VALUES OF ZETA FUNCTIONS AT $S = 0$, CONTINUED FRACTIONS AND CLASS NUMBERS

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1. Introduction

In this talk we follow Chapter 14 of [1] and we study the values at $s = 0$ of the zeta functions

$$\zeta(\mathcal{A}, s) = \sum_{\substack{\mathfrak{a} \in \mathcal{A} \\ \mathfrak{a} \subset \mathcal{D}}} \frac{1}{\mathrm{N}(\mathfrak{a})^s}$$

associated to the ideal classes $\mathcal{A}$ of a real quadratic field $\mathbb{Q}(\sqrt{D})$, and then we use these values to relate the class number of $\mathbb{Q}(\sqrt{-p})$ (where $p \neq 3$ is a prime such that $p \equiv 3 \pmod 4$ and $h_0(p) = 1$) to the representation as a continued fraction of $\sqrt{p}$. In particular, we begin by proving that for any reduced quadratic form f of discriminant $D > 0$ we have

$$Z_f(0) = \frac{1}{24}\left(\frac{b}{a} + \frac{b}{c} - 6\right),$$

where Z_f is the analytic continuation of the function defined by

$$Z_f(s) = \sum_{x,y>0} \frac{1}{f(x,y)^s} + \frac{1}{2}\sum_{x>0} \frac{1}{f(x,0)^s} + \frac{1}{2}\sum_{y>0} \frac{1}{f(0,y)^s} \tag{1.1}$$

for $\mathrm{Re}(s) > 1$. Then we use the relation

$$\zeta(\mathcal{A}, s) = \sum_{j=1}^{r} Z_{f_j}(s),$$

which was proved in the last talk and can be found in Chapter 13 of [1], to show that the zeta function $\zeta(\mathcal{A}, s)$ satisfies

$$\zeta(\mathcal{A}, 0) = \frac{1}{12}\sum_{j=1}^{r}(n_j - 3),$$

where $n_1, ..., n_r$ are the numbers from the minimal period of the continued fraction representation of $\frac{b+\sqrt{D}}{2a}$ and $f(x,y) = ax^2 + bxy + cy^2$ is any quadratic form in the equivalence class corresponding to $\mathcal{A}$. Lastly, we use these results to prove that the class number of $\mathbb{Q}(\sqrt{-p})$ is given by

$$h(-p) = \frac{1}{3}\sum_{j=1}^{r} n_j - r,$$

where $p \neq 3$ is a prime number such that $p \equiv 3 \pmod 4$ and the class number of $\mathbb{Q}(\sqrt{p})$ in the broader sense is 1 and $n_1, ..., n_r$ are the numbers from the minimal period of the continued fraction representation of

$$\sqrt{p} = n_0 - \cfrac{1}{n_1 - \cfrac{1}{\ddots\, n_{r-1} - \cfrac{1}{n_r - \cfrac{1}{n_1 - \cfrac{1}{\ddots}}}}}.$$

2. Previous results

In this section we briefly recap some results from the previous talks that we will need in order to prove the theorems in Section 3 and Section 4. For the proof of Theorem 3.2, we will need a modified version of Theorem 1 from Chapter 7 of [1], which we report below together with one of its remarks.

Theorem 2.1. *Let $\varphi(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$ be a Dirichlet series which converges for at least one value of $s \in \mathbb{C}$ and let $f(t) = \sum_{n\geq 1} a_n e^{-nt}$ be the corresponding exponential series. If f has an asymptotic expansion*

$$f(t) \approx \sum_{n\geq 0} c_n t^n \text{ for } t \to 0,$$

then φ can be extended to a holomorphic function on $\mathbb{C}$. Moreover, if f has an asymptotic expansion

$$f(t) \approx \sum_{n\geq -1} c_n t^n \text{ for } t \to 0,$$

then φ can be extended to a meromorphic function on $\mathbb{C}$ and $\varphi(s) - \frac{c_1}{s-1}$ is an entire function. In both cases we have

$$\varphi(-n) = (-1)^n n! c_n$$

for any $n \in \mathbb{Z}_{\geq 0}$.

Remark 2.2. *As one can see in the proof (which can be found on page 48 of [1]), Theorem 2.1 is also true for general Dirichlet series of the form $\varphi(s) = \sum_{n\geq 0} \frac{a_n}{\lambda_n^s}$ (assuming that the series converges for at least one value of $s \in \mathbb{C}$) and in this case one has to consider the exponential series $f(t) = \sum_{n\geq 0} a_n e^{-\lambda_n t}$.*

The proof of Theorem 3.3 is based on the relation between the zeta function $\zeta(\mathcal{A}, s)$ of an ideal class $\mathcal{A}$ and the functions Z_f associated to the reduced forms f in the cycle corresponding to $\mathcal{A}$, which was proved in Theorem 2′ from Chapter 13 of [1] and which we report in the following theorem.

Theorem 2.3. *Let $\mathcal{A}$ be an ideal class in a real quadratic field. Then for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ we have*

$$\zeta(\mathcal{A}, s) = \sum_f Z_f(s),$$

where the sum runs over the cycle of reduced forms f which correspond to the class $\mathcal{A}$.

We now recall some results that will be useful for the proof of Theorem 4.1. The following theorem (which we report from Theorem 2 on page 111 of [1]) provides a one-to-one correspondence between the genus characters of a quadratic field K and the decompositions of the discriminant of K as a product of two fundamental discriminants. In addition, below we also report the corollary of this theorem, which will allow us to determine if the class number of a quadratic field is even or odd.

Theorem 2.4. *Let D be the discriminant of a quadratic field K. Then there is a bijective correspondence between the genus characters of K and the decompositions $D = D' \cdot D''$ of D as the product of two fundamental discriminants D' and D'', where we identify the decompositions $D = D' \cdot D''$ and $D = D'' \cdot D'$. In particular, the genus character corresponding to the partition $D = D' \cdot D''$ is given by*

$$\chi(\mathfrak{p}) = \begin{cases} \chi_{D'}(N_\mathfrak{p}), & \text{if } (N_\mathfrak{p}, D') = 1, \\ \chi_{D''}(N_\mathfrak{p}), & \text{if } (N_\mathfrak{p}, D'') = 1. \end{cases}$$

for a prime ideal $\mathfrak{p}$ and by

$$\chi\left(\prod_{j=1}^{k} \mathfrak{p}_j^{n_j}\right) = \prod_{j=1}^{k} \chi(\mathfrak{p}_j)^{n_j}$$

for an arbitrary ideal, where $\mathfrak{p}_j$ are prime ideals and $n_j \in \mathbb{Z}$. Moreover, the L-function corresponding to χ is given by

$$L_K(s, \chi) = L_{D'}(s) L_{D''}(s).$$

Corollary 2.5. *Let C be the ideal class group of the quadratic field K and let D be the discriminant of K. Then the group C/C^2 is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{t-1}$, where t is the number of different prime numbers which divide D. Moreover, the class number $h(D)$ is divisible by 2^{t-1}, and it is odd if and only if D is a prime discriminant.*

Lastly, we recall the following results, which we will use to evaluate the L-series $L_K(\mathcal{A}, 0)$. In particular, the theorem below and its corollary correspond to Theorem 2 on page 51 of [1] and its corollary, and Theorem 2.8 can be found in Theorem 3 on page 79 of [1].

Theorem 2.6. *Let χ be a Dirichlet character modulo N and $L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}$ $(\mathrm{Re}(s) > 1)$ the corresponding L-function. Then $L(s,\chi)$ can be extended to a meromorphic function on $\mathbb{C}$ with at most a simple pole at $s = 1$. Moreover, for any $n \in \mathbb{Z}_{\geq 0}$ we have*

$$L(-n,\chi) = -\frac{N^n}{n+1} \sum_{j=1}^{N} \chi(j) B_{n+1}\left(\frac{j}{N}\right),$$

where B_n is the n-th Bernoulli polynomial.

Corollary 2.7. *Except in the case $N = 1$ and $n = 0$, for every Dirichlet character χ modulo N and every $n \geq 0$ we have*

$$\chi(-1) = (-1)^n \implies L(-n,\chi) = 0.$$

Theorem 2.8. *Let D be a fundamental discriminant. If $D < 0$, then*

$$h(D) = -\frac{w}{2\,|D|} \sum_{n=1}^{|D|-1} \chi_D(n) n,$$

where

$$w(D) = \begin{cases} 6 & if\ D = -3, \\ 4 & if\ D = -4, \\ 2 & if\ D < -4. \end{cases}$$

is the number of units of the field $\mathbb{Q}(\sqrt{D})$. If $D > 0$, then

$$h(D) = -\frac{1}{\log(\varepsilon_0)} \sum_{n=1}^{D-1} \chi_D(n) \log\left(\sin\left(\frac{\pi n}{D}\right)\right),$$

where ε_0 is the fundamental unit of $\mathbb{Q}(\sqrt{D})$.

3. Values of zeta functions at $s = 0$

This section is devoted to the proofs of Theorem 3.2 and Theorem 3.3. In order to prove Theorem 3.2, we will use the general result about the analytical continuation of Dirichlet series given in Theorem 2.1. In particular, Theorem 2.1 can be modified so that, if f satisfies the weaker condition of having an asymptotic expansion of the form $f(t) \approx \frac{c_{-1}}{t} + c_0 + O(t^{\frac{1}{2}})$ for $t \to 0$, then it follows that φ can be extended to a meromorphic function on the half-plane $\mathrm{Re}(s) > -\frac{1}{2}$ and the formula for $\varphi(-n)$ still holds in the case $n = 0$. This is exactly the content of the following proposition, of which we omit the proof since it is almost identical to the proof of Theorem 2.1.

Proposition 3.1. *Let $\varphi(s) = \sum_{n\geq 0} \frac{a_n}{\lambda_n^s}$ be a general Dirichlet series which converges for at least one value of $s \in \mathbb{C}$ and let $f(t) = \sum_{n\geq 0} a_n e^{-\lambda_n t}$ be the corresponding exponential series. If f has an asymptotic expansion*

$$f(t) \approx \frac{c_{-1}}{t} + c_0 + O(t^{\frac{1}{2}}) \text{ for } t \to 0,$$

then φ can be extended to a meromorphic function on the half-plane $\mathrm{Re}(s) > -\frac{1}{2}$ with at most a simple pole at the point $s = 1$. Moreover, we have $\varphi(0) = c_0$.

We are now ready to prove Theorem 3.2, which we report (together with its proof) from Theorem 1 in Chapter 14 of [1].

Theorem 3.2. *Let $f(x,y) = ax^2 + bxy + cy^2$ be an indefinite quadratic form with positive coefficients and let $Z_f(s)$ be as in (1.1). Then Z_f can be extended to a holomorphic function on the half-plane $\mathrm{Re}(s) > -\frac{1}{2}$ with at most a simple pole at the point $s = 1$. Moreover, we have*

$$Z_f(0) = \frac{1}{24}\left(\frac{b}{a} + \frac{b}{c} - 6\right).$$

Proof. Note that the exponential series corresponding to f is given by

$$V_f(t) = \sum_{x,y>0} e^{-f(x,y)t} + \frac{1}{2}\sum_{x>0} e^{-f(x,0)t} + \frac{1}{2}\sum_{y>0} e^{-f(0,y)t}$$

and that this series converges absolutely for any $t > 0$. Hence by Proposition 3.1 it suffices to show that V_f has an asymptotic expansion $V_f(t) \approx \frac{c_{-1}}{t} + c_0 + O(t^{\frac{1}{2}})$ for $t \to 0$ with $c_0 = \frac{1}{24}\left(\frac{b}{a} + \frac{b}{c} - 6\right)$. We use the notation $F_t(x,y) = e^{-f(x,y)t}$ and we define the function G_{F_t} by

$$G_{F_t}(x,y) = \frac{1}{4}\left(F_t(x,y) + F_t(x-1,y) + F_t(x,y-1) + F_t(x-1,y-1)\right) - \int_{x-1}^{x}\int_{y-1}^{y} F_t(u,v)dudv$$
$$+ \frac{1}{12}\int_{x-1}^{x}\left(\frac{\partial F_t}{\partial y}(u,y-1) - \frac{\partial F_t}{\partial y}(u,y)\right)du + \frac{1}{12}\int_{y-1}^{y}\left(\frac{\partial F_t}{\partial x}(x-1,v) - \frac{\partial F_t}{\partial x}(x,v)\right)dv.$$

Since

$$\sum_{x>0}\sum_{y>0}\int_{x-1}^{x}\left(\frac{\partial F_t}{\partial y}(u,y-1) - \frac{\partial F_t}{\partial y}(u,y)\right)du = \sum_{x>0}\int_{x-1}^{x}\frac{\partial F_t}{\partial y}(u,0)du = \int_0^{+\infty}\frac{\partial F_t}{\partial y}(u,0)du$$

and similarly

$$\sum_{x>0}\sum_{y>0}\int_{y-1}^{y}\left(\frac{\partial F_t}{\partial x}(x-1,v) - \frac{\partial F_t}{\partial x}(x,v)\right)dv = \int_0^{+\infty}\frac{\partial F_t}{\partial x}(0,v)dv,$$

we have

$$\sum_{x>0}\sum_{y>0} G_{F_t}(x,y) = \frac{1}{4}F_t(0,0) + \frac{1}{2}\sum_{x>0}F_t(x,0) + \frac{1}{2}\sum_{y>0}F_t(0,y) + \sum_{x,y>0}F_t(x,y) - \int_0^{+\infty}\int_0^{+\infty}F_t(u,v)dudv$$
$$+ \frac{1}{12}\int_0^{+\infty}\frac{\partial F_t}{\partial y}(u,0)du + \frac{1}{12}\int_0^{+\infty}\frac{\partial F_t}{\partial x}(0,v)dv.$$

With the substitution $(u',v') = \left(u\sqrt{t}, v\sqrt{t}\right)$ we find that

$$\int_0^{+\infty}\int_0^{+\infty} e^{-f(u,v)t}dudv = \frac{c_{-1}}{t},$$

where $c_{-1} = \int_0^{+\infty}\int_0^{+\infty} e^{-f(u,v)}dudv$. Since

$$\int_0^{+\infty}\frac{\partial F_t}{\partial y}(u,0)du = -bt\int_0^{+\infty} ue^{-au^2t}du = -\frac{b}{2a}$$

and

$$\int_0^{+\infty}\frac{\partial F_t}{\partial x}(0,v)dv = -bt\int_0^{+\infty} ve^{-cv^2t}dv = -\frac{b}{2c},$$

we can write $\sum_{x>0}\sum_{y>0} G_{F_t}(x,y)$ as

$$\sum_{x>0}\sum_{y>0} G_{F_t}(x,y) = \frac{1}{4} + V_f(t) - \frac{c_{-1}}{t} - \frac{b}{24a} - \frac{b}{24c} = V_f(t) - \frac{c_{-1}}{t} - c_0,$$

where $c_0 = \frac{1}{24}\left(\frac{b}{a} + \frac{b}{c}\right) - \frac{1}{4}$. We claim that $\sum_{x>0}\sum_{y>0} G_{F_t}(x,y) = O(t^{\frac{1}{2}})$ for $t \to 0$. By Taylor's Theorem, for fixed $x, y > 0$ we can write $F_t = P + R$, where P is a polynomial of degree at most 2 in u and v and R and its partial derivatives $\frac{\partial R}{\partial u}$ and $\frac{\partial R}{\partial v}$ are bounded in the rectangle

$$\{(u,v) \mid x-1 \le u \le x,\ y-1 \le v \le y\}$$

by some constant C times

$$M(x,y) = \max_{\substack{x-1\le u\le x\\ y-1\le v\le y}} \max_{0\le j\le 3}\left|\frac{\partial^3 F_t}{\partial u^j \partial v^{3-j}}(u,v)\right|.$$

Note that $G_p(u,v) = 0$ for any polynomial p in two variables (one can easily verify this by computing $G_{u^i v^j}(u,v)$ for the six pairs (i,j) with $i,j \ge 0$ and $i+j \le 2$) and therefore

$$G_{F_t}(u,v) = G_P(u,v) + G_R(u,v) = G_R(u,v).$$

Since

$$\begin{aligned}|G_R(x,y)| &\le \frac{1}{4}\left(|R(x,y)|+|R(x-1,y)|+|R(x,y-1)|+|R(x-1,y-1)|\right)+\int_{x-1}^{x}\int_{y-1}^{y}|R(u,v)|\,dudv\\&+\frac{1}{12}\int_{x-1}^{x}\left(\left|\frac{\partial R}{\partial y}(u,y-1)\right|+\left|\frac{\partial R}{\partial y}(u,y)\right|\right)du+\frac{1}{12}\int_{y-1}^{y}\left(\left|\frac{\partial R}{\partial x}(x-1,v)\right|+\left|\frac{\partial R}{\partial x}(x,v)\right|\right)dv\\&\le \frac{7C}{3}M(x,y),\end{aligned}$$

it follows that $G_{F_t}(x,y)$ is also bounded by some constant times $M(x,y)$. For any $0\le j\le 3$ we have

$$\frac{\partial^3 F_1}{\partial u^j\partial v^{3-j}}(u,v)=p(u,v)F_1(u,v),$$

where p is a polynomial of degree at most 3, and thus using that $F_t(u,v)=F_1(u\sqrt{t},v\sqrt{t})$ we obtain

$$\frac{\partial^3 F_t}{\partial u^j\partial v^{3-j}}(u,v)=t^{\frac{3}{2}}p\left(u\sqrt{t},v\sqrt{t}\right)F_t(u,v).$$

Therefore

$$M(x,y)\le K\cdot\max_{i+j\le 3}x^iy^jt^{\frac{3+i+j}{2}}e^{-f(x,y)t}$$

with a constant $K>0$ which depends only on a, b and c. Since

$$\int_{x-1}^{x}\int_{y-1}^{y}u^iv^je^{-f(u,v)t}dudv=t^{-1-\frac{i+j}{2}}\int_{(x-1)\sqrt{t}}^{x\sqrt{t}}\int_{(y-1)\sqrt{t}}^{y\sqrt{t}}u^iv^je^{-f(u,v)}dudv$$

by the substitution $(u',v')=(u\sqrt{t},v\sqrt{t})$, we have

$$\begin{aligned}\sum_{x>0}\sum_{y>0}\max_{i+j\le 3}x^iy^jt^{\frac{3+i+j}{2}}e^{-f(x,y)t}&=O\left(\sum_{x>0}\sum_{y>0}\int_{x-1}^{x}\int_{y-1}^{y}\max_{i+j\le 3}u^iv^jt^{\frac{3+i+j}{2}}e^{-f(u,v)t}dudv\right)\\&=O\left(t^{\frac{1}{2}}\int_0^{+\infty}\int_0^{+\infty}\max_{i+j\le 3}u^iv^je^{-f(u,v)}dudv\right)\\&=O(t^{\frac{1}{2}}).\end{aligned}$$

Therefore we obtain that

$$\sum_{x>0}\sum_{y>0}G_{F_t}(x,y)=O(t^{\frac{1}{2}}),$$

proving that V_f has the desired asymptotic expansion. □

One can also prove that the function Z_f satisfies the more strict assumptions of Theorem 2.1 and therefore it extends to a meromorphic function on $\mathbb{C}$ with at most a simple pole at $s=1$. However, since we are only interested in the value of Z_f at $s=0$, in this case it is sufficient to prove that V_f satisfies the weaker assumptions of Proposition 3.1. We now turn our attention to the proof of Theorem 3.3, which follows directly from Theorem 3.2 and the formula given in Theorem 2.3. The following theorem and its proof can be found in Theorem 2, Chapter 14 of [1].

Theorem 3.3. *Let $\mathcal{A}$ be an ideal class in a real quadratic field and $f(x,y)=ax^2+bxy+cy^2$ be any quadratic form in the equivalence class corresponding to $\mathcal{A}$. Assume that the continued fraction representation of $\frac{b+\sqrt{D}}{2a}$ is $[[m_0,...,m_k,\overline{n_1,...,n_r}]]$, where $n_1,...,n_r$ are the numbers from the minimal period. Then the value of the zeta function of the ideal class $\mathcal{A}$ at the point $s=0$ is given by*

$$\zeta(\mathcal{A},0)=\frac{1}{12}\sum_{j=1}^{r}(n_j-3).$$

Proof. Using Theorem 3.2 and the Identity Theorem, it's easy to see that the formula in Theorem 2.3 holds for any $s\neq 1$ with $\mathrm{Re}(s)>-\frac{1}{2}$. Recall that in the previous talk we have seen that $n_1,...,n_r$ are the numbers in the cycle of reduced quadratic forms corresponding to the equivalence class of f. Let $f_1,...,f_r$ be this cycle of reduced forms (where we consider the indices modulo r) and assume that they are enumerated so that for any $1\le j\le r$ we have $S_{n_j}f_j=f_{j+1}$, where

$$S_{n_j}=\left(\begin{smallmatrix}n_j & 1\\ -1 & 0\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z}).$$

Write $f_j(x,y) = a_jx^2 + b_jxy + c_jy^2$ with $a_j, b_j, c_j \in \mathbb{Z}$. Then by Theorem 3.2 we have

$$\zeta(\mathcal{A}, 0) = \sum_{j=1}^{r} Z_{f_j}(0) = \frac{1}{24}\sum_{j=1}^{r}\left(\frac{b_j}{a_j} + \frac{b_j}{c_j} - 6\right).$$

Since $S_{n_j} f_j = f_{j+1}$, we have $f_{j+1}(x,y) = (a_jn_j^2 - n_jb_j + c_j)x^2 + (2n_ja_j - b_j)xy + a_jy^2$ and thus $c_{j+1} = a_j$ and $b_{j+1} = 2n_ja_j - b_j$. This implies that $\frac{b_j}{a_j} + \frac{b_{j+1}}{c_{j+1}} = \frac{2n_ja_j}{a_j} = 2n_j$ and therefore we obtain

$$\frac{1}{24}\sum_{j=1}^{r}\left(\frac{b_j}{a_j} + \frac{b_j}{c_j} - 6\right) = \frac{1}{24}\sum_{j=1}^{r}\frac{b_j}{a_j} + \frac{1}{24}\sum_{j=1}^{r}\frac{b_{j+1}}{c_{j+1}} - \frac{r}{4} = \frac{1}{12}\sum_{j=1}^{r}(n_j - 3),$$

concluding the proof. □

4. Continued fractions and class numbers

In this last section we use the values of $\zeta(\mathcal{A}, 0)$ which we have found in Section 3 to relate the class number of $\mathbb{Q}(\sqrt{-p})$ to the continued fraction representation of $\sqrt{p}$, where $p \neq 3$ is any prime such that $p \equiv 3 \pmod 4$ and the class number of $\mathbb{Q}(\sqrt{p})$ in the broader sense is 1. We prove this result in the following theorem, which we report from Theorem 3 in Chapter 14 of [1].

Theorem 4.1. *Consider a prime number $p \neq 3$ such that $p \equiv 3 \pmod 4$ and the class number of $\mathbb{Q}(\sqrt{p})$ in the broader sense is $h_0(p) = 1$. Let*

$$\sqrt{p} = n_0 - \cfrac{1}{n_1 - \cfrac{1}{\ddots\, n_{r-1} - \cfrac{1}{n_r - \cfrac{1}{n_1 - \cfrac{1}{\ddots}}}}}$$

be the continued fraction representation of p, where $n_1, ..., n_r$ are the numbers from the minimal period. Then the class number of $\mathbb{Q}(\sqrt{-p})$ is

$$h(-p) = \frac{1}{3}\sum_{j=1}^{r} n_j - r.$$

Proof. Let $K = \mathbb{Q}(\sqrt{p})$, let C be the ideal class group of K and consider a genus character $\chi : C \to \{\pm 1\}$. Then Theorem 2.3 and Theorem 3.2 imply that the L-function

$$L_K(s, \chi) = \sum_{\mathcal{A}\in C} \chi(\mathcal{A})\zeta(\mathcal{A}, s)$$

can be extended to a holomorphic function on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > -\frac{1}{2} \text{ and } s \neq 1\}$. Moreover, by Theorem 3.3 the value of $L_K(s, \chi)$ at $s = 0$ is given by

$$L_K(0, \chi) = \frac{1}{12}\sum_{\mathcal{A}\in C} \chi(\mathcal{A}) \sum_{j=1}^{r(\mathcal{A})} (n_j(\mathcal{A}) - 3),$$

where $\{n_j(\mathcal{A})\}_{1\leq j\leq r(\mathcal{A})}$ are the numbers from the minimal period of the continued fraction representation of the largest root of $f(x, -1) = 0$ for an arbitrary quadratic form f in the equivalence class corresponding to $\mathcal{A}$. Since the class number $h(p)$ can only be $h_0(p) = 1$ or $2h_0(p) = 2$ and it must be even by Corollary 2.5, we have $h(p) = 2$. Hence let E and Θ be the elements of C, where E is the identity element of this group. Note that the discriminant of K is $D = 4p$ and that there are only two ways in which we can decompose D into the product of two fundamental discriminants $D = D' \cdot D''$, namely $D = 1 \cdot 4p$ and $D = (-4)\cdot(-p)$. By Theorem 2.4 this implies that there are exactly two genus characters $\chi : C \to \{\pm 1\}$, which are given by $\chi_0(E) = 1$, $\chi_0(\Theta) = 1$ and $\chi_1(E) = 1$, $\chi_1(\Theta) = -1$. We first evaluate $L_K(0, \chi_0)$. Note that $L_K(s, \chi_0) = L_1(s)L_{4p}(s)$ by Theorem 2.4. Since $L_{4p}(0) = 0$ by Corollary 2.7, we therefore have

$$L_K(0, \chi_0) = L_1(0)L_{4p}(0) = 0.$$

Now we compute the value of $L_K(0,\chi_1)$. If $\widetilde{D}$ is negative, then by Theorem 2.6 and Theorem 2.8 we have

$$L_{\widetilde{D}}(0) = -\frac{1}{\left|\widetilde{D}\right|}\sum_{j=1}^{|\widetilde{D}|-1} \chi_{\widetilde{D}}(j)j = \frac{2h(\widetilde{D})}{w(\widetilde{D})},$$

where

$$w(\widetilde{D}) = \begin{cases} 6 & if\ \widetilde{D} = -3, \\ 4 & if\ \widetilde{D} = -4, \\ 2 & if\ \widetilde{D} < -4. \end{cases}.$$

Since $L_K(s,\chi_1) = L_{-4}(s)L_{-p}(s)$ by Theorem 2.4 and $h(-4) = 1$, we therefore obtain

$$L_K(0,\chi_1) = \frac{2h(-4)}{w(-4)} \cdot \frac{2h(-p)}{w(-p)} = \frac{1}{2}h(-p).$$

Using the values of $L_K(0,\chi_0)$ and $L_K(0,\chi_1)$ we just computed, we obtain that

$$\begin{cases} \zeta(E,0) + \zeta(\Theta,0) = L_K(0,\chi_0) = 0 \\ \zeta(E,0) - \zeta(\Theta,0) = L_K(0,\chi_1) = \frac{1}{2}h(-p) \end{cases}.$$

Adding together the two equations yields

$$h(-p) = 4\zeta(E,0).$$

Note that $\sqrt{p}$ is the largest solution of $f(x,-1) = 0$, where $f(x,y) = x^2 - py^2$ is a quadratic form of discriminant $D = 4p$ which corresponds to the ideal $\mathbb{Z} + \sqrt{p}\cdot\mathbb{Z} = (1)$. Therefore the numbers $n_1, ..., n_r$ are as in Theorem 3.3 and it follows that

$$\zeta(E,0) = \frac{1}{12}\sum_{j=1}^{r}(n_j - 3) = \frac{1}{12}\sum_{j=1}^{r} n_j - \frac{r}{4},$$

which implies

$$h(-p) = \frac{1}{3}\sum_{j=1}^{r} n_j - r$$

and thus concludes the proof. □

As an application of the theorem, we compute the class number of $\mathbb{Q}(\sqrt{-7})$. First of all, note that $7 \equiv 3 \pmod 4$ and the class number of $\mathbb{Q}(\sqrt{7})$ in the broader sense is 1. Now we need to find the continued fraction representation of $\sqrt{7}$. There is an easy algorithm that can be used to compute the continued fraction representation of any given real number x. Indeed, it is sufficient to define $x_0 = x$ and $n_0 = \lceil x_0 \rceil$ and then recursively $x_j = \frac{1}{n_{j-1} - x_{j-1}}$ and $n_j = \lceil x_j \rceil$. Then the continued fraction representation of x is given by $x = [[n_0, n_1, ...]]$ and if there exist $j \geq 0$ and $r \geq 1$ such that $x_j = x_{j+r}$, then $x = [[n_0, ..., n_{j-1}, \overline{n_j, ..., n_{j+r-1}}]]$. Note that

$$\left\lceil \sqrt{7} \right\rceil = 3,$$

$$\left\lceil \frac{1}{3-\sqrt{7}} \right\rceil = \left\lceil \frac{3+\sqrt{7}}{2} \right\rceil = 3$$

and

$$\left\lceil \frac{1}{3 - \frac{3+\sqrt{7}}{2}} \right\rceil = \left\lceil 3+\sqrt{7} \right\rceil = 6.$$

Since $\frac{1}{6-(3+\sqrt{7})} = \frac{1}{3-\sqrt{7}}$, the numbers 3 and 6 will be repeated infinitely many times and so they must be the numbers in the minimal period of the continued fraction. Hence the continued fraction representation of $\sqrt{7}$ is $\sqrt{7} = [[3, \overline{3,6}]]$ and therefore by Theorem 4.1 the class number of $\mathbb{Q}(\sqrt{-7})$ is given by

$$h(-7) = \frac{1}{3}(3+6) - 2 = 1.$$

References

[1] Don Bernard Zagier. *Zetafunktionen und quadratische Körper: Eine Einführung in die höhere Zahlentheorie*. Springer Berlin Heidelberg, 1981.