Characters

Student Seminar in Number Theory: L-functions

Sami Fawaz

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Definition 1 (Character)

Let G be a finite group. A *character* on G is a group homomorphism

$$\chi : G \to \mathbb{C}^{\times}.$$

The set of characters on G form a group under the operations

$$\chi\chi'(g) \coloneqq \chi(g)\chi'(g),\ \chi^{-1}(g) \coloneqq \chi(g)^{-1}$$

which we denote by $\hat{G}$.

Definition 1 (Dirichlet character)

Let $m \in \mathbb{N}_{>0}$. For a group $G = (\mathbb{Z}/m\mathbb{Z})^\times = \{n \mod m \mid (n, m) = 1\}$ we call a character χ on G a *Dirichlet character of modulus m*.

Aliter

A Dirichlet character of modulus m is a function $\chi : \mathbb{Z} \to \mathbb{C}$ with

1. $\forall n : \chi(n) = 0 \Leftrightarrow (n, m) \neq 1$
2. $\forall n,\ n' : \chi(nn') = \chi(n)\chi(n')$
3. $\forall n : \chi(n + m) = \chi(n)$.

Example 2 (Principal character of modulus m)

$$\chi_0(n) : \begin{cases} 1, & (n,m) = 1 \\ 0, & (n,m) \neq 1. \end{cases}$$

This corresponds to the neutral element in $\hat{G}$.

Example 3 (Legendre symbol)

For a prime p the *Legendre symbol* is

$$\left(\frac{n}{p}\right) := \begin{cases} 0, \text{ if } p | n \\ 1, \text{ if } p \not| n, \; n \equiv x^2 \pmod{p}, \text{ for some } x \\ -1, \text{ else} \end{cases}$$

Theorem 4

Let G be a finite abelian Group. Then the group of G characters $\hat{G}$ is isomorphic to G. In particular

$$\varphi(m) = |\widehat{(\mathbb{Z}/m\mathbb{Z})^\times}|,$$

that is the Euler phi function yields the number of Dirichlet characters.

Recall

$$\varphi(m) = |\{n \pmod{m} \mid (n, m) = 1\}| = m \prod_{p|m} (1 - \frac{1}{p})$$

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Theorem 5

Let χ be a Dirichlet character of modulus m. Then

$$\sum_{n \pmod{m}} \chi(n) = \begin{cases} \varphi(n), \text{ if } \chi = \chi_0 \\ 0, \text{ else} \end{cases}$$

holds, where $\sum\limits_{n \pmod{m}}$ is the sum over a representing system of $\mathbb{Z}/m\mathbb{Z}$.

Theorem 5

Let χ be a Dirichlet character of modulus m. Then

$$\sum_{n \pmod m} \chi(n) = \begin{cases} \varphi(n), & if\ \chi = \chi_0 \\ 0, & else \end{cases}$$

holds, where $\sum_{n \pmod m}$ *is the sum over a representing system of* $\mathbb{Z}/m\mathbb{Z}$.

Theorem 6

Let $n \in \mathbb{Z}$. Then

$$\sum_{\chi} \chi(n) = \begin{cases} \varphi(m), \text{ if } n \equiv 1 \pmod{m} \\ 0, \text{ else} \end{cases}$$

holds, where $\sum\limits_{\chi}$ is the sum over all Dirichlet characters of modulus m.

Theorem 6

Let $n \in \mathbb{Z}$. *Then*

$$\sum_{\chi} \chi(n) = \begin{cases} \varphi(m), & if\ n \equiv 1 \pmod{m} \\ 0, & else \end{cases}$$

holds, where $\sum_{\chi}$ *is the sum over all Dirichlet characters of modulus* m.

Definition 7 (Primitive and imprimitive character)

Let $m' \neq m$ and $m'|m$, let χ' be a character of modulus m'. The character resulting from the composition

$$\chi : (\mathbb{Z}/m\mathbb{Z})^{\times} \xrightarrow{\pmod{m'}} (\mathbb{Z}/m'\mathbb{Z})^{\times} \xrightarrow{\chi'} \mathbb{C}^{\times}$$

is called *imprimitive*. A character that cannot be obtained this way is called *primitive*.

Definition 8 (Fundamental discriminant)

A *fundamental discriminant* is an integer D with

$$D \equiv 1 \pmod 4, \text{ and } D \text{ is square-free, or}$$

$$D = 4m, \text{ with } m \equiv 2 \text{ or } 3 \pmod 4, \text{ and } m \text{ is square-free,}$$

Define a function $\chi_D : \mathbb{N} \to \mathbb{Z}$ as follows

$$\chi_D(p) = \left(\frac{D}{p}\right), \text{ for p odd prime}$$

$$\chi_D(2) = \begin{cases} 0, \text{ if } D \equiv 0 \pmod 4 \\ 1, \text{ if } D \equiv 1 \pmod 8 \\ -1, \text{ if } D \equiv 5 \pmod 8 \end{cases}$$

$$\chi_D(p_1^{r_1} \cdots p_k^{r_k}) = \chi_D(p_1)^{r_1} \cdots \chi_D(p_k)^{r_k}$$

Theorem 9

Let D be a fundamental discriminant. The map χ_D defines a primitive Dirichlet character of modulus $|D|$ denoted by χ_D as well. In particular

$$\chi_D(-1) = \text{sign}(D)$$

Moreover every real primitive Dirichlet character is of this type χ_D for some fundamental discriminant D.

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Characters of modulus 4

$n \pmod 4$	0	1	2	3
$\chi_0(n)$	0	1	0	1
$\varepsilon_4(n)$	0	1	0	-1

Characters of modulus 8

$n \pmod 8$	0	1	2	3	4	5	6	7
$\chi_0(n)$	0	1	0	1	0	1	0	1
$\varepsilon_8(n)$	0	1	0	1	0	-1	0	-1
$\varepsilon_8'(n)$	0	1	0	-1	0	-1	0	1
$\varepsilon_8''(n)$	0	1	0	-1	0	1	0	-1