GAMMA FUNCTION

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1. Interpolation of the Gamma function

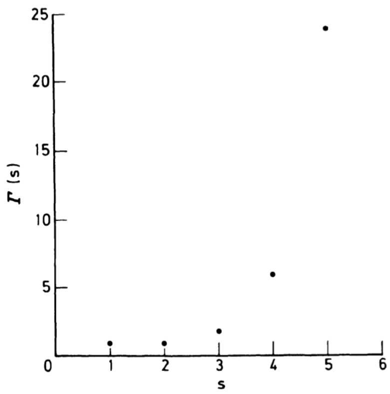
The Gamma function Γ is one of the most important mathematical functions, which is a relatively easy but at the same time a non trivial function. The Gamma function plays an important role for the study of Dirichlet series. In order to find a correct definition of the Gamma function we try to find the interpolation function of the function

$$f : n \mapsto n!, \tag{1.1}$$

i.e we want to find a continuous function $\Pi(x)$ such that $\Pi(n) = n!$ for all natural numbers n. Equivalently we may find a function $\Gamma(s) = \Pi(s-1)$, i.e. a continuous function $\Gamma(s)$ such that $\Gamma(n) = (n-1)!$ for all natural numbers. This function must also fulfill another important property:

$$\Gamma(s+1) = s \cdot \Gamma(s) \qquad s \neq 0. \tag{1.2}$$

This because the factorial function fulfills $n! = n \cdot (n-1)!$.
For small s the gamma function $\Gamma(s)$ must go through the following points s, but it is not clear



how to interpolate. Instead for bigger numbers s the function f is more uniform continuous, and for this reason it should be easier to find an interpolation. Repeating (1.2) more and more times we obtain

$$\Gamma(s+N) = s \cdot (s+1) \ldots (s+N+1) \cdot \Gamma(s).$$

And from this we can set

$$\Gamma(s) = \lim_{N\to\infty} \frac{\Gamma(s+N)}{s(s+1)\ldots(s+N-1)}.$$

For a natural number $s \in \mathbb{N}$ we have

$$\begin{aligned}\Gamma(N+s) &= (N+s-1)! = (N+s-1)\cdot(N+s-2)\ldots(N+1)\cdot N\cdot(N-1)!\\ &= N^s \cdot \left(1+\frac{s-1}{N}\right)\cdot\left(1+\frac{s-2}{N}\right)\ldots\left(1+\frac{1}{N}\right)\cdot(N-1)!.\end{aligned}$$

Therefore we obtain $\Gamma(s+N) \sim N^s \cdot (N-1)!$ for $N \to \infty$. From this result if the limit exists it seems obvious to define the Gamma function as follows.

Definition 1.

$$\Gamma(s) = \lim_{N\to\infty} \frac{N^s (N-1)!}{s(s+1)\ldots(s+N-1)}, \qquad s \notin -\mathbb{N}. \tag{1.3}$$

Definition 2. *For the case $s \notin -\mathbb{N}$ and $N \in \mathbb{N}$ we have*

$$\Gamma_N(s) = \frac{N^s(N-1)!}{s(s+1)\dots(s+N-1)}.$$

It remains to prove that the Gamma function is well defined.

Proof. We want to prove that the limit $\lim_{N\to\infty}\Gamma_N(s)$ exists. We have

$$\frac{\Gamma_{N+1}(s)}{\Gamma_N(s)} = \left(\frac{N+1}{N}\right)^s \cdot \frac{N}{s+N} = \left(1+\frac{1}{N}\right)^s \cdot \left(1+\frac{s}{N}\right)^{-1}.$$

Then using the Taylor expansion of $(1+\frac{1}{N})^s$, i.e. $(1+\frac{1}{N})^s = (1+\frac{s}{N}+o(\frac{1}{N^2}))$ and of $(1+\frac{s}{N})^{-1}$, i.e. $(1+\frac{s}{N})^{-1} = (1-\frac{s}{N}+o(\frac{1}{N^2}))$ we obtain

$$\frac{\Gamma_{N+1}(s)}{\Gamma_N(s)} = \left(1+\frac{s}{N}+o\left(\frac{1}{N^2}\right)\right)\cdot\left(1-\frac{s}{N}+o\left(\frac{1}{N^2}\right)\right) = \left(1+o\left(\frac{1}{N^2}\right)\right).$$

This proves that

$$\prod_{N\geqslant 1}\frac{\Gamma_{N+1}(s)}{\Gamma_N(s)}$$

converges, which means that the limit exists. And therefore $\Gamma_N(s)$ is well defined. □

We also obtain

$$\Gamma_N(s) = \Gamma_1(s)\cdot\prod_{n=1}^{N-1}\frac{\Gamma_{n+1}(s)}{\Gamma_n(s)} = \frac{1}{s}\prod_{n=1}^{N-1}\left[\left(1+\frac{1}{n}\right)^s\cdot\left(1+\frac{s}{n}\right)^{-1}\right].$$

Definition 3. *The Gamma function defined as an infinite product due to Euler*

$$\Gamma(s+1) = s\Gamma(s) = \prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^s}{\left(1+\frac{s}{n}\right)}. \tag{1.4}$$

Proof. We check that this definition satisfies the desired properties, i.e that $\Gamma(s+1) = s\Gamma(s)$ for all $s \neq -\mathbb{N}$ and that $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned}\Gamma(s+1) &= \lim_{N\to\infty}\left[\frac{N^{s+1}(N-1)!}{(s+1)\cdot(s+2)\dots(s+N)}\right]\\ &= \lim_{N\to\infty}\left[s\cdot\frac{N}{N+s}\cdot\frac{N^s(N-1)!}{s(s+1)\cdot(s+2)\dots(s+N-1)}\right]\\ &= \lim_{N\to\infty}\left[s\cdot\frac{N}{N+s}\cdot\Gamma_N(s)\right] = s\Gamma(s).\end{aligned}$$

To prove the second property we use induction. We have that $\Gamma(1) = 1$ by the previous formula. So the statement holds for all n, it remains to prove it also for $n+1$. We have using the first property and the induction step that

$$\Gamma(n+1) = n\Gamma(n) = n\cdot(n-1)! = n!.$$

□

2. Weierstrass representation of the Gamma function

We can now write the following

$$\Gamma(s+1) = s\Gamma(s) = \lim_{N\to\infty}\left[\frac{N^s}{(1+\frac{s}{1})\cdot(1+\frac{s}{2})\dots(1+\frac{s}{N-1})}\right]. \tag{2.1}$$

Then we take the logarithm and we find for $|s| < 1$ that

$$\begin{aligned}\log\Gamma(s+1) &= \lim_{N\to\infty}\left[s\log N - \sum_{n=1}^{N-1}\log\left(1+\frac{s}{n}\right)\right]\\ &= \lim_{N\to\infty}\left[s\log N - \sum_{n=1}^{N-1}\left(\frac{s}{n}-\frac{s^2}{2n^2}+\frac{s^3}{3n^2}-\dots\right)\right]\\ &= \lim_{N\to\infty}\left[s\left(\log N - \left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{N-1}\right)\right)\right.\\ &+ \frac{s^2}{2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{(N-1)^2}\right)\\ &\left.- \frac{s^3}{3}\left(\frac{1}{1^3}+\frac{1}{2^3}+\dots+\frac{1}{(N-1)^3}\right)+\dots\right].\end{aligned}$$

Definition 4. *The Euler's constant γ is defined as*

$$\gamma := \lim_{N\to\infty} 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{N-1}-\log N.$$

Moreover the series $1+\frac{1}{2^r}+\frac{1}{3^r}+\dots+\frac{1}{(N-1)^r}$ with $r \geqslant 2$, goes to the limit $\sum_{n=1}^{\infty}\frac{1}{n^r} = \zeta(r)$, where we recall that $\zeta(r)$ represents the Riemann zeta function. Exchanging the limit with the sum (using dominated convergence) we obtain

$$\log\Gamma(1+s) = -\gamma s + \frac{\zeta(2)}{2}s^2 - \frac{\zeta(3)}{3}s^3+\dots, \qquad |s|<1.$$

Where the Riemann zeta function evaluated at integers represents the coefficients of the Taylor series of $\log\Gamma(s+1)$, where the expansion is made at the point $s = 1$. We can deduce another product formula for $\Gamma(s)$. Using the following property

$$1+\frac{1}{2}+\dots+\frac{1}{N-1} = \log(N)+\gamma+o(1),$$

we can write

$$N^s = e^{s\log N} = e^{s(1+\frac{1}{2}+\dots+\frac{1}{N-1}-\gamma-o(1))} \sim e^{-\gamma s}e^{s(1+\frac{1}{2}+\dots+\frac{1}{N-1})}.$$

Inserting this last formula in (2.1) we obtain

$$\begin{aligned}\Gamma(s) &= \lim_{N\to\infty}\left[\frac{N^s}{s\cdot(1+\frac{s}{1})\cdot(1+\frac{s}{2})\dots(1+\frac{s}{N-1})}\right] = \frac{e^{-\gamma s}}{s}\lim_{N\to\infty}\left[\frac{e^{s(1+\frac{1}{2}+\dots+\frac{1}{N-1})}}{(1+\frac{s}{1})\cdot(1+\frac{s}{2})\dots(1+\frac{s}{N-1})}\right]\\ &= \frac{1}{se^{\gamma s}}\lim_{N\to\infty}\prod_{n=1}^{N-1}\left[\frac{e^{\frac{s}{n}}}{(1+\frac{s}{n})}\right]\end{aligned}$$

Definition 5. *The function $\frac{1}{\Gamma(s)}$ defined as an infinite product due to Weierstrass*

$$\frac{1}{\Gamma(s)} = se^{\gamma s}\prod_{n=1}^{\infty}\left[\left(1+\frac{s}{n}\right)e^{\frac{-s}{n}}\right]. \tag{2.2}$$

From this representation we can see that $\frac{1}{\Gamma(s)}$ is defined in the whole complex plane and that this function is holomorphic, since the product formula converge. Further we can see that $\frac{1}{\Gamma(s)}$ has simple zeros at the points $s = 0, -1, -2, \dots$ and otherwise is always different from zero.

Proposition 6. *The Gamma function $\Gamma(s)$ as seen in (1.3), (1.4) represents a meromorphic function and is defined on the entire complex plane. The Gamma function has simple poles at $s = 0, -1, -2, \dots$ and is otherwise holomorphic. Moreover, the Gamma function is never equal to 0, i.e the function $\frac{1}{\Gamma(s)}$ is always holomorphic.*

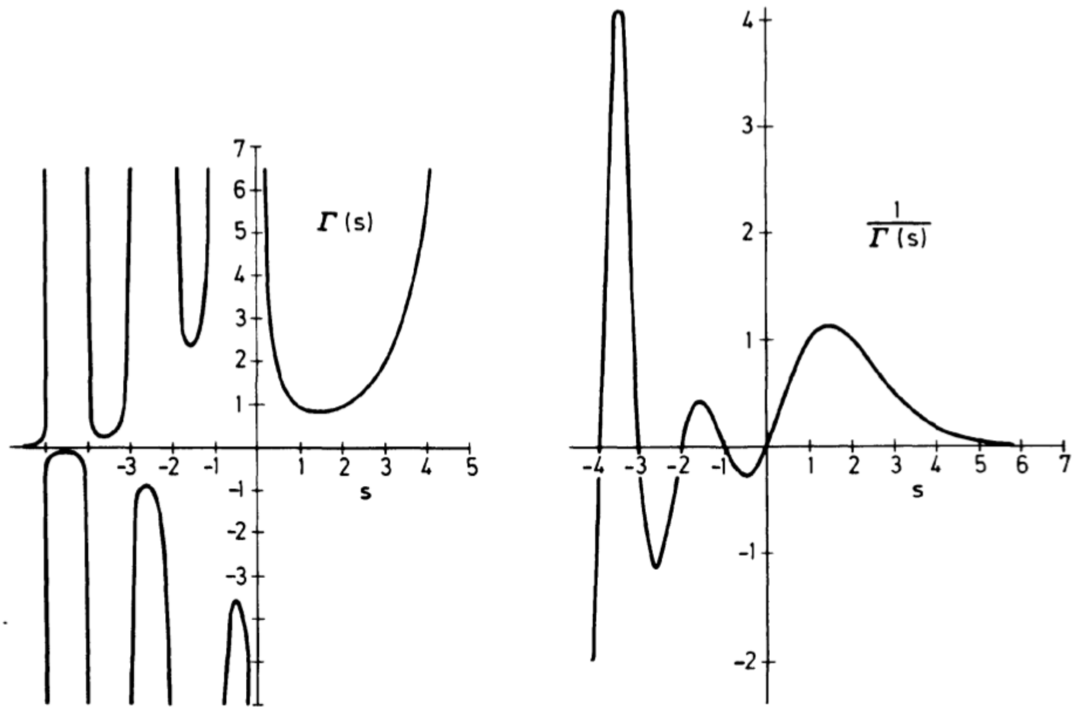


FIGURE 1. Graph of $\Gamma(s)$ and of $\frac{1}{\Gamma(s)}$

3. Three new functions defined using the Gamma function

3.1. **Example 1.** We define $f(s) = \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)$. Then we have

$$f(s+1) = \Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s}{2}+1\right) = \frac{s}{2}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s}{2}\right) = \frac{s}{2}f(s),$$

and

$$2^{s+1}f(s+1) = s \cdot 2^s f(s).$$

Proposition 7. *We have $2^s f(s) = C \cdot \Gamma(s)$ where C is a constant.*

Proof.

$$\begin{aligned} 2^s \cdot \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) &= \lim_{N\to\infty} 2^s \left[\frac{N^{\frac{s}{2}}(N-1)!}{\frac{s}{2}\left(\frac{s}{2}+1\right)\dots\left(\frac{s}{2}+N-1\right)} \cdot \frac{N^{\frac{s+1}{2}}(N-1)!}{\frac{s+1}{2}\left(\frac{s+1}{2}+1\right)\dots\left(\frac{s+1}{2}+N-1\right)}\right] \\ &= \lim_{N\to\infty}\left[\frac{2^{2N+s}\cdot N^{s+\frac{1}{2}}\cdot(N-1)!^2}{s(s+2)(s+4)\dots(s+2N-2)\times(s+1)(s+3)\dots(s+2N-1)}\right] \\ &= \lim_{N\to\infty}\left[2^{2N}\cdot N^{\frac{1}{2}}\cdot\frac{(N-1)!^2}{(2N-1)!}\cdot\frac{(2N)^s(2N-1)!}{s(s+1)(s+2)\dots(s+2N-1)}\right] \\ &= C\cdot\Gamma(s). \end{aligned}$$

Then we find

$$C = \lim_{N\to\infty}\left[2^{2N}N^{\frac{1}{2}}\frac{(N-1)!^2}{(2N-1)!}\right] = 2\sqrt{\pi}.$$

□

Therefore we obtain the Legendre doubling formula

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = 2^{1-s}\sqrt{\pi}\Gamma(s).$$

In particular we obtain $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ by putting $s = 1$ in the above formula.

3.2. **Example 2.** We define $g(s) = \frac{1}{\Gamma(s)\Gamma(1-s)}$. The function $g(s)$ is holomorphic, since $\frac{1}{\Gamma(s)}$ is holomorphic, and has zeros at $s \in \mathbb{Z}$, since $\frac{1}{\Gamma(s)}$ has zeros only at $s = 0, -1, -2, \dots$. Moreover we have

$$g(s+1) = \frac{1}{\Gamma(1+s)\Gamma(-s)} = \frac{1}{s\Gamma(s)\Gamma(-s)} = \frac{-1}{\Gamma(1-s)\Gamma(s)} = -g(s).$$

In particular the function $g(s)$ is a periodic function with period 2.

Proposition 8. *The function g can be written as*

$$g(s) = C \cdot \sin \pi s \tag{3.1}$$

where C is a constant. Moreover we have $C = \frac{1}{\pi}$.

Proof sketch. We have

$$\lim_{s \to 0} \left[\frac{g(s)}{s} \right] = \lim_{s \to 0} \left[\frac{1}{\Gamma(1+s)\Gamma(1-s)} \right] = 1$$

and therefore we obtain by the Taylor series that $C = \frac{1}{\pi}$. So we have

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

□

Formula (3.1) can also be obtained using (2.2) and the relation

$$\frac{\sin \pi s}{\pi s} = \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right).$$

3.3. **Example 3.** We now define a new function. Let

$$h(s) = \int_0^\infty t^{s-1} e^{-t} \, dt.$$

This integral converges for $\sigma > 0$ and using partial integration we have that

$$h(s+1) = \int_0^\infty t^s \, d(-e^{-t}) = \int_0^\infty -e^{-t} \, d(t^s) = s \int_0^\infty t^{s-1} e^{-t} \, dt = sh(s)$$

and that

$$h(1) = \int_0^\infty e^{-t} \, dt = 1.$$

The function $h(s)$ is a candidate for the original Gamma function, i.e. a function that fulfills these two properties: $\Gamma(n) = (n-1)!$ for all natural numbers and $\Gamma(s+1) = s\Gamma(s)$ for all $s \neq -\mathbb{N}$.

Proposition 9. *We have that $\Gamma(s) = h(s)$.*

This means that we can also define the Gamma function as an integral

Definition 10.

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \, dt \quad \textit{for } \sigma > 0.$$

From this formula we can explain the importance of the Gamma function for the theory of Dirichlet's series. If fact we have using $u = nt$

$$\int_0^\infty t^{s-1} e^{-nt} \, dt = n^{-s} \int_0^\infty u^{s-1} e^{-u} \, du = \Gamma(s) n^{-s}.$$

Moreover in the area of absolute convergence we have

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty \left(\sum_{n=1}^{\infty} a_n e^{-nt} \right) t^{s-1} \, dt. \tag{3.2}$$

This means that the ordinary Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and the power series $F(z) = \sum_{n=1}^{\infty} a_n z^n$, using the same coefficients, are connected to each other through the Mellin transform

$$f(s) = \frac{1}{\Gamma(s)} \int_0^\infty F(e^{-t}) t^{s-1} \, dt.$$

This allows to transform the properties of the Dirichlet series in properties of power series and vice versa. Also for general Dirichlet series we found an analog property

$$\sum_{n=1}^{\infty} a_n \lambda_n{}^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \left(\sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \right) t^{s-1} \, dt.$$

We make now an example for the use of (3.2). We have $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, i.e $a_n = 1$. Then we find $\sum_{n=1}^{\infty} a_n e^{-nt} = \frac{1}{e^t - 1}$ and we obtain

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} \, dt, \quad \sigma > 1.$$

Take the function $\psi(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} = (1 - 2^{1-s} \zeta(s))$. We have $a_n = (-1)^{n-1}$ and therefore $\sum_{n=1}^{\infty} a_n e^{-nt} = \frac{1}{e^t + 1}$. We find out

$$1 - 2^{1-s} \zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t + 1} \, dt, \quad \sigma > 0.$$

References

[1] D. Zagier, Zetafunktionen und quadratische Körper, Springer, Berlin, 1981

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