Hecke Operators II

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22.05.24

During this handout, we abuse notation in the following ways:

- We use "Elements of a quotient group" and "representatives of the quotient group" interchangeably. So e.g. by $L \in \Gamma_\infty \backslash \Gamma$ we will mostly mean L to be a representative of one of the elements of $\Gamma_\infty \backslash \Gamma$.
- By $b \pmod{d}$ we mean $b \in \mathbb{Z}/p\mathbb{Z}$; or also in accordance with the previous point $b \in \{0, 1, \ldots, p-1\}$.

Hecke-Operators

Reminder, for $n \geq 1$ the Hecke operator T_n on M_k is given by

$$T_n f = n^{k-1} \sum_{M \in \Gamma \backslash \mathcal{M}_n} f|_k M = n^{k-1} \sum_{\substack{ad=n \\ d>0}} \sum_{b \bmod d} f|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

where $\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, ad = n, d > 0, b \pmod{d} \right\}$ is a representative system of $\Gamma \backslash \mathcal{M}_n$.

In the previous week we also saw how the Hecke operator is an Endomorphism of S_k and M_k for all even $k \geq 12$. We also saw the following theorem.

Lemma 0.1. *For $f = \sum_{m=0}^{\infty} a_f(m) q^m \in M_k$, the Fourier series of $T_n f$ is given by*

$$T_n f = \sum_{m=0}^{\infty} a_{T_n f}(m) q^m, \quad a_{T_n f}(m) = \sum_{d|(m,n)} d^{k-1} a_f\left(\frac{mn}{d^2}\right).$$

1 Simultaneous eigenform

A modular form $f \in M_n$ is called a simultaneous eigenform of all Hecke-operators T_n if

$$T_n f = \lambda_f(n) f$$

for all $n \in N$, with an appropriate eigenvalue $\lambda_f(n) \in \mathbb{C}$.

Lemma 1.1. *For a non-constant $f \in M_k$ the following are equivalent:*

1. *f is a simultaneous eigenform for all T_n.*

2. *We have that* $a_f(1) \neq 0$ *and for all* $m \in \mathbb{N}_0, n \in \mathbb{N}$ *the following equality holds:*

$$a_f(m)a_f(n) = a_f(1) \sum_{d|(m,n)} d^{k-1} a_f(mn/d^2).$$

For such f*, the eigenvalues are given by* $\lambda_f(n) = a_f(n)/a_f(1)$*. For coprime* m, n*, we then also have*

$$a_f(m)a_f(n) = a_f(1)a_f(mn).$$

Proof. Assume that f is a simultaneous eigenform of all T_n. Comparing the m-th coefficient of $\lambda_f(n)f$ and T_nf (see Lemma 0.1) gives us the equation

$$\lambda_f(n)a_f(m) = a_{T_nf}(m) = \sum_{d|(m,n)} d^{k-1} a_f(mn/d^2).$$

For $m = 1$ this gives us $\lambda_f(n)a_f(1) = a_f(n)$. So if we had $a_f(1) = 0$, then $a_f(n) = 0$ for all n, which contradicts the assumption that f is non-constant. So we must have $a_f(1) \neq 0$ and $\lambda_f(n) = a_f(n)/a_f(1)$. Plugging this into the above equation gives us

$$a_f(n)a_f(m) = a_f(1) \sum_{d|(n,m)} d^{k-1} a_f(mn/d^2),$$

as wanted.

For the other direction, equation 2. together with lemma 0.1, gives us that $T_nf = \frac{a_f(n)}{f(1)} f$ for all $n \in \mathbb{N}$. So f is indeed a simultaneous eigenform of all T_n. □

For a *normed* simultaneous eigenform f, that is $a_f(1) = 1$, the coefficients are multiplicative by the above lemma and fulfil

$$a_f(m)a_f(n) = a_f(nm)$$

for $(n.m) = 1$. With this we can show that the Fourier coefficients of the Δ-function are multiplicative.

Theorem 1.2. *The* Δ*-function is a normed eigenform of all* T_n *with*

$$T_n\Delta = \tau(n)\Delta.$$

For the Fourier coefficients $\tau(n)$ *of* Δ*, we have the identity*

$$\tau(m)\tau(n) = \sum_{d|(n,m)} d^{11} \tau(mn/d^2)$$

for all $n, m \in \mathbb{N}$*. So in particular we have*

$$\tau(m)\tau(n) = \tau(nm)$$

for $(n, m) = 1$*.*

Proof. Since T_n is an endomorphism of $S_{12}\mathbb{C}\Delta$, the cusp form $T_n\Delta$ must be a multiple of Δ, that is $T_n\Delta = \lambda_\Delta(n)\Delta$ for an appropriate $\lambda_\Delta(n) \in \mathbb{C}$. So we have that Δ is a simultaneous eigenform of all T_n. Since Δ is normed, the result follows from lemma 1.1 $\square$

We can also show that the Eisenstein series E_k (and even more general, the Poincaré series $P_{m,k}$) are simultaneous eigenforms as well. For the proof of that we will need a quick lemma.

Lemma 1.3. *The set*

$$S := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} L \;\middle|\; ad = n, d > 0, b \pmod{d}, L \in \Gamma_\infty \backslash \Gamma \right\}$$

is a representative system of $\Gamma_\infty \backslash \mathcal{M}_n$.

Proof. For S to be a representative system, we need that no two elements in S are similar in respect to Γ_∞ and that every Matrix $M \in \mathcal{M}_n$ can be written as a matrix in Γ_∞ times one of the matrices in S. Note that the matrices $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ commute. we have

$$T^n \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} L = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} L'$$

$$\Leftrightarrow \frac{1}{n} \begin{pmatrix} \delta & -\beta \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = L'L^{-1}T^{-n} \in \Gamma \Leftrightarrow \quad \frac{1}{n} \begin{pmatrix} \delta & -\beta \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} T^n L = L'.$$

And since the matrix on the LHS of the second equation should have integer entries, we must have that $a = \alpha = \frac{n}{d} = \frac{n}{\delta}$, which makes the matrix actually be in Γ_∞. So with that in mind, from the third line we get that $L = L' \mod \Gamma_\infty$, proving the first part.

For the second part, let $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a matrix in $\mathcal{M}_n$. Let $g := \gcd(C, D)$, $C' := \frac{C}{g}$ and $D' := \frac{D}{g}$. Since C' and $D'$are coprime, we can pick $\alpha, \beta \in \mathbb{Z}$ such that $\begin{pmatrix} \alpha & \beta \\ C' & D' \end{pmatrix}$ is in Γ. Now note that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ C' & D' \end{pmatrix}^{-1} = \begin{pmatrix} AD' - BC' & -A\beta + B\alpha \\ 0 & -C\beta + D\alpha, \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

which is an upper triangular matrix with integer entries and determinant n. There exists a unique n, such that $b = b' + nd$ with $0 \leq b' < d$. So multiplication of the matrix $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with T^{-n} will lead to a matrix as in the set S as desired, finishing the proof. $\square$

Theorem 1.4. *For even $k \geq 4$ the Eisenstein-series E_k is a simultaneous eigenform of all T_n with*

$$T_n E_k = \sigma_{k-1}(n) E_k.$$

The following relation holds for all $n, m \in \mathbb{N}$:

$$\sigma_{k-1}(n)\sigma_{k-1}(m) = \sum_{d|(n,m)} d^{k-1} \sigma_{k-1}(mn/d^2).$$

In particular σ_{k-1} is multiplicative.

Proof. Plugging in the series formulation of E_k into the definition of the Hecke-operator, gives us

$$T_n E_k = n^{k-1} \sum_{M \in \Gamma \backslash \mathcal{M}_n} \sum_{L \in \Gamma_\infty \backslash \Gamma} 1|_k LM = n^{k-1} \sum_{A \in \Gamma_\infty \backslash \mathcal{M}_n} 1|_k A.$$

We can now show that the set

$$S := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} L \;\middle|\; ad = n, d > 0, b \pmod d, L \in \Gamma_\infty \backslash \Gamma \right\}$$

is also a set of representatives of $\Gamma \backslash \mathcal{M}_n$.
Using this, we have

$$\begin{aligned} T_n E_k =& n^{k-1} \sum_{\substack{ad=n \\ d>0}} \sum_{b \bmod d} \sum_{L \in \Gamma_\infty \backslash \Gamma} 1 \Big|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} L \\ =& n^{k-1} \sum_{d|n} d^{1-k} \sum_{L \in \Gamma_\infty \backslash \Gamma} 1|_k L = \sigma_{k-1}(n) E_k. \end{aligned}$$

So the function $-\frac{B_k}{2k} E_k$ is a normed simultaneous eigenform, whose n-th Fourier-coefficient is equal to $\sigma_{k-1}(n)$. The desired relations follow from lemma 1.1. □

With a similar proof, one can also describe the action of the Hecke operator on the Poincaré-series by the following theorem.

Theorem 1.5. *For all $n, m \in \mathbb{N}$ we have*

$$T_n P_{m,k} = \sum_{d|(n,m)} (n/d)^{k-1} P_{mn/d^2,k}.$$

Proof. Note that theorem 1.4 is just the special case $m = 0$ of this theorem. Analogously to the proof of theorem 1.4, we have

$$\begin{aligned} T_n P_{m,k} =& n^{k-1} \sum_{A \in \Gamma_\infty \backslash \mathcal{M}_n} e^{2\pi i m \tau}|_k A \\ =& n^{k-1} \sum_{\substack{ad=n \\ d>0}} \sum_{b \bmod d} \sum_{L \in \Gamma_\infty \backslash \Gamma} e^{2\pi i m \tau} \Big|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} L \\ =& n^{k-1} \sum_{d|n} \sum_{b \bmod d} \sum_{L \in \Gamma_\infty \backslash \Gamma} d^{-k} e^{2\pi i m \frac{b}{d} \tau} e^{2\pi i m \frac{n}{d^2} \tau}|_k L. \end{aligned}$$

Now we use the fact that

$$\sum_{b \bmod d} e^{2\pi m \frac{b}{d} \tau} = \begin{cases} d & \text{if } d \mid m \\ 0 & \text{if } d \nmid m, \end{cases}$$

giving us

$$\begin{aligned} T_n P_{m,k} =& n^{k-1} \sum_{d|(n,m)} d^{1-k} \sum_{L \in \Gamma_\infty \backslash \Gamma} e^{2\pi i \frac{mn}{d^2} \tau}|_k L \\ =& \sum_{d|(n,m)} (n/d))^{k-1} P_{mn/d^2,k}, \end{aligned}$$

as desired. □

2 The algebra of Hecke-operators

In the previous talk, it was shown that the Hecke operators define an Endomorphism on the Vector space M_k. Let H_k be the subalgebra of all endomorphisms on M_k, generated by the Hecke operators T_n, that is all endomorphisms equal to a polynomial expression in the T_i. We call H_k the Hecke algebra in weight k. The focus of this section will be to prove the following theorem.

Theorem 2.1. *The Hecke algebra H_K is a commutative subalgebra of End(M_k) which is generated by all T_p with p prime. Furthermore, for all $m, n \in \mathbb{N}$, we have*

$$T_m T_n = \sum_{d|(n,m)} d^{k-1} T_{mn/d^2}. \tag{1}$$

In particular, for m, n coprime we have

$$T_m T_n = T_{mn}, \tag{2}$$

and for every prime p and for all $r \in \mathbb{N}$ we have

$$T_{p^r} T_p = T_{p^{r+1}} + p^{k-1} T_{p^{r-1}}. \tag{3}$$

With this theorem, we can expand on lemma 1.1:

Korollar 2.2. *For a non-constant $f \in M_k$, the following are equivalent:*

1. *f is a simultaneous eigenform of all T_n, where $n \in \mathbb{N}$.*
2. *f is a simultaneous eigenform of all T_p, where p prime.*
3. *For every prime p and all $m \in \mathbb{N}_0$, we have*

$$a_f(p)a_f(m) = a_f(1)\left(a_f(mp) + p^{k-1}a_f(m/p)\right)$$

where we take $a_f(m/p) = 0$ if $p \nmid m$.

The proof of theorem 2.1 is split into four parts: Equation (2), then equation (3), followed by its generalization Cor. 2.7 and then finally the full form of equation (1). The fact that the subalgebra is commutative follows immediately from (1), as the right-hand side is symmetric in m and n, so we must have $T_m T_n = T_n T_m$.

Lemma 2.3. *Let $m, n \in \mathbb{N}$ be coprime and $a_1, a_2, d_1, d_2 \in \mathbb{N}$ such that $a_1 d_1 = m$, $a_2 d_2 = n$. Then the map*

$$\psi \colon (b_1, b_2) \mapsto b_{12} := a_2 b_1 + b_2 d_1,$$

defines a bijection between all pairs of integers $b_1 \pmod{d_1}$, $b_2 \pmod{d_2}$ and all $b_{12} \pmod{d_1 d_2}$.

Proof. Since the two sets are of equal finite size, it suffices to show that ψ is injective.
Assume that $\varphi(b_1, b_2) = \varphi(b_1', b_2')$. Looking at the equation mod d_1 gives us:

$$a_2 b_1 \equiv a_2 b_1' \pmod{d_1}$$

And since $\gcd(a_2, d_1) \mid \gcd(m, n) = 1$, we can divide by a_2, leading us to $b_1 = b_1'$. By looking at the remaining equality mod d_2, gives us $b_2 d_1 = b_2' d_1 \pmod{d_2}$. Noting that $\gcd(d_1, d_2) = 1$, leads to $b_2 = b_2'$ analogously to the above. □

Theorem 2.4. *For $(n.m) = 1$ we have $T_m T_n = T_{mn}$.*

Proof. For $f \in M_k$ we have

$$T_m T_n f = (mn)^{k-1} \sum_{a_1 d_1 = m} \sum_{a_2 d_2 = n} (d_1 d_2)^{-k} \sum_{b_1 \pmod{d_1}} \sum_{b_2 \pmod{d_2}} f\left(\frac{a_1 a_2 \tau + a_2 b_1 + b_2 d_1}{d_1 d_2}\right)$$
$$= (mn)^{k-1} \sum_{ad=mn} d^{-k} \sum_{b \bmod d} f \frac{a\tau + b}{d} = T_{mn} f,$$

where we used $a = a_1 a_2$, $d = d_1 d_2$ and $b = a_2 b_1 + b_2 d_1$. □

For the second part, we will need a similar lemma as 2.3.

Lemma 2.5. *1. The function $\psi\colon (b_\nu, a) \mapsto c_\nu := b_\nu + ap^\nu$, defines a bijection between all pair of integers $b_\nu \pmod{p^\nu}$, $a \pmod{p}$ and all $c_\nu \pmod{p^{\nu+1}}$.*

2. If $b_n u$ goes through a representative system of $\pmod{p^\nu}$, then b_ν runs through a representative system of $p^{\nu-1}$ exactly p times.

Proof. Since the map ψ is between two finite sets, it suffices to show injectivity. Assume that we have $\psi(b_\nu, a) = \psi(b'_\nu, a')$.
Reducing the equality down to $\pmod{p)^\nu}$ instantly gives us $b_\nu = b'_\nu$, leaving us with $p^\nu a = p^\nu a' \pmod{p)^{\nu+1}}$. This is equivalent to $\Leftrightarrow p^\nu(a - a') = 0$, which can only be true if $a - a' = \pmod{p}$ implying that $a = a'$.

Part 2 can be seen, by noting that for any given b_ν there are exactly $p - 1$ other $b'_\nu \pmod{p^\nu}$, which have the same residue $\bmod p^{\nu-1}$. More precisely, the $p - 1$ possibilities are $\{b_\nu + kp^{\nu-1} \mid 1 \leq k \leq p - 1\}$. □

Theorem 2.6. *For every prime p and all $r \in \mathbb{N}$, we have*

$$T_{p^r} T_p = T_{p^{r+1}} + p^{k-1} T_{p^{r-1}}.$$

Proof. For $f \in M_k$ we have

$$T_{p^r} T_p f = p^{r(k-1)} \sum_{\nu=0}^{r} p^{-\nu k} \sum_{b_\nu \bmod p^\nu} T_p f\left(\frac{p^{r-\nu}\tau + b_\nu}{p^\nu}\right) \tag{1}$$
$$= p^{(r+1)(k-1)} f(p^{r+1}\tau) + p^{r(k-1)-1} \sum_{a \bmod p} f\left(\frac{p^r \tau + a}{p}\right) + \tag{2}$$
$$p^{r(k-1)} \sum_{\nu=1}^{r} p^{-\nu k} \sum_{b_\nu \bmod p^\nu} \left[p^{k-1} f\left(\frac{p^{r-\nu}\tau + b_\nu}{p^{\nu-1}}\right) + \frac{1}{p} \sum_{a \bmod p} f\left(\frac{p^{r-\nu}\tau + c_\nu}{p^{\nu+1}}\right)\right], \tag{3}$$

where $c_\nu = b_\nu + ap^\nu$. Note that the two lines (2) and (3) come from separating the cases $\nu = 0$ and $\nu > 0$. With the previous lemma, this is equal to

$$T_{p^r} T_p f = p^{(r+1)(k-1)} f(p^{r+1}\tau) + \tag{4}$$
$$p^{r(k-1)} \sum_{\nu=1}^{r} p^{-(\nu-1)k} \sum_{b_\nu \bmod p^{\nu-1}} f\left(\frac{p^{(r-1)-(\nu-1)}\tau + b_\nu}{p^{\nu-1}}\right) + \tag{5}$$
$$p^{r(k-1)} \sum_{\nu=0}^{r} p^{-\nu k-1} \sum_{c_\nu \bmod p^{\nu+1}} f\left(\frac{p^{(r+1)-(\nu+1)}\tau + c_\nu}{p^{\nu+1}}\right) \tag{6}$$
$$= p^{(r+1)(k-1)} f(p^{r+1}\tau) + p^{r(k-1)-(r-1)(k-1)} T_{p^{r-1}} f + \tag{7}$$
$$p^{(r+1)(k-1)} \sum_{\nu=0}^{r} p^{-(\nu+1)k} \sum_{c_\nu \bmod p^{\nu+1}} f\left(\frac{p^{(r+1)-(\nu+1)}\tau + c_\nu}{p^{\nu+1}}\right) \tag{8}$$
$$= p^{k-1} T_{p^{r-1}} f + T_{p^{r+1}} f. \tag{9}$$

Note that line (2) is the case $\nu = 0$ in line (6). Similarly the first term in line (4) and (7) is the case $\nu = -1$ is the sum of line (8), creating $T_{p^{r+1}} f$. □

Korollar 2.7. *For a prime p and $r, s \in \mathbb{N}_0$, we have*

$$T_{p^r} T_{p^s} = \sum_{\nu=0}^{\min(r,s)} p^{\nu(k-1)} T_{p^{r+s-2\nu}}.$$

Proof. We shall show the statement by induction on s, where the base case $s = 0$ is trivial and $s = 1$ is given by the theorem 2.6.
Assume that the statement is true for all $s' < s$. We have

$$T_{p^r} T_{p^{s-1}} T_p = \sum_{\nu=0}^{\min(r,s-1)} p^{\nu(k-1)} T_{p^{r+s-1-2\nu}} T_p = \sum_{\nu=0}^{\min(r,s-1)} p^{(\nu+1)(k-1)} T_{p^{r+s-2(\nu+1)}} + p^{\nu(k-1)} T_{p^{r+s-2\nu}},$$

but also

$$T_{p^r} T_{p^{s-1}} T_p = T_{p^r} p^{k-1} T_{s-2} + T_{p^r} T_s = \sum_{\nu=0}^{\min(r,s-2)} p^{(\nu+1)(k-1)} T_{p^{r+s-2(\nu+1)}} + T_{p^r} T_{p^s},$$

where $T_{p^{-1}}$ is taken to be 0. So in the case $r \leq s-2 < s$, we have

$$T_{p^r} T_{p^s} = \sum_{\nu=0}^{r} p^{\nu(k-1)} T_{p^{r+s-2\nu}}$$

as wanted. In the other case we have

$$T_{p^r} T_{p^s} = p^{s(k-1)} T_{p^{r+s-2s}} + \sum_{\nu=0}^{s-1} p^{\nu(k-1)} T_{p^{r+s-2\nu}} = \sum_{\nu=0}^{\min(s,r)} p^{\nu(k-1)} T_{p^{r+s-2\nu}},$$

where we used that $T_{p^{r+s-2s}} = 0$, if $r = s-1$. □

Now all that is left is to show equation (1) in Thm. 2.1. We do this by induction on the amount of prime divisors of $\gcd(n, m)$. The base case is when $\gcd(n, m) = 1$ and is given by thm. 2.4.

For the induction step, assume that p is a prime divisor of n and m and $m = m'p^r$, $n = n'p^s$, where of course $(n',p) = (m',p) = 1$. We then have

$$\begin{aligned}
T_mT_n =& T_{m'}T_{p^r}T_{n'}T_{p^s}\\
=& T_{m'}T_{n'}T_{p^r}T_{p^s}\\
=& \left(\sum_{d|(n',m')} d^{k-1}T_{mn/d^2}\right)\left(\sum_{d|p^{\min(r,s)}} d^{k-1}T_{p^{r+s}/d^2}\right)\\
=& \sum_{\substack{d_1|(m',n')\\ d_2|(p^r,p^s)}} (d_1d_2)^{k-1}T_{n'm'/d_1^2}T_{p^{r+s}/d_2^2}\\
=& \sum_{d|(n,m)} d^{k-1}T_{mn/d^2}.
\end{aligned}$$

3 Self-adjointness of the Hecke-operators

Reminder, we know the following about the Petersson inner product.

Lemma 3.1. *Let $k \geq 4$ be even and $m \in \mathbb{N}$. For $f = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k$ the following formula holds.*

$$\langle f, P_{m,k}\rangle = \frac{(k-2)!}{(4\pi m)^{k-1}} a_f(m).$$

With this we can show that the Hecke operator T_n is self-adjoint regarding the Petersson inner product.

Theorem 3.2. *For $f, g \in S_k$ and $n \in \mathbb{N}$, we have*

$$\langle T_nf, g\rangle = \langle f, T_ng\rangle,$$

Proof. Beacause of $S_k = 0$ for $k < 12$ and k odd, we can assume that $k \geq 12$ and that k is even. Since in this case, S_k is generated by the Poincaré series $P_{m,k}$, it suffices to show the equality for $g = P_{m,k}$. Using lemma 3.1, 0.1 and 1.5, we have

$$\begin{aligned}
\langle T_nf, P_{m,k}\rangle &= \frac{(k-2)!}{(4\pi m)^{k-1}} a_{T_nf}(m)\\
&= \frac{(k-2)!}{(4\pi m)^{k-1}} \sum_{d|(m,n)} d^{k-1}a_f(mn/d^2)\\
&= \frac{(k-2)!}{(4\pi m)^{k-1}} \sum_{d|(m,n)} d^{k-1}\frac{(4\pi mn/d^2)^{k-1}}{(k-2)!}\langle f, P_{mn/d^2,k}\rangle\\
&= \left\langle f, \sum_{d|(m,n)} (n/d)^{k-1}P_{mn/d^2,k}\right\rangle\\
&= \langle f, T_nP_{m,k}\rangle
\end{aligned}$$

□

It is known from linear algebra that a family of commutative self-adjoint operators can be simultaneously diagonalized on finite-dimensional spaces, i.e. that there is a basis consisting of eigenvectors of all operators. This means that there is a basis consisting of simultaneous eigenforms of all operators. Moreover the eigenvalues of self-adjoint operators are real, and eigenvectors to different eigenvalues are orthogonal. From this we obtain:

Theorem 3.3. *The space S_k has an orthonormal basis sonsisting of simutaneous eigenforms, with all real fourier coefficients.*

Proof. According to the above statement from linear algebra, the spacee S_k has a basis consisting of simultaneous eigenforms with real eigenvalues. Wlog we can assume that these eigenforms are normed.
If two normed forms f, g had the same eigenvalues $\lambda_f(n) = \lambda_g(n)$, then they would be equal to each other. Indeed, we would have $a_f(n) = \frac{\lambda_f(n)}{a_f(1)} = \lambda_f(n) = \frac{\lambda_g(n)}{a_g(1)} = a_g(n)$. And since eigenvectors to different eigenvalues are orthogonal, the normed basis is indeed an orthonormal one. Also by the same equation above $a_f(n) = \lambda_f(n) \in \mathbb{R}$ all fourier coefficents are real. □

References

[1] Schwagenscheidt *Modulformen*, lecture notes