Seminar in Number Theory: L-Functions
Special values of Dirichlet series*

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1 Values at negative integers

Dirichlet series are a generalization of the Riemann zeta function $\zeta(s)$. So it is quite natural to try to generalize results about the latter to them. One such result is the fact that the values of $\zeta(s)$ at negative integers are rational and can be expressed in a closed form. While it may seem unlikely, one can also calculate values at negative integers of general Dirichlet series fairly easily, under quite reasonable conditions.

Definition 1.1 *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a function and let $(b_n)_n \subseteq \mathbb{C}$. We say that f has the asymptotic expansion*

$$f(t) \approx b_{-1}t^{-1} + b_0 + b_1 t + b_2 t^2 + \cdots, \quad t \to 0,$$

if for any $N \in \mathbb{N}$ there exists a $t_0 > 0$ such that

$$\left| f(t) - \sum_{n=-1}^{N} b_n t^n \right| \leq Ct^N, \quad 0 < t < t_0. \tag{1}$$

It is important to note, that in the definition we do not assume that $\sum b_n t^n$ converges. If it converges we also do not assume that it has the same value as $f(t)$. That said we can easily see that condition (1) is true for any function which is analytic at 0, by using its Taylor expansion.

Theorem 1.2 *Let $\varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be a Dirichlet series, which converges for at least one value of $s \in \mathbb{C}$, and let $f(t) = \sum_{n=1}^{\infty} a_n e^{-nt}$ be the corresponding exponential series. If f has the asymptotic expansion*

$$f(t) \approx b_0 + b_1 t + b_2 t^2 + \cdots, \quad t \to 0,$$

then $\varphi(s)$ has an holomorphic extension to all $\mathbb{C}$ and

$$\varphi(-n) = (-1)^n \, n! \, b_n, \quad n = 0, 1, 2, \ldots \tag{2}$$

If, more generally, f has the asymptotic expansion

$$f(t) \approx b_{-1}t^{-1} + b_0 + b_1 t + b_2 t^2 + \cdots, \quad t \to 0,$$

then $\varphi(s)$ has an meromorphic extension to all $\mathbb{C}$, $\varphi(s) - \frac{b_{-1}}{s-1}$ is an entire function and (2) still holds.

*Based on Chapter 7 in D.B.Zaiger [1]

Proof Given that condition (1) holds, it is clear that the exponential series $f(t)$ is absolutely convergent for $t > 0$, since the a_n's can grow at most like a polynomial. Therefore the Mellin transform of f in the range of convergence is equal to $\Gamma(s)\varphi(s)$, or more explicitly

$$\Gamma(s)\varphi(s) = \int_0^\infty f(t)t^{s-1}dt.$$

Let us divide the integral as

$$g + h, \quad g(s) := \int_0^1 f(t)t^{s-1}dt, \quad h(s) := \int_1^\infty f(t)t^{s-1}dt.$$

Since $f(t) \to 0$ exponentially as $t \to \infty$, the integral $h(s)$ converges for all s, in particular it actually converges absolutely and uniformly on compact sets. Therefore h must be an entire function. For $Re(s) > 1$ we have that

$$\int_0^1 \left(\sum_{n<N} b_n t^n \right) t^{s-1}dt = \sum_{n<N} b_n \frac{t^{n+s}}{n+s} \bigg|_0^1 = \sum_{n<N} \frac{b_n}{n+s},$$

so that in particular

$$g(s) = \sum_{n<N} \frac{b_n}{n+s} + \int_0^1 \left(f(t) - \sum_{n<N} b_n t^n \right) t^{s-1}dt.$$

If we have $Re(s) > -N$, then condition (1) implies that the integral in the above expression converges absolutely and uniformly on compact sets. Therefore $g(s) = \sum_{n<N} \frac{b_n}{n+s} + g'(s)$, where g' is holomorphic for $Re(s) > -N$, in particular $\Gamma(s)\varphi(s) - \sum_{n<N} \frac{b_n}{n+s}$ has an holomorphic continuation on $Re(s) > -N$. Since this holds for any N, we get a meromorphic continuation of $\Gamma(s)\varphi(s)$ on all $\mathbb{C}$, whose simple poles can only possibly be at $s = -n$ for $n \in \{-1, 0, 1, 2, \ldots\}$ and it has residue b_n at them. Since $1/\Gamma(s)$ is holomorphic on all $\mathbb{C}$ and has zeros at $s \in \{0, -1, -2, \ldots\}$, $\varphi(s)$ must be holomorphic except for possibly a simple pole at $s = 1$ with residue b_{-1}. It is well known that $Res(\Gamma, -n) = (-1)^n/n!$, so for all $n \in \mathbb{N}$ we have

$$b_n = Res(\Gamma\varphi, -n) = Res(\Gamma, -n)\varphi(-n) = \frac{(-1)^n}{n!}\varphi(-n),$$

which implies

$$\varphi(-n) = (-1)^n \, n! \, b_n. \qquad \square$$

Remark 1.3 Theorem 1.2 is also true for general Dirichlet series $\varphi(s) = \sum_{n=1}^\infty \frac{a_n}{\lambda_n^s}$ which converge for at least one value s, by using $f(s) = \sum_{n=1}^\infty a_n e^{-\lambda_n t}$ and the corresponding Mellin transform formula.

Example 1.4 We will now consider the example of the Riemann zeta function $\varphi(s) = \zeta(s)$. In particular we have $a_n = 1$ for all $n \in \mathbb{N}$, so

$$f(t) = \sum_{n=1}^\infty e^{-nt} = \frac{1}{e^t - 1}$$

and, with B_n being the nth Bernoulli number we have the asymptotic expansion

$$f(t) \approx \frac{1}{t} + \sum_{n=0}^{\infty} \frac{B_{n+1}}{(n+1)!} t^n,$$

which actually converges for $0 < t < 2\pi$, so Theorem 1.2 gives us immediately the holomorphic continuation of $\zeta(s) - 1/(s-1)$ on the whole complex plane and the values

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} = \begin{cases} -1/2 & n = 0 \\ -B_{n+1}/(n+1) & n \geq 1 \text{ odd} \\ \ \ 0 & n \geq 2 \text{ even} \end{cases}$$

which could also have been obtained directly.

We now want to consider Dirichlet L-series, so let $\mathcal{X}$ be a Dirichlet character mod N and consider $\varphi(s) = L(s, \mathcal{X}) = \sum_{n=1}^{\infty} \frac{\mathcal{X}(n)}{n^s}$, $Re(s) > 1$. By the periodicity of $\mathcal{X}$ we get directly that

$$f(t) = \sum_{n=1}^{\infty} \mathcal{X}(n) e^{-nt} = \sum_{n=1}^{N} \mathcal{X}(n) \sum_{m=0}^{\infty} e^{-(n+mN)t} = \sum_{n=1}^{N} \mathcal{X}(n) \frac{e^{-nt}}{1 - e^{-Nt}}.$$

Given that as $t \to 0$ we have the asymptotic expansions

$$e^{-nt} \approx \sum_{k=0}^{\infty} \frac{(-n)^k}{k!} t^k$$

$$\frac{1}{1 - e^{-Nt}} \approx \sum_{r=0}^{\infty} \frac{(-1)^r B_r}{r!} (Nt)^{r-1},$$

the asymptotic expansion of f as $t \to 0$ is given by

$$f(t) \approx \sum_{n=1}^{N} \mathcal{X}(n) \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^{k+r} n^k N^{r-1} B_r}{k!\ r!} t^{k+r-1}.$$

So in particular for $m \in \{-1, 0, 1, 2, \dots\}$ we have

$$b_m = \sum_{n=1}^{N} \mathcal{X}(n) \sum_{\substack{k+r-1=m \\ k,r \geq 0}} \frac{(-1)^{k+r} n^k N^{r-1} B_r}{k!\ r!}, \tag{3}$$

which gives us the value

$$b_{-1} = \frac{1}{N} \sum_{n=1}^{N} \mathcal{X}(n). \tag{4}$$

But of course the sum (4) is zero whenever $\mathcal{X} \neq \mathcal{X}_0$, that is whenever $\mathcal{X}$ is not the principal character, so in this case Theorem 1.2 gives us a holomorphic continuation of $L(s, \mathcal{X})$ for all $s \in \mathbb{C}$. In the case of the principal character we get

$$b_{-1} = \frac{1}{N} \sum_{\substack{n=1 \\ (n,N)=1}}^{N} 1 = \frac{\phi(N)}{N},$$

where ϕ denotes the Euler totient function, so $L(s, \chi_0)$ has a simple pole at $s = 1$ with residue $\phi(N)N^{-1}$, which could also have been proven directly. To rewrite the expression (3) in a simpler way we can use the Bernoulli polynomials, namely

$$B_n(x) := \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k, \tag{5}$$

in particular the first few are given by

$$\begin{aligned} B_0(x) &= 1, \\ B_1(x) &= x - \frac{1}{2}, \\ B_2(x) &= x^2 - x + \frac{1}{6}, \\ B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{6}x, \\ &\vdots \end{aligned}$$

Given this polynomials we have the more readable expression

$$b_m = \frac{(-1)^{m+1} N^m}{(m+1)!} \sum_{n=1}^{N} \chi(n) B_{m+1}\left(\frac{n}{N}\right).$$

To put it all together we get:

Theorem 1.5 *Let χ be a Dirichlet character mod N and let $L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$, $Re(s) > 1$, be its corresponding L-series. Then $L(s, \chi)$ can be extended to a meromorphic function on all of $\mathbb{C}$. In particular the extension is holomorphic except if $\chi = \chi_0$, in which case there is a simple pole by $s = 1$ with residue $\phi(N)/N$. We also have*

$$L(-n, \chi) = -\frac{N^n}{n+1} \sum_{m=1}^{N} \chi(m) B_{n+1}\left(\frac{m}{N}\right), \quad n = 0, 1, 2, \ldots$$

Using the fact that $B_1(x) = x - \frac{1}{2}$ and that $\sum_{m=1}^{N} \chi(m) = 0$ when $\chi \neq \chi_0$, an example of the Theorem is

$$\begin{aligned} L(0, \chi) &= -\sum_{m=1}^{N} \chi(m) \left(\frac{m}{N} - \frac{1}{2}\right) \\ &= -\frac{1}{N} \sum_{m=1}^{N} \chi(m) m + \frac{1}{2} \sum_{m=1}^{N} \chi(m) = -\frac{1}{N} \sum_{m=1}^{N} \chi(m) m. \end{aligned}$$

2 Properties of the Bernoulli polynomials

The Bernoulli polynomials defined before have very nice properties. Thanks the definition we can deduce

$$B_n(0) = B_n \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} B_n(x) = nB_{n-1}(x) \tag{7}$$

Proof

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} B_n(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\Big(\sum_{k=0}^{n}\binom{n}{k}B_{n-k}x^k\Big) = \sum_{k=1}^{n}\frac{n!}{k!(n-k)!}B_{n-k}kx^{k-1} \\ &= \sum_{k=0}^{n-1} n\frac{(n-1)!}{k!(n-k-1)!}B_{n-k-1}x^k = nB_{n-1}(x). \qquad \square\end{aligned}$$

We notice that (6) and (7) together give us a second and inductive definition of the Bernoulli polynomials. Another definition that we can consider for the $B_n(x)$ is

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{te^{xt}}{e^t-1} \tag{8}$$

Proof

$$\begin{aligned}\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}B_{n-k}x^k\frac{t^n}{n!} = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{1}{k!(n-k)!}B_{n-k}t^{n-k}t^kx^k \\ &= \sum_{k=0}^{\infty}\frac{t^kx^k}{k!}\sum_{n=k}^{\infty}\frac{B_{n-k}t^{n-k}}{(n-k)!} = e^{tx}\frac{t}{e^t-1} = \frac{te^{xt}}{e^t-1} \quad \text{where } (|t| \le 2\pi). \qquad \square\end{aligned}$$

The generating function gives us two further properties of the polynomial $B_n(x)$: symmetry and a recursion formula.

$$B_n(1-x) = (-1)^n B_n(x) \tag{9}$$

$$B_n(x+1) = B_n(x) + nx^{n-1} \tag{10}$$

Proof Using the property (8) we get

$$\begin{aligned}\sum_{n=0}^{\infty} B_n(1-x)\frac{t^n}{n!} &= \frac{te^{t-tx}}{e^t-1} = \frac{te^te^{-xt}}{e^t-1} = \frac{te^{-xt}}{1-e^{-t}} = \frac{(-t)e^{x(-t)}}{e^{-t}-1} \\ &= \sum_{n=0}^{\infty} B_n(x)\frac{(-t)^n}{n!} = \sum_{n=0}^{\infty}(-1)^nB_n(x)\frac{t^n}{n!}.\end{aligned}$$

Since $\{1, t, t^2, \cdot\}$ forms an independent system, then we get (9). In the same way we calculate

$$\begin{aligned}\sum_{n=0}^{\infty} B_n(x+1)\frac{t^n}{n!} &= \frac{te^{xt+t}}{e^t-1} = \frac{te^{xt}e^t - te^{xt} + te^{xt}}{e^t-1} = \frac{te^{xt}(e^t-1)+te^{xt}}{e^t-1} \\ &= te^{xt} + \frac{te^{xt}}{e^t-1} = t\sum_{k=0}^{\infty}\frac{(xt)^k}{k!} + \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty}\frac{x^k t^{k+1}}{k!} + \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \sum_{k=1}^{\infty}\frac{x^{k-1}t^k}{(k-1)!} + \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty}\frac{kx^{k-1}t^k}{k!} + \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(B_n(x) + nx^{n-1}\right)\frac{t^n}{n!}.\end{aligned}$$

□

finally due to (10) it follows the formula for the power sums that Jakob Bernoulli introduced the numbers B_k, named after him.

$$\begin{aligned}1^n + 2^n + \cdots + N^n &= \frac{B_{n+1}(N+1) - B_{n+1}(0)}{n+1} \\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}B_k\frac{N^{n+1-k}}{n+1-k}\end{aligned}$$

Proof If we rewrite (10) we obtain

$$(n+1)x^n = B_{n+1}(x+1) - B_{n+1}(x),$$

and therefore

$$\sum_{x=0}^{N}(n+1)x^n = \sum_{x=0}^{N}\left(B_{n+1}(x+1) - B_{n+1}(x)\right) = B_{n+1}(N+1) - B_{n+1}(0).$$

Thus we get the first equality. For the second equality we use (9)

$$B_{n+1}(N+1) = (-1)^{n+1}B_{n+1}(-N).$$

Then

$$\begin{aligned}\frac{B_{n+1}(N+1) - B_{n+1}(0)}{n+1} &= \frac{(-1)^{n+1}B_{n+1}(-N) - B_{n+1}}{n+1} \\ &= \frac{1}{1+n}\left((-1)^{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}B_{n+1-k}(-N)^k - B_{n+1}\right) \\ &= \frac{1}{1+n}\left((-1)^{n+1}\sum_{k=1}^{n+1}\binom{n+1}{k}B_{n+1-k}(-N)^k\right) \\ &= \sum_{k=1}^{n+1}(-1)^{n+1-k}\frac{n!}{k!(n+1-k)!}B_{n+1-k}N^k \\ &= \sum_{i=0}^{n}(-1)^i\frac{n!}{(n+1-i)!i!}B_iN^{n+1-i} \\ &= \sum_{i=0}^{n}(-1)^i\binom{n}{i}B_i\frac{N^{n+1-i}}{n+1-i},\end{aligned}$$

where in the third equality we used the fact that $(-1)^{n+1}B_{n+1} = B_{n+1}$ for all $n \geq 1$, and in the last equalities we used the index change $i = n + 1 - k$. $\square$

As an example of the Theorem 1.5 and the properties of the Bernoulli polynomials we consider the *Hurwitz Zeta function* defined as

$$\zeta(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s} \qquad (Re(s) > 1,\ a > 0).$$

The corresponding exponential series is

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} e^{-(n+a)t} = e^{-at}\sum_{n=0}^{\infty} e^{-nt} = \frac{1}{t}\frac{te^{-at}}{e^t - 1} = \frac{1}{t}\sum_{n=0}^{\infty} B_n(a)\frac{t^n}{n!} \\ &= \frac{1}{t} + \sum_{n=1}^{\infty} B_n(a)\frac{t^{n-1}}{n!} = \frac{1}{t} + \sum_{n=0}^{\infty} B_{n+1}(a)\frac{t^n}{(n+1)!} \end{aligned}$$

Using a theorem 1.2, we get that $\zeta(s,a) - \frac{1}{s-1}$ is a holomorph extension in the all plane and for $s \in \{0, -1, -2, \cdots\}$ and takes the value

$$\zeta(-n,a) = -\frac{1}{n+1}B_{n+1}(a).$$

In particular we have that

$$\begin{aligned} L(s,\mathcal{X}) &= \sum_{n=1}^{\infty}\frac{\mathcal{X}(n)}{n^s} = \sum_{m=1}^{N}\sum_{n=1}^{\infty}\mathcal{X}(nN+m)(nN+m)^{-s} = \sum_{m=1}^{N}\mathcal{X}(m)\sum_{n=1}^{\infty}(nN+m)^{-s} \\ &= \sum_{m=1}^{N}\mathcal{X}(m)\sum_{n=1}^{\infty}(n+\frac{m}{N})^{-s}N^{-s} = \sum_{m=1}^{N}\mathcal{X}(m)\zeta(s,\frac{m}{N})N^{-s}, \end{aligned}$$

and so it follows Theorem 1.5.

Since it holds $\mathcal{X}(-1)^2 = \mathcal{X}(1) = 1$, then for all Dirichlet character $\mathcal{X}$ it holds either $\mathcal{X}(-1) = +1$ or $\mathcal{X}(-1) = -1$. So we call $\mathcal{X}$ *even* for the first case and *odd* for the second case.

Corollary 2.1 *Except in the case of $N = 1$, $n = 0$, then it holds for all $\mathcal{X}$ and all $n \geq 0$*

$$\mathcal{X}(-1) = (-1)^n \Rightarrow L(-n,\mathcal{X}) = 0,$$

i.e. the L-series of an even (respectively odd) character disappear at the negative even (respectively odd) n.

3 Functional equation of the Riemann Zeta function

Theorem 3.1

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s). \tag{11}$$

To prove the functional equation of the Riemann Zeta function, we still need to recall some information.

Definition 3.2 *A smooth function $f : \mathbb{R} \to \mathbb{C}$ is called a Schwartz function if it satisfies*

$$\sup_{x\in\mathbb{R}} \left| x^\alpha \frac{d^\beta}{dx^\beta} f(x) \right| < \infty,$$

for all α and β in $\mathbb{N}$. We denote the space of all Schwartz functions as $\mathcal{S}(\mathbb{R})$.

Recall that for a function $f \in \mathcal{S}(\mathbb{R})$, its Fourier transform is define as

$$\hat{f}(y) = \int_{\mathbb{R}} f(x) e^{-2\pi i y x} \, \mathrm{d}x.$$

Notice that also $\hat{f}$ is in the Schwartz space.

Lemma 3.3 (Poisson summation formula) *Let $f \in \mathcal{S}(\mathbb{R})$, then*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \hat{f}(k).$$

Proof We consider the function

$$g : \mathbb{R} \to \mathbb{C}, \quad g(x) = \sum_{n\in\mathbb{Z}} f(n+x).$$

Since f is in the Schwartz space, then g is smooth. Furthermore, we notice that $g(x+1) = g(x)$ by construction, so g is 1-periodic. Then

$$\begin{aligned}
g(x) &= \sum_{k\in\mathbb{Z}} \Big(\int_0^1 g(y) e^{-2\pi i k y} \, \mathrm{d}y \Big) e^{2\pi i k x} = \sum_{k\in\mathbb{Z}} \Big(\int_0^1 \sum_{n\in\mathbb{Z}} f(n+y) e^{-2\pi i k y} \, \mathrm{d}y \Big) e^{2\pi i k x} \\
&= \sum_{k\in\mathbb{Z}} \Big(\sum_{n\in\mathbb{Z}} \int_0^1 f(n+y) e^{-2\pi i k y} \, \mathrm{d}y \Big) e^{2\pi i k x} = \sum_{k\in\mathbb{Z}} \Big(\sum_{n\in\mathbb{Z}} \int_n^{n+1} f(z) e^{-2\pi i k (z-n)} \, \mathrm{d}z \Big) e^{2\pi i k x} \\
&= \sum_{k\in\mathbb{Z}} \Big(\sum_{n\in\mathbb{Z}} \int_n^{n+1} f(z) e^{-2\pi i k z} \, \mathrm{d}z \Big) e^{2\pi i k x} = \sum_{k\in\mathbb{Z}} \Big(\int_{\mathbb{R}} f(z) e^{-2\pi i k z} \, \mathrm{d}z \Big) e^{2\pi i k x} \\
&= \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{2\pi i k x}.
\end{aligned}$$

If we take $x = 0$, then the Lemma is proved. □

Recall that for a complex number τ the *Jacobi theta function* is defined as

$$\vartheta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots$$

where $q = e^{2\pi i \tau}$.

Theorem 3.4 *For $t > 0$ it holds*

$$\vartheta\Big(\frac{i}{2t}\Big) = \sqrt{t}\,\vartheta\Big(\frac{it}{2}\Big).$$

Proof We have to prove

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2/t} = \sqrt{t} \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}.$$

We want to use the Poisson summation formula for the Schwartz function $f_t(x) = e^{-\pi x^2/t}$. We define $g(x) = e^{-\pi x^2}$ and calculate

$$\begin{aligned}\hat{f}_t(n) &= \int_{\mathbb{R}} e^{-\pi x^2/t} e^{-2\pi i n x}\,\mathrm{d}x = \sqrt{t}\int_{\mathbb{R}} e^{-\pi y^2} e^{-2\pi i n\sqrt{t}y}\,\mathrm{d}y \\ &= \sqrt{t}\hat{g}(\sqrt{t}n) = \sqrt{t}g(\sqrt{t}n) = \sqrt{t}e^{-\pi n^2 t} = \sqrt{t}f_{1/t}(n),\end{aligned}$$

where in the second equality we used the substitution $x = \sqrt{t}y$ and at the end we used the property $\hat{g}(x) = g(x)$. Then it holds

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2/t} = \sum_{n\in\mathbb{Z}} f_t(n) = \sum_{n\in\mathbb{Z}} \hat{f}_t(n) = \sum_{n\in\mathbb{Z}} \sqrt{t}f_{1/t}(n) = \sqrt{t}\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}. \qquad \square$$

Proof (Theorem 3.1) For the proof we consider $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then

$$\begin{aligned}\xi(2s) &= \pi^{-s}\Gamma(s)\zeta(2s) = \pi^{-s}\int_0^\infty e^{-y}y^{s-1}\,\mathrm{d}y\sum_{n=1}^\infty n^{-2s} \\ &= \int_0^\infty \sum_{n=1}^\infty (\pi n^2)^{-s} e^{-y}y^{s-1}\,\mathrm{d}y = \int_0^\infty \sum_{n=1}^\infty e^{-\pi n^2 z}z^{s-1}\,\mathrm{d}z \\ &= \frac{1}{2}\int_0^\infty \sum_{n\in\mathbb{Z},n\neq 0} e^{-\pi n^2 z}z^{s-1}\,\mathrm{d}z = \frac{1}{2}\int_0^\infty (\vartheta(iz/2)-1)z^{s-1}\,\mathrm{d}z \\ &= \frac{1}{2}\int_1^\infty (\vartheta(iz/2)-1)z^{s-1}\,\mathrm{d}z + \frac{1}{2}\int_0^1 (\vartheta(iz/2)-1)z^{s-1}\,\mathrm{d}z,\end{aligned}$$

where we used the substitution $zn^2\pi = y$. Now for the second integral we make the substitution $x = \frac{1}{z}$ and the Theorem 3.4 and we obtain

$$\begin{aligned}\frac{1}{2}\int_0^1 (\vartheta(iz/2)-1)z^{s-1}\,\mathrm{d}z &= \frac{1}{2}\int_1^\infty (\vartheta(i/2x)-1)x^{-s-1}\,\mathrm{d}x \\ &= \frac{1}{2}\int_1^\infty (\sqrt{x}\vartheta(ix/2)-1)x^{-s-1}\,\mathrm{d}x \\ &= \frac{1}{2}\int_1^\infty (\vartheta(ix/2)-1)x^{-s-1/2}\,\mathrm{d}x + \frac{1}{2}\int_1^\infty x^{-s-1/2} - x^{-s-1}\,\mathrm{d}x \\ &= \frac{1}{2}\int_1^\infty (\vartheta(ix/2)-1)x^{-s-1/2}\,\mathrm{d}x + \frac{1}{2s-1} - \frac{1}{2s}.\end{aligned}$$

Therefore,

$$\xi(2s) = \frac{1}{2}\int_1^\infty (\vartheta(ix/2)-1)(x^s + x^{-s+1/2})\,\frac{\mathrm{d}x}{x} + \frac{1}{2s-1} - \frac{1}{2s}.$$

If we substitute $2s$ with s we get

$$\xi(s) = \frac{1}{2}\int_1^\infty (\vartheta(ix/2)-1)(x^{s/2} + x^{(1-s)/2})\,\frac{\mathrm{d}x}{x} + \frac{1}{s-1} - \frac{1}{s}.$$

We notice that it holds immediately

$$\xi(1-s) = \xi(s),$$

and the theorem follows. □

Without proving it, we can state a general equation, the *functional equation* of the L-series $L(s, \chi)$, for a primitive character χ:

$$\pi^{-s/2} N^{s/2} \Gamma\left(\frac{s+\delta}{2}\right) L(s, \chi) = \frac{G}{i^{\delta}\sqrt{N}} \pi^{-(1-s)/2} N^{(1-s)/2} \Gamma\left(\frac{1-s+\delta}{2}\right) L(1-s, \tilde{\chi}), \tag{12}$$

where $\tilde{\chi}$ is the conjugated character of χ, δ is equal to 0 when χ is even and equal to 1 when χ is odd, and G is the *Gauss sum* $\sum_{n=1}^{N} \chi(n) e^{2\pi i n/N}$. Notice that the factor $\frac{G}{i^{\delta}\sqrt{N}}$ has absolute value equal 1.

Example 3.5 For χ primitive and odd character it holds

$$L(1, \chi) = -\frac{\pi i G}{N^2} \sum_{m=1}^{N} \hat{\chi}(m) m.$$

The Riemann zeta function is a special case of L-series. If we consider the principal character χ_0 mod N, with $N = 1$, then $\chi_0(n) = 1$ for all $n \in \mathbb{N}$. Therefore,

$$L(s, \chi_0) = \sum_{n=1}^{\infty} \frac{\chi_0(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s).$$

References

[1] Don Bernard Zagier. *Zetafunktionen und quadratische Körper*. Springer, 1981.