SEMINAR

INTRODUCTION TO THE RIEMANN ZETA FUNCTION

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1 Introduction and preliminaries results

In the following we want to analyse the Riemann Zeta Function, which in the category of Dirichlet Series is the easiest and most important one.
Let s be a complex number and $\sigma,\ t$ respectively its real and imaginary parts. Then we define for $\sigma > 1$ the ζ-function as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\sigma > 1). \tag{1}$$

There are also others representations of the zeta function, which, depending on the situation, can be very useful. Two of them are the well known Euler product and the Integral representation

$$\zeta(s) = \prod_{p\,prime} \frac{1}{1-p^{-s}} \tag{2}$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt \tag{3}$$

where both formulas hold, as in (1), for $\sigma > 1$.
We want now to see some properties of the zeta function and in particular to find out what are the values that ζ attains at the natural numbers.

1.1 Continuation of the zeta function for positive real part

We want to use the formula for computing the abscissa of convergence of a Dirichlet series , hence

$$\sigma_0 = \limsup_{n\to\infty} \frac{\log\left(|\sum_{k=1}^{n} a_k|\right)}{\lambda_n} \qquad \text{for } \sum a_n e^{-\lambda_n s} \tag{4}$$

where we take the series $\psi(s) = (1 - \frac{1}{2^s} + \frac{1}{3^s} - \cdots)$. Because for ψ we get $\sigma_0 = 0$ and because of

$$\psi(s) = \zeta(s) - 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \cdots\right) = (1 - 2^{1-s}) \cdot \zeta(s) \tag{5}$$

we can conclude that ζ can be extended as meromorphic function to any s such that $\sigma = Re(s) > 0$ and that the only points where ζ can have a singularity (poles for instance) are the zeros of $(1 - 2^{1-s})$, hence

$$s \in \{1, 1 \pm \frac{2\pi i}{log(2)}, 1 \pm \frac{4\pi i}{log(2)} + \cdots\} \tag{6}$$

By slightly changing ψ to $\tilde{\psi} = (1 + \frac{1}{2^s} - \frac{2}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} - \frac{2}{6^s} - \cdots)$ and noticing that this series is equal to $(1 - 3^{1-s}) \cdot \zeta(s)$, we can conclude, using a similar analysis as before, that the poles of ζ (with positive real part) have also to lie in the set

$$\{1, 1 \pm \frac{2\pi i}{log(3)}, 1 \pm \frac{4\pi i}{log(3)} + \cdots\} \tag{7}$$

Moreover one can notice that the intersection of the sets (6) and (7) contains only the point $s = 1$, because of $\frac{\log(3)}{\log(2)} = \log_2(3) = \{x | 2^x = 3\} \notin \mathbb{Q}$.

We will see later that actually ζ is holomorphic in the whole complex plane except to the point $s = 1$.

1.2 Bernoulli Numbers

Before coming to the main theorem our analysis we introduce the *Bernoulli numbers* B_n. These are defined as the real coefficients such that:

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k, \qquad \text{where } (|t| < 2\pi). \tag{8}$$

Expanding the left hand side by a Taylor expansion of the exponential we can compute (at least) the first few coefficients by taking the inverse of the power series in the denominator

$$\frac{t}{e^t-1} = \frac{t}{t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\dots} = 1-\frac{t}{2}+\frac{t^2}{12}+0\cdot t^3-\frac{t^4}{720}+\dots \tag{9}$$

Finding therefore, by comparing (8) and (9)

$$B_0=1,\quad B_1=-\frac{1}{2},\quad B_2=\frac{1}{6},\quad B_3=0,\quad B_4=-\frac{1}{30},\quad \dots \tag{10}$$

More in general, observing the Taylor coefficients (see (8)) of the equation

$$\frac{t}{e^t-1}-\frac{-t}{e^{-t}-1} = \frac{te^{-t}-t+te^t-t}{e^te^{-t}-e^t-e^-t+1} = -t \tag{11}$$

we see that it must holds

$$\left(\frac{B_k}{k!}-(-1)^k\frac{B_k}{k!}\right)t^k = 0\cdot t^k \qquad \text{for } k\geq 2 \tag{12}$$

hence we conclude that for any odd $k \geq 3$ it holds $B_k = 0$.
Note that for $k=1$ the relation (11) leads to $B_1 = -\frac{1}{2}$.

We have thus the simple but very useful relation

$$(-1)^n B_n = B_n, \qquad n\geq 2 \tag{13}$$

One last relation which is very interesting is the following recursive formula

Lemma 1. *For the Bernoulli numbers B_n it holds*

$$\sum_{r=0}^{n}\binom{n}{r}B_r = (-1)^n B_n \tag{14}$$

Proof. We can obtain the claimed result by using the generating series (8)

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r}B_r\right)\frac{t^n}{n!} &= \sum_{0\leq r\leq n}\frac{B_r t^n}{r!(n-r)!} = \sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{B_r t^{r+k}}{r!k!} = \left(\sum_{r=0}^{\infty}\frac{B_r t^r}{r!}\right)\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}\right) = \\ &= \frac{t}{e^t-1}\cdot e^t = \frac{-t}{e^{-t}-1} = \sum_{n=0}^{\infty}(-1)^n B_n\frac{t^n}{n!}\end{aligned} \tag{15}$$

where we used the index change $n \to (k=n-r)$ and the fact that $\mathbb{1}_{\{r\leq n\}} = \mathbb{1}_{\{r\leq k+r\}} = \mathbb{1}_{\{0\leq k\}}$ □

2 Basic properties of the zeta function

We can now formulate and prove the following theorem:

Theorem 1. *For the function defined (for $\sigma > 1$) by (1) there exist a meromorphic continuation in the whole complex plane, with just one unique pole. This appears at the point s=1 and is a simple pole with residue 1. Moreover the values of ζ at the non-positive Integers are rational and it holds:*

- $\zeta(0) = -\frac{1}{2}$
- $\zeta(-2n) = 0$
- $\zeta(1-2n) = -\frac{B_{2n}}{2n}$
- $\zeta(2n) = \frac{(-1)^{n-1}2^{2n-1}\pi^{2n}B_{2n}}{(2n)!}$

where $n \in \mathbb{Z}_{\geq 1}$ and B_n are the Bernoulli Numbers introduced in the chapter 1.

Proof. We divide the proof in three parts

- Part 1, meromorphic continuation and poles

For computing $\Gamma(s)\zeta(s)$ we first fix $n > 0$ and define the function

$$f_n(t) = \sum_{k=0}^{n} (-1)^k \frac{B_k}{k!} t^k \tag{16}$$

Then we divide $\Gamma \cdot \zeta$ in two integrals that we analyse separately

$$\begin{aligned}
\Gamma(s)\zeta(s) &= \int_0^\infty \frac{te^t}{e^t - 1} e^{-t} t^{s-2} dt \qquad (17)\\
&= \int_0^\infty \left(\frac{te^t}{e^t-1} - f_n(t)\right) e^{-t} t^{s-2} dt + \int_0^\infty f_n(t) e^{-t} t^{s-2} dt\\
&= I_1(s) + I_2(s) \qquad (18)
\end{aligned}$$

For the first integral we note that $\left(\frac{te^t}{e^t-1} - f_n(t)\right)$ simplifies to the sum

$$\sum_{k=n+1}^{\infty} (-1)^k \frac{B_k}{k!} t^k = O(t^{n+1}) \tag{19}$$

because

$$\frac{te^t}{e^t-1} = \frac{-t}{e^{-t}-1} = \sum_{k=0}^{\infty} (-1)^k \frac{B_k}{k!} t^k \tag{20}$$

by (8).

We then get an integral of the form

$$\int_0^\infty e^{-t} O(t^{n+s-1}) dt \tag{21}$$

which converges for $t \to \infty$ because of e^{-t} and for $t \to 0$ if $\sigma > -n$ (by integrability of $\frac{1}{t^{1-\epsilon}}$ on $[0,1]$ for $\epsilon > 0$).
The second integral is, by definition of f_n equal to

$$\int_0^\infty \left(1 + \frac{t}{2} + \sum_{k=2}^{n} \frac{B_k}{k!} t^k\right) e^{-t} t^{s-2} dt = \int_0^\infty e^{-t} t^{s-2} dt + \frac{1}{2}\int_0^\infty e^{-t} t^{s-1} dt + \sum_{k=2}^{n} \frac{B_k}{k!} \int_0^\infty e^{-t} t^{s+k-2} dt \tag{22}$$

$$= \Gamma(s-1) + \frac{1}{2}\Gamma(s) + \sum_{k=2}^{n} \frac{B_k}{k!} \Gamma(s+k-1) \tag{23}$$

That means that I_2 has poles at most, in $(\mathbb{Z}_{\leq 1}) \cup (\mathbb{Z}_{\leq 0}) \cup (\mathbb{Z}_{\leq n+1})$ and is therefore a meromorphic function.
Therefore by (18) we get, using the functional equation for Γ:

$$\begin{aligned}
\zeta(s) &= \frac{I_2(s)}{\Gamma(s)} + \frac{I_1(s)}{\Gamma(s)} \qquad (24)\\
&= \frac{\Gamma(s-1)}{\Gamma(s)} + \frac{1}{2} + \sum_{k=2}^{n} \frac{B_k}{k!} \frac{\Gamma(s+k-1)}{\Gamma(s)} + \frac{I_1(s)}{\Gamma(s)} \qquad (25)\\
&= \frac{1}{s-1} + \frac{1}{2} + \sum_{k=2}^{n} \frac{B_k}{k!} (s)(s+1)\cdots(s+k-2) + \frac{I_1(s)}{\Gamma(s)} \qquad (26)
\end{aligned}$$

where, knowing that $\frac{1}{\Gamma(s)}$ is an holomorphic function and $I_1(s)$ is holomorphic for $\sigma > -n$, we can deduce that

$$\zeta(s) - \frac{1}{s-1} = \frac{1}{2} + \sum_{k=2}^{n} \frac{B_k}{k!} (s)(s+1)\cdots(s+k-2) + \frac{I_1(s)}{\Gamma(s)} \tag{27}$$

is holomorphic for $\sigma > -n$. But, because n was arbitrary, we can also conclude that $\zeta(s) - \frac{1}{s-1}$ is holomorphic in the complex plane. By taking $s = 1$ in (26) one sees moreover that the (unique) pole of ζ is exactly the one claimed.

- Part 2, values of ζ for negative integers

  We choose $s = -k \in (\mathbb{Z}_{\leq 0})$ where we fix n such that $-n < (-k) \leq 0$. Using the fact that Γ has poles at the points $s \in \{0, -1, -2, ...\}$ (thus in this cases $\frac{1}{\Gamma(s)}$ is equal to 0) and that $I_1(s)$ is holomorphic, we get, by (27):

$$\begin{aligned}
\zeta(-k) &= \frac{1}{-k-1} + \frac{1}{2} + \sum_{r=2}^{n} \frac{B_r}{r!}(-k)(-k+1)\cdots(-k+r-2) && (28)\\
&= -\frac{1}{k+1} + \frac{1}{2} + \sum_{r=2}^{k+1} (-1)^{r-1} \frac{B_r}{r!} \frac{k!}{(k+1-r)!} && (29)\\
&= -\frac{1}{k+1} + \frac{1}{2} + \sum_{r=2}^{k+1} \frac{-B_r}{r!} \frac{1}{k+1} \frac{(k+1)!}{(k+1-r)!r!} && (30)\\
&= -\frac{1}{k+1} \sum_{r=0}^{k+1} \binom{k+1}{r} B_r && (31)\\
&= -\frac{B_{k+1}}{k+1} && (32)
\end{aligned}$$

  using in line (30) the property (13) and in the last 2 lines the properties (10) and (14) respectively. Therefore, because of B_k vanishing for all odd integers k, we get the desired result and the second part of the proof is complete.

- Part 3, values of ζ for positive odd integers

  We now look for $\zeta(2n)$, $n \in \mathbb{N}$.
  For the next computation we will use two useful properties of the Γ-function

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \tag{33}$$

$$\log(\Gamma(1+s)) = -\gamma s + \frac{\zeta(2)}{2}s^2 - \frac{\zeta(3)}{3}s^3 + \cdots \qquad |s| < 1, \tag{34}$$

  and the facts that

$$\frac{x}{\tan(x)} = ix\frac{\cos(x) + i\cdot\sin(x) + \cos(x) - i\cdot\sin(x)}{\cos(x) + i\cdot\sin(x) - \cos(x) + i\cdot\sin(x)} = ix\frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}}, \tag{35}$$

$$\frac{s}{2}\frac{d}{ds}\log\left(\frac{\pi s}{\sin(\pi s)}\right) = \frac{s}{2}\frac{\sin(\pi s)}{(\pi s)}\left(\frac{\pi \sin(\pi s) - \pi^2 s\cdot\cos(\pi s)}{\sin^2(\pi s)}\right) = \frac{1}{2}\left(1 - \frac{\pi s}{\tan(\pi s)}\right). \tag{36}$$

  Then if we take the series (for $|s| < 1$)

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n-1}\pi^{2n}B_{2n}}{(2n)!}s^{2n} &= -\frac{1}{2}\left(\frac{2\pi is}{e^{2\pi is}-1} - 1 + \frac{2\pi is}{2}\right) && (37)\\
&= \frac{1}{2} - \frac{\pi is}{2}\frac{e^{\pi is} + e^{-\pi is}}{e^{\pi is} - e^{-\pi is}} && (38)\\
&= \frac{1}{2}\left(1 - \frac{\pi s}{\tan(\pi s)}\right) && (39)\\
&= \frac{s}{2}\frac{d}{ds}\log\left(\frac{\pi s}{\sin(\pi s)}\right) && (40)\\
&= \frac{s}{2}\frac{d}{ds}\log(\Gamma(1+s)\Gamma(1-s)) && (41)\\
&= \frac{s}{2}\frac{d}{ds}\big(\log(\Gamma(1+s)) + \log(\Gamma(1-s))\big) && (42)\\
&= \frac{s}{2}\frac{d}{ds}\left(\zeta(2)s^2 + \frac{\zeta(4)}{2}s^4 + \cdots\right) && (43)\\
&= \sum_{n=1}^{\infty} \zeta(2n)\cdot s^{2n} && (44)
\end{aligned}$$

Where we used, in the order, the properties (8) (with $t = 2\pi i s$ and subtracting the term for $n = 0$), (35), (36), (33), (34). Therefore the proof of the third part, and consequently of the whole theorem, is concluded.

□

Looking the results of Theorem 1 one notice that the values of $\zeta(2n)$ and $\zeta(1-2n)$ are similar, in the sense that both contains the same Bernoulli numbers. Therefore it make sense to look for a relation between $\zeta(s)$ and $\zeta(1-s)$. In this sense, starting from

$$\zeta(1-2n) = -\frac{B_{2n}}{2n} \qquad \text{and} \qquad \zeta(2n) = \frac{(-1)^{n-1} 2^{2n-1} \pi^{2n} B_{2n}}{(2n)!} \tag{45}$$

we get, by $\zeta(-2\mathbb{N}) = 0$

$$\frac{2^{k-1}\pi^k}{(k-1)!}\zeta(1-k) = \begin{cases} (-1)^{\frac{k}{2}}\zeta(k) & k > 0, \text{even} \\ 0 & k > 1, \text{odd} \end{cases} \tag{46}$$

To interpolate everything we take, as a natural choice, for $k \to (k-1)!$ the Γ-function and for

$$k \to \begin{cases} (-1)^{\frac{k}{2}} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

the function $\cos(\frac{\pi k}{2})$. That leads to the relation

$$\frac{2^{s-1}\pi^s}{\Gamma(s)} \cdot \zeta(1-s) = \cos(\frac{\pi s}{2}) \cdot \zeta(s) \tag{47}$$

Obviously this derivation of (47) does not give a proof. A formal and complete proof takes a much more complex work (which we are not going to do).

3 Functional equation and zeros of the zeta functions

To analyse the zeros of ζ it's very useful to rewrite (47) in a symmetric form. To do that we need (33) and

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = 2^{1-s}\sqrt{\pi} \cdot \Gamma(s) \tag{48}$$

then by $\cos(\frac{\pi s}{2}) = \sin(\frac{\pi s}{2} + \frac{\pi}{2}) = \sin(\pi \frac{s+1}{2})$ we get

$$\frac{2^{s-1}\pi^s}{\Gamma(s)} \cdot \zeta(1-s) = \sin(\pi\frac{s+1}{2}) \cdot \zeta(s)$$

$$\frac{\pi}{\sin(\pi\frac{s+1}{2})} \cdot \frac{2^{s-1}\pi^{s-1}}{\Gamma(s)} \cdot \zeta(1-s) = \zeta(s)$$

$$2^{s-1}\pi^{s-1}\frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s+1}{2}\right)}{\Gamma(s)} \cdot \zeta(1-s) = \zeta(s)$$

$$\pi^{s-\frac{1}{2}} \cdot \Gamma\left(\frac{1-s}{2}\right) \cdot \zeta(1-s) = \zeta(s) \cdot \Gamma\left(\frac{s}{2}\right)$$

$$\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right) \cdot \zeta(1-s) = \pi^{-\frac{s}{2}} \cdot \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s)$$

Let's now analyse this last equation for different sigmas.

- $\sigma > 1$
  Then the right hand side is finite (beacuse both Γ and ζ have no poles in this part of the complex plane) and different from zero because of ζ being equal (1) and Γ unequal zero.
- $\sigma < 0$
  In this case is the left hand side finite and unequal zero and therefore the zeros of $\zeta(s)$ have to be exactly where the poles of $\Gamma\left(\frac{s}{2}\right)$ are, hence in $\{-2, -4, -6, \cdots\}$.

Therefore to summarize $\zeta(s)$ has a pole only at $s = 1$, zeros at $s \in \{-2, -4, -6, \cdots\}$ and, possibly, more zeros in the region $0 < \sigma < 1$.

In the past years, thanks to computers and new software, mathematicians were able to compute more and more zeros of the ζ function, finding out that at least the first 150 000 000 lies of the critical line $\frac{1}{2} + it$ (it was also proven that on this line the zeta function has infinite zeros). However this is obviously still not a sufficient proof of the well known *Riemann hypothesis*, which claims that all non trivial zeros of ζ fulfills $\sigma = \frac{1}{2}$.

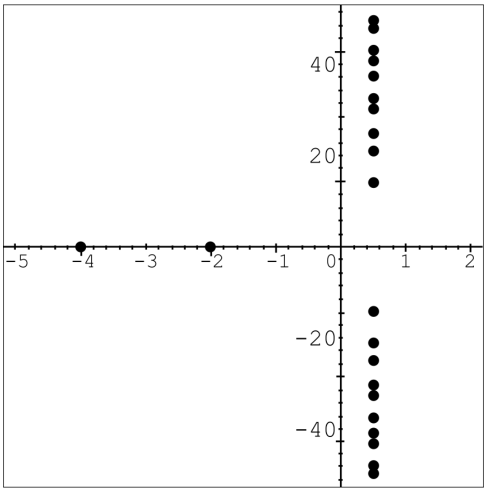

Figure 1: *zeros of* ζ