty}\sum_{l=0}^{k}\sum_{i\in I} b_l\\ &= \sum_{l=0}^{\infty}\sum_{i\in I} b_l\end{aligned}$$

where the second equality is due to addition being continuous, and the third we swap two finite sums.

Applying this claim to the above equality yields us

$$\begin{aligned}&= n^{k-1}\sum_{\substack{ad=n\\ d>0}} d^{-k}\sum_{m=0}^{\infty}\sum_{b\ (mod\ d)} a_f(m)\, exp\left(2\pi i\left(\frac{a\tau+b}{d}\right)\right)^m\\ &= n^{k-1}\sum_{\substack{ad=n\\ d>0}} d^{-k}\sum_{m=0}^{\infty}\sum_{b\ (mod\ d)} a_f(m)\, exp\left(\frac{2\pi i m a\tau}{d}\right) exp\left(\frac{2\pi i m b}{d}\right) \qquad (2)\\ &= n^{k-1}\sum_{\substack{ad=n\\ d>0}} d^{-k}\sum_{m=0}^{\infty}\left(a_f(m)\, exp\left(\frac{2\pi i m a\tau}{d}\right)\sum_{b\ (mod\ d)} exp\left(\frac{2\pi i m b}{d}\right)\right)\end{aligned}$$

We now have 2 Cases:

Claim 2. Let $d|m$, then $\exists l \in \mathbb{Z}$ such that $ld = m$. Insert this above to get

$$\begin{aligned}&= \sum_{b\ (mod\ d)} exp\left(\frac{2\pi i l d b}{d}\right)\\ &= \sum_{b\ (mod\ d)} exp\,(2\pi i l b)\end{aligned}$$

and since $l, b \in \mathbb{Z}$ every term is equal to 1 and there are d terms the sum simplifys to d.
Now let $d \nmid m$. It is well known that $exp\left(\frac{2\pi i b}{d}\right)$ for $b \in \{0, 1, ..., d-1\}$ are the solutions for the equation

$$z^d = 1 \qquad (3)$$

Another way to write this is ζ^b for $b \in \{0, 1, ..., d-1\}$ and $\zeta = exp\left(\frac{2\pi i}{d}\right)$. This looks almost like our terms, we just have an additional m. We now notice that for any solution for (3 ζ^b it holds $\zeta^{mb} = 1$ aswell, since $\zeta^{mb} = (\zeta^b)^m = (1)^m = 1$.

We thus know that such an m maps solutions of 3 to solutions of 3. Further notice that we can assume $m < d$ because for $a \equiv b mod d$ it holds: Let $a = b+ld$ for some $0 \le b < dl \in Z$, then $\zeta^a = \zeta^{b+ld} = \zeta^b \zeta^{ld} = \zeta^b$. But we cant yet say if this is bijectiv, which would allow us to drop the m entirely in the sum, because it would only permute the order of terms.
To now see how $exp\left(\frac{2\pi imb}{d}\right)$ and $exp\left(\frac{2\pi ib}{d}\right)$ are related, note that $\{\zeta^b \mid b \in \{0, 1, 2, ..., d-1\}\ \}$ with multiplication $\zeta^{b_1}\zeta^{b_2} = \zeta^{b_1+b_2 \ mod \ d}$ is a group and by $\zeta^b \mapsto b$ is isomorphic to the cyclic group of order d $\mathbb{Z}/d\mathbb{Z}$. Now given an $m \in Z/d\mathbb{Z}$ (i.e. $m < d$ as above) we can define the left-multiplication map $l_m(b) = mb \ \forall b \in \mathbb{Z}/d\mathbb{Z}$. Note that since m has an inverse as a element of a group and $l_{m^{-1}} l_m(b) = m^{-1}mb = b \ \forall b \in \mathbb{Z}/d\mathbb{Z}$. We have shown l_m has an inverse map and thus is bijectiv, thus such an m just acts as a permutation on the group. Taking our isomorphismus back we conclude $\sum_{b \ (mod \ d)} exp\left(\frac{2\pi imb}{d}\right) = \sum_{b \ (mod \ d)} exp\left(\frac{2\pi ib}{d}\right)$, since we know that m just permutes the terms and we sum over all of them.

This allows us now to proof that such a sum is zero. This is geometricaly intuitiv since the d-unit roots are just evenly spread out on the unit circle and the complex sum is expected to be in the "middle" at zero. But we want to give a more formal, algebraic proof:

Claim 3. Let P be a polynomial equation $a_nX^n + a_{n-1}X^{n-1} + ... + a_1x + a_0 = 0$ such that $a_n \neq 0$. Then the sum of the roots of P is $-\frac{a_{n-1}}{a_n}$

Proof. By the fundamental theorem of algebra we know such a polynomial has n roots $x_1, ..., x_n$. Then we can rewrite P in factored form as: $a_n\Pi_{k=1}^n(x - x_k) = a_n(x - x_1)(x - x_2)\cdots(x - x_n)$. Multiplying this out, P can be expressed as: $a_n(x^n - (x_1 + x_2 + ... + x_n)x^{n-1} + ... + (-1)^n x_1x_2x \cdots x_n) = 0$, where the coefficients $x^{n-2}, x^{n-3}...$ are irrelevant. Equating the powers of x, it follows that: $-a_n(x_1 + x_2 + ... + x_n) = a_{n-1}$ and thus the claim follows. ☐

We can continue at Claim 2: Since we have the d-solutions to the polynomial $z^d - 1 = 0$ applying to above formula gives us directly that the sum has to vanish, which finishes Claim 2.

Utilising this result we can now finally simplify (2) further. Writing $m = dr$ we get

$$n^{k-1} \sum_{\substack{ad=n \\ d>0}} d^{-k} \sum_{m=0}^{\infty} \left(a_f(m)\, exp\left(\frac{2\pi ima\tau}{d}\right) \sum_{b \ (mod \ d)} exp\left(\frac{2\pi imb}{d}\right) \right)$$

$$= n^{k-1} \sum_{\substack{ad=n \\ d>0}} d^{1-k} \sum_{r=0}^{\infty} a_f(dr)\, exp(2\pi iar\tau)$$

$$= \sum_{a|n} a^{k-1} \sum_{r=0}^{\infty} a_f(rn/a) exp(2\pi iar\tau)$$

and for $m = ar$ we finally get

$$T_n f = \sum_{m=0}^{\infty} \sum_{a|(m,n)} a^{k-1} a_f(mn/a^2) exp(2\pi i m\tau)$$

which finishes the proof. $\square$

We now want to use this for the following theorem.

Theorem 0.13. For $n \in \mathbb{N}$ the Hecke-Operator T_n defines an Endomorphism on M_k and S_k

Proof. It is clear that T_n is linear. As a finite sum of holomorphic functions $T_n f$ is also holomorphic. Now let $L \in \Gamma$, then it holds

$$\begin{aligned} (T_n f)|_k L &= n^{k-1} \sum_{M \in \Gamma \backslash \mathcal{M}_n} f|_k M|_k L \\ &= n^{k-1} \sum_{M \in \Gamma \backslash \mathcal{M}_n} f|_k ML = n^{k-1} \sum_{M \in \Gamma \backslash \mathcal{M}_n} f|_k M = T_n f \end{aligned} \tag{4}$$

where the second equality follows by the cocycle relation from chapter 2.2 in [3], so that it holds $f|_k M|_k L = f|_k ML$ for all $M, L \in \mathrm{SL}_2(\mathbb{R})$ and the third since ML for $M \in \Gamma \backslash \mathcal{M}_n$ also goes through a representation system of $\Gamma \backslash \mathcal{M}_n$, so $T_n f$ satisfies invariance under the k-slash operator. From lemma 0.12 we also get the fourier expansion of $T_n f$, starting by $n = 0$. Also if $f \in S_k$ then $a_f(0) = 0$ and by lemma 0.12 again $a_{T_n f} = 0$ and thus $T_n f \in S_k$, which concludes the proof. $\square$

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