The Riemann zeta function

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Outline

Introduction and alternative representations

Let s be a complex number and σ, t respectively his real and imaginary parts. Then we define for $\sigma > 1$ the ζ-function as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\sigma > 1) \tag{1}$$

having that it holds

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt \tag{2}$$

Analytic continuation for positive real part

We use the abscissa of convergence formula for Dirichlet series:

$$\sigma_0 = \limsup_{n\to\infty} \frac{\log\left(\left| \sum_{k=1}^{n} a_k \right|\right)}{\lambda_n} \qquad \text{for } \sum a_n e^{-\lambda_n s}$$

for $\psi(s) = (1 - \frac{1}{2^s} + \frac{1}{3^s} - \cdots)$, where we get as result $\sigma_0 = 0$.

Therefore because of

$$\psi(s) = \zeta(s) - 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \cdots\right) = (1 - 2^{1-s}) \cdot \zeta(s)$$

we can conclude that ζ can be extended as meromorphic function to any s such that $\sigma = Re(s) > 0$

Introduction and alternative representations

The only points where ζ can have a singularity (poles for instance) are the zeros of $(1 - 2^{1-s})$ hence

$$s \in \{1, 1 \pm \frac{2\pi i}{\log(2)}, 1 \pm \frac{4\pi i}{\log(2)} + \cdots\}$$

By slighlty changing ψ to $\tilde{\psi} = 1 + \frac{1}{2^s} - \frac{2}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} - \frac{2}{6^s} - \cdots$ and noticing

$$\tilde{\psi}(s) = (1 - 3^{1-s}) \cdot \zeta(s)$$

we conclude that $s = 1$ is the only possible pole (with positive real part).

Bernoulli numbers

The Bernoulli numbers are defined as the real coefficients such that:

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k, \qquad \text{where } (|t| < 2\pi). \tag{3}$$

Taking the inverse of the power series in the denominator

$$\frac{t}{e^t - 1} = \frac{t}{t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots} = 1 - \frac{t}{2} + \frac{t^2}{12} + 0 \cdot t^3 - \frac{t^4}{720} + \dots \tag{4}$$

one can compute recursively each B_n Therefore

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad \dots \tag{5}$$

Bernoulli numbers

By the Taylor coefficients of the equation

$$\frac{t}{e^t - 1} - \frac{-t}{e^{-t} - 1} = \frac{te^{-t} - t + te^t - t}{e^t e^{-t} - e^t - e^{-t} + 1} = -t$$

we see that for any odd $k \geq 3$ it holds $B_k = 0$.

We have thus the simple but very useful relation

$$(-1)^n B_n = B_n, \qquad n \geq 2 \tag{6}$$

One last relation which is very interesting is the recursive formula

$$\sum_{r=0}^{n} \binom{n}{r} B_r = (-1)^n B_n \tag{7}$$

Bernoulli numbers

We can obtain this result by using the generating series (3)

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{r=0}^{n}\binom{n}{r}B_r\right)\frac{t^n}{n!} &= \sum_{0\leq r\leq n}\frac{B_r t^n}{r!(n-r)!}\\ &= \sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{B_r t^{r+k}}{r!k!}\\ &= \left(\sum_{r=0}^{\infty}\frac{B_r t^r}{r!}\right)\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}\right)\\ &= \frac{t}{e^t-1}\cdot e^t = \frac{-t}{e^{-t}-1} = \sum_{n=0}^{\infty}(-1)^n B_n\frac{t^n}{n!}\end{aligned}$$

Properites of ζ

Theorem

For the function defined (for $\sigma > 1$) by $\sum_{n=1}^{\infty} \frac{1}{n^s}$, there exist a meromorphic continuation in the whole complex plane, with just one unique pole. This appears at the point s=1 and is a simple pole with residum 1. Moreover the values of ζ at the non-postive Integers are rational and it holds:

- $\zeta(0) = -\frac{1}{2}$
- $\zeta(-2n) = 0$
- $\zeta(1-2n) = -\frac{B_{2n}}{2n}$
- $\zeta(2n) = \frac{(-1)^{n-1}2^{2n-1}\pi^{2n}B_{2n}}{(2n)!}$

where $n \in \mathbb{Z}_{\geq 1}$ and B_n are the Bernoulli numbers introduced at the beginning.

Proof, part1, meromorphic continuation and poles

Proof:

For computing $\Gamma(s)\zeta(s)$ we first fix $n > 0$ and define the function

$$f_n(t) = \sum_{k=0}^{n} (-1)^k \frac{B_k}{k!} t^k \tag{8}$$

Then we divide $\Gamma \cdot \zeta$ in two integrals that we analyse separately

$$\begin{aligned}\Gamma(s)\zeta(s) &= \int_0^\infty \frac{te^t}{e^t - 1} e^{-t} t^{s-2} dt \\ &= \int_0^\infty \left(\frac{te^t}{e^t - 1} - f_n(t) \right) e^{-t} t^{s-2} dt + \int_0^\infty f_n(t) e^{-t} t^{s-2} dt \\ &= I_1(s) + I_2(s) \end{aligned} \tag{9}$$

Proof, part1, meromorphic continuation and poles

Proof:

For the first integral we note that the factor multiplied by $e^{-t}t^{s-2}$ semplifies to the sum

$$\sum_{k=n+1}^{\infty}(-1)^k\frac{B_k}{k!}t^k = O(t^{n+1}) \tag{10}$$

because

$$\tfrac{te^t}{e^t-1} = \tfrac{-t}{e^{-t}-1} = \textstyle\sum_{k=0}^{\infty}(-1)^k\tfrac{B_k}{k!}t^k \tag{11}$$

by the taylor expantion of $\frac{t}{e^t-1}$. We then get an integral of the form

$$\int_0^{\infty} e^{-t}O(t^{n+s-1})dt \tag{12}$$

which converges for $t \to \infty$ because of e^{-t} and for $t \to 0$ if $\sigma > -n$ (by integrability of $\frac{1}{t^{1-\epsilon}}$ on $[0,1]$ for $\epsilon > 0$).

Proof, part1, meromorphic continuation and poles

Proof:

The second integral is, by definition of f_n equal to

$$\int_0^\infty \left(1+\frac{t}{2}+\sum_{k=2}^{n}\frac{B_k}{k!}t^k\right)e^{-t}t^{s-2}dt =$$

$$\int_0^\infty e^{-t}t^{s-2}dt+\frac{1}{2}\int_0^\infty e^{-t}t^{s-1}dt+\sum_{k=2}^{n}\frac{B_k}{k!}\int_0^\infty e^{-t}t^{s+k-2}dt =$$

$$\Gamma(s-1)+\frac{1}{2}\Gamma(s)+\sum_{k=2}^{n}\frac{B_k}{k!}\Gamma(s+k-1) \tag{13}$$

That means that I_2 has poles at most, in $(\mathbb{Z}_{\leq 1})\cup(\mathbb{Z}_{\leq 0})\cup(\mathbb{Z}_{\leq n+1})$ and is therefore a meromorphic function.

Proof, part1 meromorphic continuation and poles

Proof:

Therefore by $\Gamma \cdot \zeta = I_1 + I_2$, we get, using the functional equation for Γ:

$$\begin{aligned}
\zeta(s) &= \frac{I_2(s)}{\Gamma(s)} + \frac{I_1(s)}{\Gamma(s)} \\
&= \frac{\Gamma(s-1)}{\Gamma(s)} + \frac{1}{2} + \sum_{k=2}^{n} \frac{B_k}{k!} \frac{\Gamma(s+k-1)}{\Gamma(s)} + \frac{I_1(s)}{\Gamma(s)} \\
&= \frac{1}{s-1} + \frac{1}{2} + \sum_{k=2}^{n} \frac{B_k}{k!}(s)(s+1)\cdots(s+k-2) + \frac{I_1(s)}{\Gamma(s)} \qquad (14)
\end{aligned}$$

where, knowing that $\frac{1}{\Gamma(s)}$ is an holomorphic function and $I_1(s)$ is holmorphic for $\sigma > -n$, we can deduce an equation, which is very useful to find the poles.

Proof, part1, meromorphic continuation and poles

Proof:

$$\zeta(s) - \frac{1}{s-1} = \frac{1}{2} + \sum_{k=2}^{n} \frac{B_k}{k!}(s)(s+1)\cdots(s+k-2) + \frac{I_1(s)}{\Gamma(s)} \qquad (15)$$

Thus the left hand side is holomorphic for $\sigma > -n$. But, because n was arbitrary, we can also conclude that $\zeta(s) - \frac{1}{s-1}$ is holomorphic in the complex plane. By taking $s = 1$ in one sees moreover that the (unique) pole of ζ is exactly the one claimed.

Proof, part2, values of ζ for negative integers

Proof:

We choose $s = -k \in (\mathbb{Z}_{\leq 0})$ where we fix n such that $-n < (-k) \leq 0$. Using the fact that Γ has poles at the points $s \in \{0, -1, -2, ...\}$ (thus in this cases $\frac{1}{\Gamma(s)}$ is equal to 0) and that $I_1(s)$ is holomorphic, we get, by

$$\zeta(s) - \frac{1}{s-1} = \frac{1}{2} + \sum_{k=2}^{n} \frac{B_k}{k!}(s)(s+1)\cdots(s+k-2) + \frac{I_1(s)}{\Gamma(s)}$$

a useful result for $\zeta(-k)$

Proof, part2, values of ζ for negative integers

Proof:

$$
\begin{aligned}
\zeta(-k) &= \frac{1}{-k-1} + \frac{1}{2} + \sum_{r=2}^{n} \frac{B_r}{r!}(-k)(-k+1)\cdots(-k+r-2) \\
&= -\frac{1}{k+1} + \frac{1}{2} + \sum_{r=2}^{k+1} (-1)^{r-1} \frac{B_r}{r!} \frac{k!}{(k+1-r)!} \\
&= -\frac{1}{k+1} + \frac{1}{2} + \sum_{r=2}^{k+1} \frac{-B_r}{r!} \frac{1}{k+1} \frac{(k+1)!}{(k+1-r)!r!} \\
&= -\frac{1}{k+1} \sum_{r=0}^{k+1} \binom{k+1}{r} B_r \\
&= -\frac{B_{k+1}}{k+1}
\end{aligned}
\tag{16}
$$

Proof, Part 3, values of ζ for positive odd integers

Proof:

We look now for $\zeta(2n)$, $n \in \mathbb{N}$.
For the next computation we will use two useful equalities concerning the Γ-function

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \tag{17}$$

$$\log(\Gamma(1+s)) = -\gamma s + \frac{\zeta(2)}{2}s^2 - \frac{\zeta(3)}{3}s^3 + \cdots \qquad |s| < 1 \tag{18}$$

and the facts that

$$\frac{x}{\tan(x)} = ix\frac{\cos(x)+i\cdot\sin(x)+\cos(x)-i\cdot\sin(x)}{\cos(x)+i\cdot\sin(x)-\cos(x)+i\cdot\sin(x)} = ix\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}} \tag{19}$$

$$\frac{s}{2}\frac{d}{ds}\log\left(\frac{\pi s}{\sin(\pi s)}\right) = \frac{s}{2}\frac{\sin(\pi s)}{(\pi s)}\left(\frac{\pi\sin(\pi s)-\pi^2 s\cdot\cos(\pi s)}{\sin^2(\pi s)}\right) = \frac{1}{2}\left(1-\frac{\pi s}{\tan(\pi s)}\right). \tag{20}$$

Proof, Part 3, values of ζ for positive odd integers

Proof:

Then if we take the series (for $|s| < 1$)

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n-1} \pi^{2n} B_{2n}}{(2n)!} s^{2n} &= -\frac{1}{2}\left(\frac{2\pi i s}{e^{2\pi i s}-1} - 1 + \frac{2\pi i s}{2}\right) \\
&= \frac{1}{2} - \frac{\pi i s}{2} \frac{e^{\pi i s} + e^{-\pi i s}}{e^{\pi i s} - e^{-\pi i s}} \\
&= \frac{1}{2}\left(1 - \frac{\pi s}{\tan(\pi s)}\right) \\
&= \frac{s}{2}\frac{d}{ds} \log\left(\frac{\pi s}{\sin(\pi s)}\right) \\
&= \frac{s}{2}\frac{d}{ds} \log\left(\Gamma(1+s)\Gamma(1-s)\right) \\
&= \frac{s}{2}\frac{d}{ds}\left(\zeta(2)s^2 + \frac{\zeta(4)}{2}s^4 + \cdots\right) \qquad (21)
\end{aligned}$$

Proof, Part 3, values of ζ for positive odd integers

Proof:

Thus we get the equality

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n-1} \pi^{2n} B_{2n}}{(2n)!} s^{2n} = \sum_{n=1}^{\infty} \zeta(2n) \cdot s^{2n} \tag{22}$$

(*where we used, in the order, the properites* (3), *with* $t = 2\pi i s$, *and then* (19), (20), (17), (18)).

Therefore the proof of the third part, and consequently the whole theorem, is concluded. □

Functional equations

One notice that the values of $\zeta(2n)$ and $\zeta(1-2n)$ are similar, in the sense that both contains the same Bernoulli numbers. Therefore it make sense to look for a relation between $\zeta(s)$ and $\zeta(1-s)$. In this sense we have

$$\zeta(1-2n) = -\frac{B_{2n}}{2n} \qquad \text{and} \qquad \zeta(2n) = \frac{(-1)^{n-1}2^{2n-1}\pi^{2n}B_{2n}}{(2n)!} \tag{23}$$

we get, by $\zeta(-2\mathbb{N}) = 0$

$$\frac{2^{k-1}\pi^k}{(k-1)!}\zeta(1-k) = \begin{cases} (-1)^{\frac{k}{2}}\zeta(k) & k > 0, \text{even} \\ 0 & k > 1, \text{odd} \end{cases} \tag{24}$$

Functional equations

$$\frac{2^{k-1}\pi^k}{(k-1)!}\zeta(1-k) = \begin{cases} (-1)^{\frac{k}{2}}\zeta(k) & k > 0, \text{even} \\ 0 & k > 1, \text{odd} \end{cases} \tag{25}$$

To interpolate everything we take, as a natural choice, for $k \to (k-1)!$ the Γ-function and for

$$k \to \begin{cases} (-1)^{\frac{k}{2}} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

the function $\cos(\frac{\pi k}{2})$. That leads to the relation

$$\frac{2^{s-1}\pi^s}{\Gamma(s)} \cdot \zeta(1-s) = \cos(\frac{\pi s}{2}) \cdot \zeta(s) \tag{26}$$

Zeros of ζ

To analyse the zeros of ζ it's very useful to rewrite (26) in a symmetric form. To do that we need

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \tag{27}$$

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = 2^{1-s}\sqrt{\pi}\cdot\Gamma(s) \tag{28}$$

$$\cos\left(\frac{\pi S}{2}\right) = \sin\left(\frac{\pi s}{2}+\frac{\pi}{2}\right) = \sin\left(\pi\frac{s+1}{2}\right) \tag{29}$$

Zeros of ζ

leading thus to

$$\frac{2^{s-1}\pi^s}{\Gamma(s)} \cdot \zeta(1-s) = \sin(\pi\frac{s+1}{2}) \cdot \zeta(s)$$

$$\frac{\pi}{\sin(\pi\frac{s+1}{2})} \cdot \frac{2^{s-1}\pi^{s-1}}{\Gamma(s)} \cdot \zeta(1-s) = \zeta(s)$$

$$2^{s-1}\pi^{s-1}\frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s+1}{2}\right)}{\Gamma(s)} \cdot \zeta(1-s) = \zeta(s)$$

$$\pi^{s-\frac{1}{2}} \cdot \Gamma\left(\frac{1-s}{2}\right) \cdot \zeta(1-s) = \zeta(s) \cdot \Gamma\left(\frac{s}{2}\right)$$

$$\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right) \cdot \zeta(1-s) = \pi^{-\frac{s}{2}} \cdot \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s)$$

Zeros of ζ

Let's now analyse this equation for different sigmas.

$$\pi^{-\frac{s}{2}} \cdot \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \cdot \zeta(1-s) \tag{30}$$

- $\sigma > 1$
  Then the left part is different from zero because of ζ being equal $\sum \frac{1}{n^s}$ and Γ unequal zero.
- $\sigma < 0$
  In this case is the right hand side finite and unequal zero and therefore the zeros of $\zeta(s)$ have to be exactly where the poles of $\Gamma\left(\frac{s}{2}\right)$ are, hence $s = -2, -4, -6, \cdots$.

Zeros of ζ