Dirichlet Series

Dirichlet Series: analytic theory

Alice Fritzsche

ETH Zürich

5.10.2021

Table of contents

Definitions and First examples

Notation

In the theory of Dirichlet series it is usual to denote complex variables by s, their real part by σ and their imaginary part by t.

Power and Dirichlet Series

Dirichlet series play a fundamental role in analytic number theory as power series do in complex analysis:

- Power series: represent any function as an infinite linear combination of *power functions* $z \mapsto z^n$ $(n \in \mathbb{N})$.
- Dirichlet series: take instead the *exponential functions* $z \mapsto e^{-\lambda z}$ $(\lambda \in \mathbb{R})$ as building blocks.

Dirichlet Series

Definition (Dirichlet Series)

A Dirichlet series is any series of the form

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$$

where λ_n are real numbers satisfying $\lambda_1 < \lambda_2 < \ldots, \quad \lambda_n \to \infty$, $(a_n)_{n \in \mathbb{N}}$ is a complex sequence and $s = \sigma + it$ is a complex number.

First examples

1. $\lambda_n = n$. Doesn't lead to any new theory since with the substitution $z = e^{-s}$ one gets a power series.
2. $\lambda_n = \log n$. The Dirichlet series becomes

$$\sum_{n=1}^{\infty} a_n n^{-s}.$$

This case is the most relevant for analytic number theory and series of this form are called *ordinary* Dirichlet series.

Abscissa of convergence

Radius vs Abscissa of convergence

For power series $p(z) = \sum a_n z^n$ we know that there is a non-negative real number R (radius of convergence) such that $p(z)$ converges for all z with $|z| < R$ and diverges for all z with $|z| > R$.

$\rightarrow$ Similar statement for Dirichlet series

Abscissa of convergence

Theorem (Abscissa of convergence)

If the Dirichlet series converges for s_0, then it converges uniformly on compact sets for all s with $\mathrm{Re}(s) > \mathrm{Re}(s_0)$. Therefore, there exists a real number σ_0, such that the series converges for every s with $\sigma > \sigma_0$ and diverges for every s with $\sigma < \sigma_0$.
Moreover, the function

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$$

defined for $\sigma > \sigma_0$ is holomorphic, and its derivatives are given by

$$f^{(k)}(s) = (-1)^k \sum_{n=1}^{\infty} \lambda_n^k a_n e^{-\lambda_n s}.$$

Abscissa of convergence

Definition

The number σ_0 is called the abscissa of convergence of the Dirichlet series.

Proof (Abscissa of convergence)

STEPS:

1. Notation:
   $A(N) = \sum_{n=1}^{N} a_n, \quad A(M,N) = \sum_{n=M}^{N} a_n$ for $N \geq M, \quad A(M, M-1) = 0$.
2. Consider the case of ordinary DS.
3. Uniform convergence on the domain defined by $|\arg(s - s_0)| \leq \frac{\pi}{2} - \epsilon < \frac{\pi}{2}$.
4. Bound $S(M,N) := \sum_{n=M}^{N} a_n n^{-s}$ with Abel's summation formula.
5. Conclude with Weierstrass.

Proof (Abscissa of convergence) - Abel's summation formula

$$\sum_{n=M}^{N} a_n b_n = (A(N)b_{N+1} - A(M-1)b_M) - \sum_{n=M}^{N} A(n)(b_{n+1} - b_n)$$

$$= A(0,N)b_{N+1} - A(0,M-1)b_M + \sum_{n=M}^{N} [A(0,M-1) + A(M,n)](b_n - b_{n+1})$$

$$= A(0,M-1)\underbrace{\left[\sum_{n=M}^{N}(b_n - b_{n+1}) - b_M\right]}_{=b_M - b_{N+1} - b_M = -b_{N+1}} + A(0,N)b_{N+1}$$

$$+ \sum_{n=M}^{N} A(M,n)(b_n - b_{n+1})$$

Proof (Abscissa of convergence) - Abel's summation formula

$$= b_{N+1}(A(0,N) - A(0,M-1)) + \sum_{n=M}^{N} A(M,n)(b_n - b_{n+1})$$

$$= A(M,N)b_{N+1} + \sum_{n=M}^{N-1} A(M,n)(b_n - b_{n+1}) + A(M,N)(b_N - b_{N+1})$$

$$= \sum_{n=M}^{N-1} A(M,n)(b_n - b_{n+1}) + A(M,N)b_N.$$

Formula Abscissa

Theorem (Formula Abscissa)

Let $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ be a Dirichlet series such that $\sum_{n=1}^{\infty} a_n$ diverges. Then the abscissa of convergence is given by

$$\sigma_0 = \limsup_{N\to\infty} \frac{\log |A(N)|}{\lambda_N},$$

where $A(N)$ is the sum of the coefficients.

Remark

The theorem still holds when $\sum_{n=1}^{\infty} a_n$ converges, one needs to substitute $A(N)$ with $\sum_{n=N}^{\infty} a_n$.

Example 1

The Riemann Zeta function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots$$

is a Dirichlet series with $a_n = 1$, $\lambda_n = \log n$, $A(N) = N$. Thus,

$$\sigma_0 = \limsup_{N\to\infty} \frac{\log|A(N)|}{\lambda_N} = \limsup_{N\to\infty} \frac{\log N}{\log N} = 1$$

and the series converges for $\sigma > 1$ and diverges for $\sigma < 1$.

Example 2

Let

$$\psi(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \dots .$$

Here $a_n = (-1)^{n-1}$, $\lambda_n = \log n$, $A(N)$ is equal 1 for N odd and equal 0 for N even. Thus,

$$\sigma_0 = \limsup_{N\to\infty} \frac{\log 1}{\log N} = 0$$

and the series converges for $\sigma > 0$ and diverges for $\sigma < 0$.

Differences

These examples show a big difference between the theory of (ordinary) Dirichlet series and that of power series:

- Power series: $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} |a_n| z^n$ have the same radius of convergence $\rightarrow$ absolute convergence in the interior of the disk of radius R.
- Dirichlet series: $\psi(s)$ converges for $\sigma > 0$ and the corresponding series with positive sign ($\zeta(s)$) converges only for $\sigma > 1$.

General rule

Theorem

Let $\sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with abscissa of convergence σ_0 and let σ_1 $(\geq \sigma_0)$ be the abscissa of convergence of $\sum_{n=1}^{\infty} |a_n| n^{-s}$. Then

$$\sigma_1 \leq \sigma_0 + 1.$$

Remark

This theorem only hold for ordinary Dirichlet series: for example the Dirichlet series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} (\log n)^{-s}$ converges for all s but does not converge absolutely for any s.

Landau and Uniqueness of the coefficients

Singularities

- Radius of convergence for power series is also given by the absolute value of the smallest singularity.
- For Dirichlet series only in a special case we can conclude the existence of a singularity from the abscissa of convergence.

Landau

Theorem (Landau)

Let $\sum_{n=1}^{\infty} a_n n^{-s}$ be an ordinary Dirichlet series with abscissa of convergence σ_0 and non-negative real coefficients. Then the function defined by

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \qquad (\sigma > \sigma_0)$$

has a singularity at $s = \sigma_0$.

Proof (Landau)

Without loss of generality, let $\sigma_0 = 0$ and assume that $f(s)$ is holomorphic at $s = 0$. Then it would also be holomorphic on a disk of radius ϵ around 0 and, consequently, have a Taylor expansion around $s = 1$ with radius of convergence strictly greater than 1. For a suitable $\delta > 0$ one has that the Taylor series $\sum_{k=0}^{\infty} \frac{(-\delta-1)^k}{k!} f^{(k)}(1)$ is convergent and equal to $f(-\delta)$.

Proof (Landau)

On other hand, we have

$$\sum_{k=0}^{\infty} \frac{(-\delta-1)^k}{k!} f^{(k)}(1) = \sum_{k=0}^{\infty} \frac{(-\delta-1)^k}{k!} \sum_{n=1}^{\infty} \frac{(\log n)^k}{n} a_n$$
$$= \sum_{n=1}^{\infty} \frac{a_n}{n} \sum_{k=0}^{\infty} \frac{(1+\delta)^k (\log n)^k}{k!}$$
$$= \sum_{n=1}^{\infty} \frac{a_n}{n} e^{(1+\delta)\log n} = \sum_{n=1}^{\infty} a_n n^{\delta}.$$

In the second equality we used absolute convergence of the series since $a_n > 0$. We thus have that $\sum_{n=1}^{\infty} a_n n^{\delta}$ converges which contradicts our assumption $\sigma_0 = 0$.

Uniqueness of the coefficients

Theorem (Uniqueness of the coefficients)

Let $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ and $\sum_{n=1}^{\infty} b_n e^{-\lambda_n s}$ be two Dirichlet series which converge in an open domain of $\mathbb{C}$ and where they define the same function. Then we have $a_n = b_n \; \forall \; n$.

Proof (Uniqueness of the coefficients)

Assume by sake of contradiction that this is false and let m be the smallest index such that $a_m \neq b_m$. For σ sufficiently large we get

$$0 = e^{\lambda_m \sigma} \left(\sum_{n=1}^{\infty} a_n e^{-\lambda_n \sigma} - \sum_{n=1}^{\infty} b_n e^{-\lambda_n \sigma} \right) = a_m - b_m + \sum_{n=m+1}^{\infty} (a_n - b_n) e^{-(\lambda_n - \lambda_m)\sigma}.$$

The coefficients of the series go to 0 since $\lambda_n - \lambda_m > 0$. Therefore, by uniform convergence the entire sum vanishes and this contradicts $a_m \neq b_m$.