Seminar Elliptic Functions and Modular Forms

Dr. Markus Schwagenscheidt
ETH Zürich
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General informations

The talks should take between 80-90 minutes. Two students share a talk. A script in latex is required. The seminar takes place Wednesdays from 10-12 in HG G26.5, starting on 28.02. until 29.05. (12 talks; no meetings on 21.02., 03.04. and 01.05.).

We follow the lecture notes [9] and [10], which are based on the book [5]. Most of the material can also be found in the book [4], which is available in german or in english. A nice and short overview over the modular forms part is given in [1]. The other references contain useful additional material and different viewpoints.

Topics

1 Elliptic functions (V. Michel, S. Ruppanner)

Recall the definition of meromorphic functions on $\mathbb{C}$. Define the set of periods of a meromorphic function and show that it is either $\{0\}$, $\mathbb{Z}w_f$, or $\mathbb{Z}w_1 + \mathbb{Z}w_2$. Define lattices in $\mathbb{C}$ and their fundamental parallelograms, and explain how the fundamental parallelogram can be viewed as a torus. If time permits, discuss the convergence of Eisenstein series, see [9, Section 2.3]. Define elliptic functions and mention that they form a field. State and prove the four theorems of Liouville.

References: [9], Section 2.1–2.2 and 3.1–3.2 (and Section 2.3 if time permits)
See also [4], Section V.1

2 The Weierstrass $\wp$-function (M. Corpataux, L. Schlemmer)

Start with [9, Section 4.1] and define the Weierstrass $\wp$-function. Mention that $\sum_{w\in\Omega}(z-w)^{-k}$ converges absolutely for $k \geq 3$, but not for $k = 2$, but omit the proof. Show that the $\wp$-function converges and defines an elliptic function. Then give its properties from [9, Section 3.3], and afterwards prove that $\wp$ and $\wp'$ generate the field of elliptic functions, see [9, Section 3.4]. Define the Eisenstein series G_k and compute the Laurent expansion of $\wp$. Show the two differential equations $\wp'^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$ and $\wp'^2 = 4\wp^3 - g_2\wp - g_3$. If time permits, give the identities of Eisenstein series from Corollary 4.3.5.

References: [9], Section 3.3–3.4 and Section 4
See also [4], Section V.2 and V.3

3 Complex elliptic curves (D. Blättler, E. Staikov)

Prove the addition theorem for the $\wp$-function. Define complex elliptic curves and show that the map $z + \Omega \mapsto (\wp(z), \wp'(z))$ defines a bijection between $\mathbb{C}/\Omega$ and the elliptic curve $\overline{E}(\Omega)$. Explain the geometric addition law on elliptic curves and show that it is compatible with the natural addition on $\mathbb{C}/\Omega$ under the above bijection.

References: [9], Section 7
See also [4], Appendix A to Section V.3, and Section V.4

4 The Weierstrass σ and ζ-functions (M. Daniele, A. David)

Briefly discuss the convergence of infinite products. Define the Weierstrass σ and ζ-functions and explain their connection to the $\wp$-function. Show that the Weierstrass η-function is a group homomorphism and prove the Legendre relation. Prove the transformation law for σ. Prove Abel's theorem on the existence of elliptic functions with prescribed zeros and poles, see [9, Theorem 6.3.1]. If time permits, define the Jacobi theta function and mention that it can also be used to construct elliptic functions with prescribed zeros and poles, [9, Lemma 6.4.1 and Corollary 6.4.3].

References: [9], Section 6.1–6.3 (Section 6.4 if time permits)
See also [4], Section V.6

5 The modular group and modular forms (S. Hadzhistoykov, B. Heim)

Introduce the modular group $\mathrm{SL}_2(\mathbb{Z})$ and its action on the upper half-plane by fractional linear transformations (Moebius transformations); prove that T and S generate $\mathrm{SL}_2(\mathbb{Z})$; sketch the fundamental domain and explain its properties; if time permits, explain elliptic points; define the factor of automorphy and modular forms; explain what Fourier expansions are and give the integral formula for their coefficients; prove the Hecke bound and that there are no modular forms of negative weight.

References: [10], Section 2.1 – 2.2
See also [4], Section VI.1 and VI.2, and the part on *Complex Fourier series* in Section III.5

6 Eisenstein series and the Delta function (G. Hüglin, J. Menzi)

Introduce the Eisenstein series G_k and relate the to the Eisenstein series associated to the lattice $\mathbb{Z}\tau+\mathbb{Z}$ defined in the talk on the $\wp$-function. Show convergence, modularity and the computation of the Fourier expansion; define the normalized Eisenstein series E_k and give its alternative definition using the slash operator; explain that one can write every modular form as an Eisenstein series plus a cusp form; give the evaluation of the Riemann zeta function at even natural numbers in terms of Bernoulli numbers, and use this to show that E_k has rational Fourier coefficients; introduce Ramanujan's Delta function and show that it is a cusp form of weight 12; mention the following things without proof: product expansion, multiplicative coefficients, some congruences, Ramanujan conjecture (Deligne bound), Lehmer's conjecture.

References: [10], Section 2.3 – 2.4

7 The valence formula and the structure of M_k (J. Schütt, J. Sommer)

Explain the order of a modular form at a point in the upper half-plane and at ∞; state, explain, and prove the valence formula; show that multiplication with Δ yields isomorphism $M_k \cong S_{k+12}$ and use this to prove the structure theorem for M_k for small k; prove the dimension formula for M_k and that it has a basis of products of Eisenstein series.

References: [10], Section 2.5
See also [4], Section VI.2 and Section VI.3

8 The j-invariant, the Eisenstein series of weight 2, and the Dedekind eta function (C. Veit, A.Ying)

Define the j-invariant and determine its orders; define modular functions and show that every modular function is a rational function in j; mention that $j : \Gamma\backslash\mathbb{H} \to \mathbb{C}$ is a bijection; define the holomorphic (but non-modular) Eisenstein series G_2 and compute its Fourier expansion; prove its modular transformation law; define the non-holomorphic (but modular) Eisenstein series G_2^* and the normalized versions E_2 and E_2^*; define the Dedekind eta function η as an infinite product, and show that $\frac{\eta'}{\eta} = \frac{i}{4\pi} G_2$; deduce the modular transformation properties of η; show that $\eta^{24} = \Delta$ and thereby prove the product expansion of Δ.

References: [10], Section 2.6 – 2.8

9 Modular forms for congruence subgroups and the four-squares-theorem (J. Roshardt, A. Weidmann)

Define congruence subgroups, in particular $\Gamma(N), \Gamma_1(N), \Gamma_0(N)$ and show that they have finite index in $\mathrm{SL}_2(\mathbb{Z})$; write down the explicit formula for their indices; introduce cusps; define modular forms for congruence subgroups, and explain what their expansion at different cusps are; introduce Eisenstein series for $\Gamma_0(N)$, in particular of weight 2; investigate the trace $\mathrm{tr}(f)$ and product $\pi(f)$; prove Sturm's bound on the dimension of spaces of modular forms for congruence subgroups; introduce the Jacobi theta function and prove its transformation under $\tau \mapsto -\frac{1}{4\tau}$ using Poisson summation; show that $\vartheta^4 \in M_2(\Gamma_0(4))$, but omit some details of the proof if necessary; state and prove the four-squares-theorem (omit some technical details, but make it clear how modular forms, in particular the theta function and Eisenstein series, come into play).

References: [10], Section 2.9– 2.10
See also [12], Section on "Sums of two and four squares".

10 The Petersson inner product and Poincaré series; Hecke operators I (N. Avci, M. Portmann)

Introduce the hyperbolic volume element and the Petersson inner product of two modular forms (one of which is a cusp form); prove its basic properties; show that the inner product is independent of the choice of fundamental domain; prove that Eisenstein series are orthogonal to cusp forms; define Poincaré series and show that they are cusp forms; prove that the inner product of a cusp form f with the m-th Poincaré series gives (essentially) the m-th coefficient of f; show that the Poincaré series span the space of cusp forms; introduce the set $\mathcal{M}_n$ of integral matrices with determinant n, and give a system of representatives for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{M}_n$; define Hecke operators and prove its action on the Fourier expansion; infer that Hecke operatos define endomorphisms of M_k and S_k.

References: [10], Section 3.1–3.3

11 Hecke operators II (F. Keta)

Recall the definition and basic properties of Hecke operators from the last talk; explain what a simultaneous Hecke eigenform is and show what this means on the level of Fourier coefficients; show that Δ is a simultaneous eigenform and prove that its coefficients are multiplicative; show that the Eisenstein series E_k is an eigenform, and state the formula for the action of Hecke operators on Poincaré series; prove that the algebra of Hecke operators is commutative and generated by the T_p for primes p, and give the precise composition laws (omit some details of the proof if necessary); show that the Hecke operators are self-adjoint with respect to the Petersson inner product; use this to show that S_k has an orthonormal basis of simultaneous Hecke eigenforms.

References: [10], Section 3.3–3.5

12 The Birch and Swinnerton-Dyer Conjecture (R. Angst, M. Olsen)

Recall the definition of elliptic curves over $\mathbb{Q}$, and state Mordell's Theorem. Explain the rank of a rational elliptic curve, and its meaning for the number of rational points; explain the content of the Birch and Swinnerton-Dyer Conjecture; give some historical overview and some remarks on the state of the Conjecture; explain the application to the Congruent Number Problem via Tunnell's Theorem. If time permits, mention the Modularity Theorem and thereby indicate the connection of the BSD Conjecture to modular forms.

References: [6, 7]

Contact

Dr. Markus Schwagenscheidt, mschwagen@ethz.ch
`https://people.math.ethz.ch/~mschwagen/ellipticfunctionsmodularforms`

Literatur

[1] Alfes-Neumann, *Modular forms*

[2] Apostol, *Modular Functions and Dirichlet Series in Number Theory*

[3] Diamond, Shurman, *A first course in modular forms*

[4] Freitag, Busam, *Complex Analysis* (or *Funktionentheorie*)

[5] Koecher, Krieg, *Elliptische Funktionen und Modulformen*

[6] Kramer, *Der große Satz von Fermat – die Lösung eines 300 Jahre alten Problems*, available online

[7] Kramer, *Die Vermutung von Birch und Swinnerton-Dyer*, available online

[8] Lang, *Introduction to modular forms*

[9] Schwagenscheidt, *Elliptic Functions*, lecture notes, available online

[10] Schwagenscheidt, *Modulformen*, lecture notes, available online

[11] Serre, *A course in arithmetic*

[12] Zagier's part of the book *The 1-2-3 of modular forms*