Dirichlet Series

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1 Dirichlet Series: analytic theory

We start by introducing the elementary properties of Dirichlet series which play a fundamental role in analytic number theory as power series do in complex analysis.
In the theory of power series we try to represent any function as infinite series of *power functions* $z \mapsto z^n$ $(n \in \mathbb{N})$. With Dirichlet series we take the *exponential functions* $z \mapsto e^{-\lambda z}$ $(\lambda \in \mathbb{R})$ as building blocks instead. Since $\mathbb{R}$ is uncountable we have to restrict ourselves to a sequence $\left(e^{-\lambda_n z}\right)_{n\in\mathbb{N}}$, where λ_n are real numbers satisfying

$$\lambda_1 < \lambda_2 < \ldots, \lambda_n \to \infty. \tag{1}$$

Notation 1.1. In the theory of Dirichlet series it is usual to denote complex variables by s, their real part by σ and their imaginary part by t.

1.1 Definitions and first examples

Definition 1.2 (Dirichlet Series)**.** A Dirichlet series is any series of the form

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \tag{2}$$

where λ_n are real numbers satisfying (1), a_n is a complex sequence and $s = \sigma + it$ is a complex number.

Example 1.3.

1. The simplest case is when $\lambda_n = n$. This doesn't lead to any new theory since with the substitution $z = e^{-s}$ in the series (2) one gets a power series.

2. By setting $\lambda_n = \log n$ one can write series (2) as

$$\sum_{n=1}^{\infty} a_n n^{-s}. \tag{3}$$

   This case is the most relevant for analytic number theory and series of this form are called *ordinary* Dirichlet series.

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1.2 Abscissa of convergence

For power series $p(z) = \sum_{n=0}^{\infty} a_n z^n$ we know that there is a non-negative real number R (radius of convergence) such that $p(z)$ converges for all z with $|z| < R$ and diverges for all z with $|z| > R$.
The following theorem gives us a similar statement for Dirichlet series.

Theorem 1.4 (Abscissa of convergence)**.** *If the Dirichlet series converges for $s = s_0$, then it converges uniformly on compact sets for all s with $\mathrm{Re}(s) > \mathrm{Re}(s_0)$. Therefore, there exists a real number σ_0, such that the series (2) converges for every s with $\sigma > \sigma_0$ and diverges for every s with $\sigma < \sigma_0$.*
Moreover, the function

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \tag{4}$$

defined for $\sigma > \sigma_0$ is holomorphic, and its derivatives are given by

$$f^{(k)}(s) = (-1)^k \sum_{n=1}^{\infty} \lambda_n^k a_n e^{-\lambda_n s} \tag{5}$$

and converge for $\sigma > \sigma_0$.

Definition 1.5. The number σ_0 on the previous theorem is the abscissa of convergence of the Dirichlet series.

Notation 1.6. In order to lighten our proof we introduce the following notation:

$$A(N) = \sum_{n=1}^{N} a_n, \qquad A(M,N) = \sum_{n=M}^{N} a_n \text{ for } N \geq M, \qquad A(M, M-1) = 0. \tag{6}$$

Proof. We consider only the case of ordinary Dirichlet series which is the one of greatest interest to us. In particular, we prove that for $\epsilon > 0$ the Dirichlet series converges uniformly on the domain defined by

$$|\arg(s - s_0)| \leq \frac{\pi}{2} - \epsilon < \frac{\pi}{2}. \tag{7}$$

This is a slightly stronger statement, since every compact set in $\{s : \sigma > \sigma_0\}$ is contained in such an angle.
Moreover, by replacing s with $s + s_0$ and a_n with $a_n n^{-s}$ we can assume $s_0 = 0$.
Then the series $\sum_{n=1}^{\infty} a_n$ converges by assumption, and by the Cauchy criterion we get that for every $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that $|A(M,N)| < \epsilon$ for every $N > M \geq N_0$. In particular, for every $\epsilon > 0$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ we can choose N_0 such that for all $N \geq M \geq N_0$ we have

$$|A(M,N)| < \frac{\epsilon}{\frac{|s|}{\sigma} + 1}. \tag{8}$$

With the help of the Abel's summation formula[1] we get for $N > M \geq N_0$ and two complex valued sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ that

[1] The first equality below is the so called Abel's summation formula. A proof can be found in *Dirichletreihen im Komplexen* by Dominik Wrazidlo.

$$
\begin{aligned}
\sum_{n=M}^{N} a_n b_n &= (A(N)b_{N+1} - A(M-1)b_M) - \sum_{n=M}^{N} A(n)(b_{n+1} - b_n) \\
&= A(0,N)b_{N+1} - A(0,M-1)b_M + \sum_{n=M}^{N} [A(0,M-1) + A(M,n)](b_n - b_{n+1}) \\
&= A(0,M-1) \underbrace{\left[\sum_{n=M}^{N} (b_n - b_{n+1}) - b_M\right]}_{=b_M - b_{N+1} - b_M = -b_{N+1}} + A(0,N)b_{N+1} + \sum_{n=M}^{N} A(M,n)(b_n - b_{n+1}) \\
&= b_{N+1}(A(0,N) - A(0,M-1)) + \sum_{n=M}^{N} A(M,n)(b_n - b_{n+1}) \\
&= A(M,N)b_{N+1} + \sum_{n=M}^{N-1} A(M,n)(b_n - b_{n+1}) + A(M,N)(b_N - b_{N+1}) \\
&= \sum_{n=M}^{N-1} A(M,n)(b_n - b_{n+1}) + A(M,N)b_N.
\end{aligned}
$$

Applying this to $S(M,N) \coloneqq \sum_{n=M}^{N} a_n n^{-s}$ gives us

$$
S(M,N) = \sum_{n=M}^{N} a_n n^{-s} = \sum_{n=M}^{N-1} A(M,n)(n^{-s} - (n+1)^{-s}) + A(M,N)N^{-s}.
$$

Moreover, we have

$$
\begin{aligned}
|n^{-s} - (n+1)^{-s}| &= \left| s \int_{\log(n)}^{\log(n+1)} e^{-su}\, du \right| \leq |s| \int_{\log(n)}^{\log(n+1)} \left| e^{-su} \right| du \\
&= |s| \int_{\log(n)}^{\log(n+1)} e^{-\sigma u}\, du = \frac{|s|}{\sigma}(n^{-\sigma} - (n+1)^{-\sigma}).
\end{aligned}
$$

Finally, for all $N \geq M \geq N_0$ and $\sigma > 0$ it holds

$$
\begin{aligned}
|S(M,N)| &\leq \sum_{n=M}^{N-1} |A(M,n)||n^{-s} - (n+1)^{-s}| + |A(M,N)||N^{-s}| \\
&\leq \frac{\epsilon}{\frac{|s|}{\sigma} + 1} \left[\sum_{n=M}^{N-1} |n^{-s} - (n+1)^{-s}| + 1 \right] \\
&\leq \frac{\epsilon}{\frac{|s|}{\sigma} + 1} \left[\frac{|s|}{\sigma}(M^{-\sigma} - N^{-\sigma}) + 1 \right] \\
&\leq \frac{\epsilon}{\frac{|s|}{\sigma} + 1} \left[\frac{|s|}{\sigma} + 1 \right] = \epsilon.
\end{aligned}
$$

Note that N_0 in (8) does depend on s, to get uniform convergence we need to get rid of this dependence. To do so, let s be such that $|\arg(s)| \leq \frac{\pi}{2} - \epsilon < \frac{\pi}{2}$ and consider a representation in polar coordinates $s = re^{i\phi}$, then $-(\frac{\pi}{2}-\epsilon) \leq \phi \leq \frac{\pi}{2}-\epsilon$. We get

$$\frac{|s|}{\sigma} = \frac{|r||e^{i\phi}|}{\mathrm{Re}(re^{i\phi})} = \frac{r}{r\cos\phi} \leq \frac{1}{\cos(\frac{\pi}{2}-\epsilon)} = C$$

and thus $\frac{|s|}{\sigma}$ is bounded by a constant independent of s. We may pick N_0 so that $|A(M,N)| < \frac{\epsilon}{C+1}$ in (8). This proves uniform convergence on every compact set in $\{s : \sigma > \sigma_0\}$. Now it is also clear that there exists $\sigma_0 \in \mathbb{R}$ with the desired properties.

Moreover, since the partial sums $(S_N)_{N\geq 1} = \left(\sum_{n=1}^{N} a_n n^{-s}\right)_{N\geq 1}$ are holomorphic and converge locally uniformly, by the theorem of Weierstrass we get that the limit function $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is also holomorphic. We can therefore differentiate every term of the series to get the derivatives of $f(s)$ given by (5).

□

Analogous as for power series, there is a formula depending on the coefficients a_n for the abscissa of convergence.

Theorem 1.7 (Formula Abscissa)**.** *Let $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ be a Dirichlet series such that $\sum_{n=1}^{\infty} a_n$ diverges. Then the abscissa of convergence is given by*

$$\sigma_0 = \limsup_{N\to\infty} \frac{\log|A(N)|}{\lambda_N}, \tag{9}$$

where $A(N)$ is the sum of the coefficients defined in (6).

Remark 1.8. The theorem still holds when $\sum_{n=1}^{\infty} a_n$ converges, one needs to substitute $A(N)$ with $\sum_{n=N}^{\infty} a_n$.

Example 1.9. 1. The Riemann Zeta function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots \tag{10}$$

is a Dirichlet series with $a_n = 1$, $\lambda_n = \log n$, $A(N) = N$. Thus,

$$\sigma_0 = \limsup_{N\to\infty} \frac{\log|A(N)|}{\lambda_N} = \limsup_{N\to\infty} \frac{\log N}{\log N} = 1$$

and the series converges for $\sigma > 1$ and diverges for $\sigma < 1$.

2. Let

$$\psi(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \dots. \tag{11}$$

Here $a_n = (-1)^{n-1}$, $\lambda_n = \log n$, $A(N)$ is equal 1 for N odd and equal 0 for N even. Thus,

$$\sigma_0 = \limsup_{N\to\infty} \frac{\log 1}{\log N} = 0$$

and the series converges for $\sigma > 0$ and diverges for $\sigma < 0$.

These examples show a big difference between the theory of (ordinary) Dirichlet series and that of power series. For power series the two series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} |a_n| z^n$ have the same radius of convergence which means that they converge absolutely in the interior of the disk of radius R. On the other hand, the Dirichlet series (11) converges for $\sigma > 0$ and the corresponding series with positive sign (10) converges only for $\sigma > 1$, thus, not absolutely.

Theorem 1.10. *Let $\sum_{n=1}^{\infty} a_n n^{-s}$ be an ordinary Dirichlet series with abscissa of convergence σ_0 and let σ_1 $(\geq \sigma_0)$ be the abscissa of convergence of $\sum_{n=1}^{\infty} |a_n| n^{-s}$. Then*

$$\sigma_1 \leq \sigma_0 + 1.$$

Remark 1.11. This theorem only holds for ordinary Dirichlet series: for example the Dirichlet series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} (\log n)^{-s}$ converges for all s but does not converge absolutely for any s.

1.3 Landau's theorem and uniqueness of the coefficients

There is an even more important difference between Dirichlet series and power series. We know that the radius of convergence for power series is also given by the absolute value of the smallest singularity. For Dirichlet series only in a special case we can conclude the existence of a singularity from the abscissa of convergence.

Theorem 1.12 (Landau). *Let $\sum_{n=1}^{\infty} a_n n^{-s}$ be an ordinary Dirichlet series with abscissa of convergence σ_0 and non-negative real coefficients. Then the function defined by*

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \qquad (\sigma > \sigma_0)$$

has a singularity at $s = \sigma_0$.

Proof. Without loss of generality, let $\sigma_0 = 0$ and assume that $f(s)$ is holomorphic at $s = 0$. Then it would also be holomorphic on a disk of radius ϵ around 0 and, consequently, have a Taylor expansion around $s = 1$ with radius of convergence strictly greater than 1. For a suitable $\delta > 0$ one has that the series $\sum_{k=0}^{\infty} \frac{(-\delta-1)^k}{k!} f^{(k)}(1)$ is convergent and equal to $f(-\delta)$. On other hand, by (5) we have

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(-\delta-1)^k}{k!} f^{(k)}(1) &= \sum_{k=0}^{\infty} \frac{(-\delta-1)^k}{k!} \sum_{n=1}^{\infty} \frac{(\log n)^k}{n} a_n \\
&= \sum_{n=1}^{\infty} \frac{a_n}{n} \sum_{k=0}^{\infty} \frac{(1+\delta)^k (\log n)^k}{k!} \\
&= \sum_{n=1}^{\infty} \frac{a_n}{n} e^{(1+\delta)\log n} = \sum_{n=1}^{\infty} a_n n^{\delta}.
\end{aligned}$$

In the second equality we used absolute convergence of the series since $a_n > 0$. We thus have that $\sum_{n=1}^{\infty} a_n n^{\delta}$ converges which contradicts our assumption $\sigma_0 = 0$. □

We conclude with a simple theorem about the uniqueness of the coefficients.

Theorem 1.13 (uniqueness of coefficients). *Let* $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ *and* $\sum_{n=1}^{\infty} b_n e^{-\lambda_n s}$ *be two Dirichlet series which converge in an open domain of* $\mathbb{C}$ *and where they define the same function. Then we have* $a_n = b_n$ *for every* n*.*

Proof. Assume by contradiction that the statement is false and let m be the smallest index such that $a_m \neq b_m$. For σ sufficiently large we get

$$0 = e^{\lambda_m \sigma} \left(\sum_{n=1}^{\infty} a_n e^{-\lambda_n \sigma} - \sum_{n=1}^{\infty} b_n e^{-\lambda_n \sigma} \right) = a_m - b_m + \sum_{n=m+1}^{\infty} (a_n - b_n) e^{-(\lambda_n - \lambda_m)\sigma}.$$

The coefficients of the series go to 0 as $\sigma \to \infty$ since $\lambda_n - \lambda_m > 0$. Therefore, by uniform convergence the entire sum vanishes and this contradicts $a_m \neq b_m$. □

2 Dirichlet Series: formal properties

2.1 Dirichlet product

We start by recalling the Cauchy product of two power series. Let

$$f(x) = \sum_{m=0}^{\infty} a_m x^m, \; g(x) = \sum_{n=0}^{\infty} b_n x^n$$

be two such power series. Then their product is the power series

$$h(x) = \sum_{k=0}^{\infty} c_k x^k$$

where

$$c_k = \sum_{m+n=k} a_m b_n.$$

We can generalise the Cauchy product to the Dirichlet series as follows: Let

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s} \text{ and } \sum_{n=1}^{\infty} b_n e^{-\mu_n s}$$

be two Dirichlet series. Then their product is $\sum c_N e^{-\nu_N s}$ where

$$c_N = \sum_{\lambda_m + \mu_n = \nu_N} a_m b_n$$

Now, for the rest of the talk, we will restrict ourselves to case where $\lambda_m = log(m)$ and $\mu_n = log(n)$, so that we have ordinary Dirichlet series. Then $\nu_N = log(N)$ and we obtain

$$c_N = \sum_{mn=N} a_m b_n = \sum_{d|N} a_d b_{\frac{N}{d}}$$

For this section, we won't worry about convergence, but only remark that the product of two absolutely convergent Dirichlet series, f and g is absolutely convergent. In fact, if f is absolutely convergent and g is convergent, then fg is convergent. For a proof, see "The General Theory of Dirichlet Series", by G.H. Hardy and M. Riesz (Theorem 53 and 54). We now give some examples.

Example 2.1. 1. Let $d(n)$ denote the number of divisors of n and let

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Then

$$\zeta(s)\zeta(s) = \sum_{n=1}^{\infty} \left(\sum_{d|n} 1 \cdot 1 \right) n^{-s} = \sum_{n=1}^{\infty} d(n) n^{-s}$$

2. Let

$$\sigma_k(n) = \sum_{d|n} d^k$$

be the sum of the k'th power of divisors of n. Then

$$\zeta(s)\zeta(s-k) = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{m=1}^{\infty} \frac{m^k}{m^s} = \sum_{N=1}^{\infty} c_N N^{-s},$$

where

$$c_N = \sum_{d|N} d^k = \sigma_k(N).$$

We note that both these functions satisfy a multiplicative property.

Definition 2.2. A function $f : \mathbb{N} \longrightarrow \mathbb{C}$ is multiplicative if $f(mn) = f(m)f(n)\ \forall m, n$ such that $(m, n) = 1$. The function f is said to be completely multiplicative when

$$f(mn) = f(m)f(n)\ \forall m, n \in \mathbb{N}.$$

Note that for any multiplicative function f, we always have $f(1) = 1$, since $f(1) = f(1 \cdot 1) = f(1)f(1) = (f(1))^2$. Moreover, if $f(1) = 0$, then the function f is the zero function.

We also have that $f(n) = f(p_1^{r_1})...f(p_k^{r_k})$ for $n = p_1^{r_1}...p_k^{r_k}$.

Now, in the region of absolute convergence, using the fundamental theorem of arithmetic, we can prove

Theorem 2.3. *Let $\sum f(n)n^{-s}$ be an absolutely converging Dirichlet series for $\sigma > \sigma_a$, where σ_a is the abscissa of convergence. If f is multiplicative, then we have*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + ... \right)$$

for $\sigma > \sigma_a$. For f completely multiplicative, we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \frac{1}{1 - f(p)p^{-s}}$$

if $\sigma > \sigma_a$

Proof. This will follow from Theorem 2.4 by taking $g(n) = \frac{f(n)}{n^s}$ □

Theorem 2.4. *Let $g(n)$ be a multiplicative arithmetical function such that the series $\sum g(n)$ is absolutely convergent. Then we can express the sum of the series as an absolutely convergent infinite product, namely*

$$\sum_{n=1}^{\infty} g(n) = \prod_p (1 + g(p) + g(p^2) + ...). \tag{12}$$

Moreover, if g is completely multiplicative, we can simplify (12) to get

$$\sum_{n=1}^{\infty} g(n) = \prod_p \frac{1}{1 - g(p)}.$$

The product is called the Euler product of the series.

Proof. Let us consider the finite product

$$P(x) = \prod_{p \leq x} (1 + g(p) + g(p^2) + ...)$$

over all primes $p \leq x$. As it is the product of a finite number of absolutely convergent series, we can multiply the series and rearrange the terms in any way we want. Thus, using fundamental theorem of arithmetic, we can write

$$P(x) = \sum_{n \in A} g(n)$$

where A is the set of all n's, whose prime factors are all $\leq x$. Using this, we can then write

$$\sum_{n=1}^{\infty} g(n) - P(x) = \sum_{n \in B} g(n),$$

where B is the set of the n's who have at least one prime factor $> x$. Hence

$$\left| \sum_{n=1}^{\infty} g(n) - P(x) \right| \leq \sum_{n \in B} |g(n)| \leq \sum_{n > x} |g(n)|$$

As $x \longrightarrow \infty$, the last sum on the right converges to 0, since $\sum |g(n)|$ is convergent. So, as $x \longrightarrow \infty$, we have that $P(x) \longrightarrow \sum g(n)$.

We know that $\prod(1+a_n)$, an infinite product, converges absolutely, if and only if $\sum a_n$ converges absolutely. In this case, we have

$$\sum_{p\leq x}|g(p)+g(p^2)+...|\leq\sum_{p\leq x}(|g(p)|+|g(p^2)|+...)\leq\sum_{n=2}^{\infty}|g(n)|<\infty.$$

Hence, all the partial sums are bounded. Being a series of positive terms, we can use monotone convergence theorem, to conclude the convergence of

$$\sum_{p}|g(p)+g(p^2)+...|.$$

This, in return, implies the convergence of the infinite product

$$\prod_{p}(1+g(p)+g(p^2)+...).$$

And when g is completely multiplicative, we know that $g(p^n)=g(p)^n$ and each series on the right of the original product is a convergent geometric series with sum $\frac{1}{1-g(p)}$. To see the convergence of the geometric series, note that $|g(p)|<1$, since $|g(p^n)|=|g(p)|^n\longrightarrow 0$.

□

Remark 2.5. Note that if a Dirichlet series $G(s)=\sum\frac{g(n)}{n^s}$ satisfies (12), then in fact g is multiplicative. To see this, first note that two Dirichlet series $\sum a_n n^{-s}$ and $\sum b_n n^{-s}$ are equal if and only if $a_n=b_n\ \forall n$.
Now, expanding the right hand side of (12) and matching the coefficients of n^{-s} for $n=p_1^{e_1}...p_k^{e_k}$, gives that

$$f(n)=f(p_1^{e_1})...f(p_k^{e_k}).$$

And hence, f is multiplicative.

Now we look at some examples.

Example 2.6. 1. The Riemann zeta function

$$\zeta(s)=\prod_{p}(1+p^{-s}+p^{-2s}+...)=\prod_{p}\frac{1}{1-p^{-s}}$$

by Theorem 2.3 with $f(n)=1$.

We have seen in Example 2.1 (1), that

$$\sum_{n=1}^{\infty}\frac{d(n)}{n^s}=\zeta(s)^2.$$

Thus, it follows that

$$\sum_{n=1}^{\infty}\frac{d(n)}{n^s}=\prod_{p}\frac{1}{(1-p^{-s})^2}.$$

On the other hand, recall that for $|x| < 1$, we have that

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + ... \tag{13}$$

Using (13) with $x = p^{-s}$, the fact that $d(p^k) = k+1$ and that $|p^{-s}| < 1$ for $\text{Re}(s) > 1$, we get

$$\zeta(s)^2 = \prod_p (1 + 2p^{-s} + 3p^{-2s} + ...) = \prod_p \left(1 + \frac{d(p)}{p^s} + \frac{d(p^2)}{p^{2s}} + ...\right).$$

2. The Möbius function
   Let

$$\mu(n) = \begin{cases} 0, & \text{if } n \text{ is divisible by a square,} \\ (-1)^k, & \text{if } n = p_1...p_k. \end{cases} \tag{14}$$

   Then

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1 + \frac{\mu(p)}{p^s} + \frac{\mu(p^2)}{p^{2s}} + ...\right) = \prod_p (1 - p^{-s}) = \frac{1}{\zeta(s)}.$$

A very important property of the Möbius function is

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases} \tag{15}$$

Equation (15) follows from the Dirichlet product of the two series

$$\zeta(s) \cdot \frac{1}{\zeta(s)} = 1 = 1 + \frac{0}{2^s} + \frac{0}{3^s} + ...$$

The Möbius function is important for the following theorem.

Theorem 2.7 (Möbius inversion formula)**.** *Let $f, g : \mathbb{N} \to \mathbb{C}$. Then*

$$f(n) = \sum_{d|n} g(d)$$

if and only if

$$g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d) \tag{16}$$

We also have that f is multiplicative if and only if g is multiplicative.

We will give two proofs:

First proof. This proof uses (15) and goes as follows:
Assume

$$f(n) = \sum_{d|n} g(d).$$

We then look at the right hand side of (16)

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) f(d) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) \tag{17}$$

Then, using our assumption, we get

$$\sum_{d|n} \mu(d) f\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d) \sum_{\lambda|\frac{n}{d}} g(\lambda) = \sum_{d\lambda|n} \mu(d) g(\lambda) \tag{18}$$

On the other hand,

$$\sum_{\lambda|n} g(\lambda) \sum_{d|\frac{n}{\lambda}} \mu(d) = \sum_{\lambda d|n} \mu(d) g(\lambda). \tag{19}$$

Hence,

$$\sum_{d|n} \mu(\frac{n}{d}) f(d) = \sum_{\lambda|n} g(\lambda) (\sum_{d|\frac{n}{\lambda}} \mu(d))$$

Now, use

$$\sum_{d|\frac{n}{\lambda}} (\mu(d)) = \begin{cases} 1, & \text{if } \frac{n}{\lambda} = 1 \\ 0, & \text{otherwise} \end{cases} \tag{20}$$

to conclude

$$\sum_{d|n} \mu(\frac{n}{d}) f(d) = g(n).$$

And every step is reversible. Hence, we have the "if and only if"-statement. □

Second proof. We use the associated Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \text{and} \quad G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}.$$

Then, the property that

$$f(n) = \sum_{d|n} g(d) \cdot 1$$

is equivalent to

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) \left(\sum_{m=1}^{\infty} \frac{g(m)}{m^s}\right) = \zeta(s) G(s).$$

Hence, $G(s) = \zeta(s)^{-1} F(s)$, and again forming their Dirichlet product, gives us

$$G(s) = \zeta(s)^{-1} F(s) = \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right) \left(\sum_{m=1}^{\infty} \frac{f(m)}{m^s}\right).$$

And hence, it follows that

$$g(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right).$$

To prove the statement about multiplicativity, we know that
$g(n)$ is multiplicative if and only if $G(s)$ has an Euler product (see Remark 2.5). On the other hand, $F(s) = \zeta(s)G(s)$, hence $F(s)$ has an Euler product if and only if $G(s)$ does. Hence, $g(n)$ is multiplicative if and only if $f(n)$ is multiplicative. $\square$