Gamma Function

Sabrina Galfetti

ETH Zürich

12 October 2021

Table of contents

Interpolation of the factorial function

The Gamma function

- One of the most important mathematical functions, which is a relatively easy but at the same time a non trivial function.
- Important role for the study of Dirichlet series.

First step

Find the interpolation function of the factorial function

$$f : n \mapsto n! \tag{1}$$

i.e a continuous function $\Pi(x)$ such that $\Pi(n) = n!$ for all natural numbers n.

Equivalently we may find a function $\Gamma(s) = \Pi(s-1)$, i.e. a continuous function $\Gamma(s)$ such that $\Gamma(n) = (n-1)!$ for all natural numbers.

Factorial property

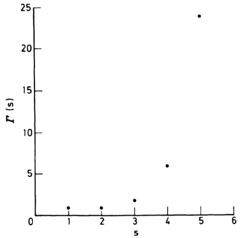
The Gamma function must also fulfill:

$$\Gamma(s+1) = s \cdot \Gamma(s) \quad \forall s \neq 0. \tag{2}$$

This because the factorial function fulfill $n! = n \cdot (n-1)!$.

First problem of interpolation

For small s is not clear how to interpolate.



Instead for bigger numbers s the function f is more uniform continuous, and for this reason it must be easier to find an interpolation.

Repeating (2) more and more times we obtain

$$\Gamma(s+N) = s \cdot (s+1) \dots (s+N+1) \cdot \Gamma(s).$$

And from this we can set

$$\Gamma(s) = \lim_{N \to \infty} \frac{\Gamma(s+N)}{s(s+1)\dots(s+N-1)}.$$

For a natural number $s \in \mathbb{N}$ we have

$$\begin{aligned}\Gamma(N+s) = (N+s-1)! &= (N+s-1)\cdot(N+s-2)\dots(N+1)\cdot N\cdot(N-1)!\\ &= N^s\cdot\left(1+\frac{s-1}{N}\right)\cdot\left(1+\frac{s-2}{N}\right)\dots\left(1+\frac{1}{N}\right)\cdot(N-1)!.\end{aligned}$$

Therefore we obtain $\Gamma(s+N) \sim N^s \cdot (N-1)!$ for $N \to \infty$.

First definition of the Gamma function

$$\Gamma(s) = \lim_{N \to \infty} \frac{N^s (N-1)!}{s(s+1)\dots(s+N-1)}, \quad s \notin -\mathbb{N}. \tag{3}$$

Definition of Γ_N

For the case $s \notin -\mathbb{N}$ and $N \in \mathbb{N}$ we have

$$\Gamma_N(s) = \frac{N^s (N-1)!}{s(s+1)\dots(s+N-1)}.$$

Proof

It rest to prove that the limit $\lim_{N\to\infty}\Gamma_N(s)$ exists. We have

$$\frac{\Gamma_{N+1}(s)}{\Gamma_N(s)} = \left(\frac{N+1}{N}\right)^s \cdot \frac{N}{s+N} = \left(1+\frac{1}{N}\right)^s \cdot \left(1+\frac{s}{N}\right)^{-1}$$
$$= \left(1+\frac{s}{N}+o\left(\frac{1}{N^2}\right)\right)\cdot\left(1-\frac{s}{N}+o\left(\frac{1}{N^2}\right)\right) = \left(1+o\left(\frac{1}{N^2}\right)\right).$$

Existence of the limit

This proves that

$$\prod_{N\geq 1} \frac{\Gamma_{N+1}(s)}{\Gamma_N(s)}$$

converges. We also obtain

$$\Gamma_N(s) = \Gamma_1(s) \cdot \prod_{n=1}^{N-1} \frac{\Gamma_{n+1}(s)}{\Gamma_n(s)} = \frac{1}{s} \prod_{n=1}^{N-1} \left[\left(1 + \frac{1}{n}\right)^s \cdot \left(1 + \frac{s}{n}\right)^{-1} \right].$$

Euler's definition as an infinite product

$$\Gamma(s+1) = s\Gamma(s) = \prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^{s}}{\left(1+\frac{s}{n}\right)}. \tag{4}$$

We check that (4) fulfill the desired properties

- $\Gamma(s+1) = s\Gamma(s) \ \ \forall s \neq 0$,
- $\Gamma(n) = (n-1)! \ \ \forall n \in \mathbb{N}$.

Proof

We have

$$\begin{aligned}\Gamma(s+1) &= \lim_{N\to\infty}\left[\frac{N^{s+1}(N-1)!}{(s+1)\cdot(s+2)\dots(s+N)}\right] \\ &= \lim_{N\to\infty}\left[s\cdot\frac{N}{N+s}\cdot\frac{N^{s}(N-1)!}{s(s+1)\cdot(s+2)\dots(s+N-1)}\right] \\ &= \lim_{N\to\infty}\left[s\cdot\frac{N}{N+s}\cdot\Gamma_N(s)\right] = s\Gamma(s).\end{aligned}$$

Proof

To prove the second property we use induction. We have that $\Gamma(1) = 1$ by the previous formula. So the statement holds for all n, it remains to prove it also for $n + 1$. We have using the first property and the induction step that

$$\Gamma(n+1) = n\Gamma(n) = n \cdot (n-1)! = n!.$$

Weierstrass representation of the Gamma function

We have

$$\Gamma(s+1) = s\Gamma(s) = \lim_{N\to\infty}\left[\frac{N^s}{(1+\frac{s}{1})\cdot(1+\frac{s}{2})\dots(1+\frac{s}{N-1})}\right]. \tag{5}$$

Then we take the logarithm and we find for $|s| < 1$ that

$$
\begin{aligned}
\log\Gamma(s+1) &= \lim_{N\to\infty}\left[s\log N - \sum_{n=1}^{N-1}\log\left(1+\frac{s}{n}\right)\right]\\
&= \lim_{N\to\infty}\left[s\log N - \sum_{n=1}^{N-1}\left(\frac{s}{n}-\frac{s^2}{2n^2}+\frac{s^3}{3n^2}-\dots\right)\right]\\
&= \lim_{N\to\infty}\left[s\left(\log N-\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{N-1}\right)\right)\right.\\
&+\frac{s^2}{2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{(N-1)^2}\right)\\
&\left.-\frac{s^3}{3}\left(\frac{1}{1^3}+\frac{1}{2^3}+\dots+\frac{1}{(N-1)^3}\right)+\dots\right].
\end{aligned}
$$

Euler's constant γ

- the series $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N-1} - \log N$ has a limit for $N \to \infty$.
- the series $1 + \frac{1}{2^r} + \frac{1}{3^r} + \ldots + \frac{1}{(N-1)^r}$ with $r \geq 2$, goes to the limit $\sum_{n=1}^{\infty} \frac{1}{n^r} = \zeta(r)$, where we recall that $\zeta(r)$ represents the Riemann zeta function.

Exchanging the limit with the sum we obtain

$$\log \Gamma(1+s) = -\gamma s + \frac{\zeta(2)}{2} s^2 - \frac{\zeta(3)}{3} s^3 + \dots, \qquad |s| < 1.$$

- the Riemann zeta function evaluated at integers represents the coefficients of the Taylor series of $\log \Gamma(s+1)$, where the expansion is made at the point $s = 1$.

deducing another product formula for $\Gamma(s)$

Using the following property:

$$1 + \frac{1}{2} + \ldots + \frac{1}{N-1} = \log(N) + \gamma + o(1),$$

we can write

$$N^s = e^{s \log N} = e^{s(1+\frac{1}{2}+\ldots+\frac{1}{N-1}-\gamma-o(1))} \sim e^{-\gamma s} \cdot e^{s(1+\frac{1}{2}+\ldots+\frac{1}{N-1})}.$$

Definition of the Gamma function due to Weierstrass

Inserting the previous result in (5) we obtain

$$\frac{1}{\Gamma(s)} = se^{\gamma s} \prod_{n=1}^{\infty} \left[\left(1 + \frac{s}{n} \right) e^{\frac{-s}{n}} \right]. \tag{6}$$

Properties

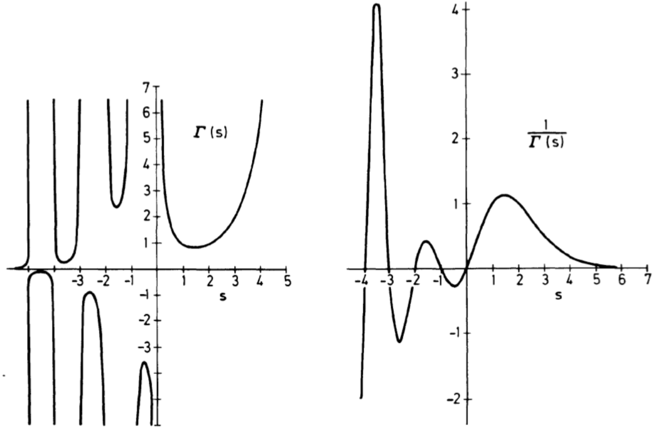
The function $\frac{1}{\Gamma(s)}$

- is defined in the whole complex plane,
- is holomorphic (since both product formula converge).
- has simple zeros at the points $s = 0, -1, -2, \ldots$ and otherwise is always different from zero.

Proposition

The Gamma function $\Gamma(s)$ as seen in (3)or (4) represents a meromorphic function and is defined on the entire complex plane. The Gamma function has simple poles by $s = 0, -1, -2, \ldots$ and otherwise is holomorphic. Moreover the Gamma function is never equal 0, i.e the function $\frac{1}{\Gamma(s)}$ is always holomorphic.

Graph of $\Gamma(s)$ and of $\frac{1}{\Gamma(s)}$



Three new functions defined using the Gamma function

Definition

We define $f(s) = \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)$. We find

1. $f(s+1) = \Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s}{2}+1\right) = \frac{s}{2}\cdot\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s}{2}\right) = \frac{s}{2}\cdot f(s)$,
2. $2^{s+1}\cdot f(s+1) = s\cdot 2^s f(s)$.

Proposition

We have $2^s f(s) = C \cdot \Gamma(s)$ where C is a constant.

Proof

$$
\begin{aligned}
2^s \cdot \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) &= \lim_{N\to\infty} 2^s \left[\frac{N^{\frac{s}{2}}(N-1)!}{\frac{s}{2}\left(\frac{s}{2}+1\right)\dots\left(\frac{s}{2}+N-1\right)} \cdot \frac{N^{\frac{s+1}{2}}(N-1)!}{\frac{s+1}{2}\left(\frac{s+1}{2}+1\right)\dots\left(\frac{s+1}{2}+N-1\right)}\right] \\
&= \lim_{N\to\infty} \left[\frac{2^{2N+s}\cdot N^{s+\frac{1}{2}}\cdot (N-1)!^2}{s(s+2)(s+4)\dots(s+2N-2)\times(s+1)(s+3)\dots(s+2N-1)}\right] \\
&= \lim_{N\to\infty} \left[2^{2N}\cdot N^{\frac{1}{2}} \cdot \frac{(N-1)!^2}{(2N-1)!} \cdot \frac{(2N)^s(2N-1)!}{s(s+1)(s+2)\dots(s+2N-1)}\right] \\
&= C\cdot\Gamma(s).
\end{aligned}
$$

Constant C

We find

$$C = \lim_{N\to\infty} \left[2^{2N} \cdot N^{\frac{1}{2}} \cdot \frac{(N-1)!^2}{(2N-1)!} \right] = 2\sqrt{\pi}.$$

Legendre doubling formula

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = 2^{1-s}\sqrt{\pi}\,\Gamma(s).$$

In particular we obtain $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Definition

We define $g(s) = \frac{1}{\Gamma(s)\Gamma(1-s)}$.

We have that $g(s)$

- is holomorphic, since the function $\frac{1}{\Gamma(s)}$ is holomorphic,
- has zeros at $s = \ldots, -2, -1, 0, 1, 2, \ldots$, since $\frac{1}{\Gamma(s)}$ has zeros only at $s = 0, -1, -2, \ldots$.

Properties of $g(s)$

We find that

$$g(s+1) = \frac{1}{\Gamma(1+s)\Gamma(-s)} = \frac{1}{s\Gamma(s)\Gamma(-s)} = \frac{-1}{\Gamma(1-s)\Gamma(s)} = -g(s).$$

So $g(s)$ is a periodic function with period 2.

Alternative definition of $g(s)$

We have

$$g(s) = C \cdot \sin \pi s,$$

where C is a constant.

Constant C

It holds

$$\lim_{s\to 0}\left[\frac{g(s)}{s}\right] = \lim_{s\to 0}\left[\frac{1}{\Gamma(1+s)\Gamma(1-s)}\right] = 1.$$

Therefore $C = \frac{1}{\pi}$.

Alternative formula for the Gamma function

So we have

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

This formula can also be obtained using (6) and the important relation

$$\frac{\sin \pi s}{s\pi} = \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right).$$

Definition

We define the function

$$h(s) = \int_0^\infty t^{s-1} e^{-t} \, dt.$$

This integral converges for $\sigma > 0$ and using partial integration we have that

$$h(s+1) = \int_0^\infty t^s \, d(-e^{-t}) = \int_0^\infty -e^{-t} \, d(t^s) = s \int_0^\infty t^{s-1} e^{-t} \, dt = sh(s)$$

and that

$$h(1) = \int_0^\infty e^{-t} \, dt = 1.$$

Alternative definition of the Gamma function

The function $h(s)$ is a candidate for the original Gamma function, i.e a function that fulfill these two properties: $\Gamma(n) = (n-1)!$ for all natural numbers and $\Gamma(s+1) = s\Gamma(s)\ \forall s \neq 0$.

In fact we have that $\Gamma(s) = h(s)$.

Definition of the Gamma function using integrals

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \, dt, \quad \text{for } \sigma > 0.$$

Importance of the Gamma function for the theory of Dirichlet's series

Using $u = nt$ we obtain

$$\int_0^\infty t^{s-1}e^{-nt}\,dt = n^{-s}\int_0^\infty u^{s-1}e^{-u}\,du = \Gamma(s)n^{-s}.$$

Moreover in the area of absolute convergence we have

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \left(\sum_{n=1}^{\infty} a_n e^{-nt} \right) t^{s-1} \, dt. \tag{7}$$

Mellin transformation

This means that the ordinary Dirichlet's series $f(s) = \sum a_n n^{-s}$ and the power series $F(z) = \sum a_n z^n$, using the same coefficients, are connect to each other through the Mellin transformation

$$f(s) = \frac{1}{\Gamma(s)} \int_0^\infty F(e^{-t}) t^{s-1} \, dt.$$

Transform properties of the Dirichlet's series in properties of power series

Also for general Dirichlet's series we found an analog property

$$\sum_{n=1}^{\infty} a_n {\lambda_n}^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \left(\sum_{n=1}^{\infty} a_n e^{-\lambda_n t} \right) t^{s-1} \, dt.$$

Example

We have $\zeta(s) = \sum n^{-s}$, i.e $a_n = 1$. Then we find $\sum a_n e^{-nt} = \frac{1}{e^t - 1}$ and we obtain

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1}\, dt \quad \sigma > 1.$$

Take the function $\psi(s) = \sum (-1)^{n-1} n^{-s} = (1 - 2^{1-s}\zeta(s))$. We have $a_n = (-1)^{n-1}$ and therefore $\sum a_n e^{-nt} = \frac{1}{e^t + 1}$. We find out

$$1 - 2^{1-s}\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t + 1}\, dt \quad \sigma > 0.$$