Seminar on L-functions: Dirichlet L-series, Part Two

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Definition of L-series

Definition

Let χ be a Dirichlet character of modulus m. The *Dirichlet L-series* corresponding to χ is given by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{1}$$

Remark 2

For $\sigma > 1$, the Dirichlet L-series can be written as an Euler product as follows,

$$\prod_p \frac{1}{1 - \frac{\chi(p)}{p^s}}, \tag{2}$$

where the product runs over all primes p.

Example 3

We have

$$L(s, \chi_0) = \prod_{p|m} (1 - p^{-s}) \zeta(s). \tag{3}$$

As $L(s, \chi_0)$ differs from the Riemann zeta function $\zeta(s)$ only by a simple multiplicative factor, it follows that $L(s, \chi_0)$ has only one singularity, namely a simple pole at $s = 1$.

Remark 4

If χ is a Dirichlet character different from χ_0, then the abscissa of convergence of $L(s, \chi)$ is 0.

Theorem 5

Theorem

If χ is a Dirichlet character different from χ_0, then

$$L(1, \chi) \neq 0. \tag{4}$$

Corollary 6

Corollary (Dirichlet's theorem on arithmetic progressions.)

Let $a, m \in \mathbb{Z}_{>0}$, such that a and m are mutually prime. Then the arithmetic progression $(mk + a)_{k \in \mathbb{Z}_{\geq 0}}$ contains infinitely many primes. Furthermore, we have

$$\sum_{\substack{p \text{ prime}, \\ p \equiv a \pmod{m}}} \frac{1}{p} = \infty. \tag{5}$$

Remark 7

Let $a, b \in \mathbb{Z}$ such that b, m are mutually prime. Then

$$\frac{1}{\phi(m)} \sum_{\chi} \chi(a)\overline{\chi}(b) = \begin{cases} 1, \text{ if } a \equiv b \pmod{m} \\ 0, \text{ if } a \not\equiv b \pmod{m}. \end{cases} \tag{6}$$

References

- D.B. Zagier, *Zetadfunktionen und quadratische Körper (pp. 41-47)*, Springer 1981.