Class number formulas

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1 Computation of $L(1, \chi)$

In the previous chapter, the theory of quadratic forms led to an expression for the class numbers $h(D)$ in terms of $L(1, \chi_D)$ where χ_D is the Dirichlet character (defined as the Kronecker symbol $\chi_D(n) = \left(\frac{D}{n}\right)$ as follows:

$$h(D) = \begin{cases} \frac{w\sqrt{|D|}}{2\pi} L(1, \chi_D), & D < 0 \\ \frac{\sqrt{D}}{\log \varepsilon_0} L(1, \chi_D), & D > 0 \end{cases}$$

where w is the well-understood number of automorphisms of primitive quadratic forms with discriminant D

$$w = \begin{cases} 2, D < -4 \\ 4, D = -4 \\ 6, D = -3 \end{cases}$$

and $\varepsilon_0 = (t + u\sqrt{d})/2$ where (t, u) is a solution of the corresponding Pell equation $t^2 - du^2 = 4$ with minimal u. In this section, we focus on evaluating $L(1, \chi_D)$.

We start off with two lemmas, which try to make sense of the *Gauss sums* of the form

$$G(\chi) = \sum_{n=1}^{N} \chi(n) e^{2\pi i n/N} \tag{1}$$

where χ is a primitive Dirichlet character modulo N.

Lemma 1.1. *Let G be defined as in (1). Then we have:*

$$\sum_{n=1}^{N} \chi(n) e^{2\pi i k n/N} = \overline{\chi(k)} G(\chi) \tag{2}$$

$$|G(\chi)| = \sqrt{N} \tag{3}$$

for all integers k.

Proof:

Consider first the case where $(N, k) = 1$. In this case, k has an inverse in $\mathbb{Z}/N\mathbb{Z}$, denoted by k^{-1}. Left multiplication by k^{-1} is a bijection from $\mathbb{Z}/N\mathbb{Z}$ to itself. We can then write (as $\chi(k)\chi(k^{-1}) = 1 \implies \chi(k^{-1}) = \overline{\chi(k)}$)

$$\sum_{n=1}^{N} \chi(n) e^{2\pi i k n/N} = \sum_{n=1}^{N} \chi(n k^{-1}) e^{2\pi i n/N} = \overline{\chi(k)} G(\chi)$$

which proves the statement in this case. We are left with the case $(N, k) = d > 1$. Here, we have $\chi(k) = 0$ so we need to prove that

$$\sum_{n=1}^{N} \chi(n) e^{2\pi i k n/N} = 0$$

as well.

If we write $k = dk_1$ and $N = dN_1$ we can write

$$\begin{aligned} \sum_{n=1}^{N} \chi(n) e^{2\pi i k n/N} &= \sum_{n=1}^{N} \chi(n) e^{2\pi i k n_1/N_1} \\ &= \sum_{n_1=1}^{N_1} \left(\sum_{j=1}^{d} \chi(jN_1 + n_1) \right) e^{2\pi i k n_1/N_1} \end{aligned}$$

We will show that the inner sum vanishes. Consider the elements $c \in \mathbb{Z}/N\mathbb{Z}$ such that $(c, N) = 1$ and $c \equiv 1 \pmod{N_1}$.

Assume we have $\chi(c) = 1$ for all of them. We then have that χ is 1 on the kernel of the quotient map from $(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/N_1\mathbb{Z})^\times$, so χ factorises over $(\mathbb{Z}/N_1\mathbb{Z})$ which contradicts primitiveness.

Pick, then, some such c such that $\chi(c) \neq 1$. We can write:

$$(1-\chi(c))\sum_{j=1}^{d}\chi(jN_1+n_1) = \sum_{j=1}^{d}\chi(jN_1+n_1) - \sum_{j=1}^{d}\chi(c(jN_1+n_1)) = 0$$

where we use that multiplication by c is a bijection on the set of all $n \equiv n_1 \pmod{N_1}$ in $\mathbb{Z}/N\mathbb{Z}$.

To prove (3), we use (2):

$$\begin{aligned}
|G|^2 = G\overline{G} &= G\sum_{k=1}^{N}\overline{\chi(k)}e^{-2\pi ik/N} \\
&= \sum_{k=1}^{N}\left(\sum_{n=1}^{N}\chi(n)e^{2\pi ikn/N}\right)e^{-2\pi ik/N} \\
&= \sum_{n=1}^{N}\chi(n)\sum_{k=1}^{N}e^{2\pi ik(n-1)/N}
\end{aligned}$$

For $n = 1$ the inner sum is exactly N, and for $n \neq 1$ it vanishes, as we have

$$e^{2\pi i(n-1)/N}\cdot\sum_{k=1}^{N}e^{2\pi ik(n-1)/N} = \sum_{k=2}^{N+1}e^{2\pi ik(n-1)/N} = \sum_{k=1}^{N}e^{2\pi ik(n-1)/N}$$

so multiplication by $e^{2\pi i(n-1)/N} \neq 1$ preserves it (just shifting the index of summation by 1) so we get $|G(\chi)|^2 = \chi(1)N = N$ as desired. □

An easy corollary is that by taking (2) and conjugating both sides we get

$$\chi(k) = \frac{1}{\overline{G}}\sum_{n=1}^{N}\overline{\chi(n)}e^{-2\pi ikn/N} \tag{4}$$

which we will use in further computations.

To compute $L(1,\chi)$, we will also need another, simpler lemma.

Lemma 1.2. *For $0 < \theta < 2\pi$ we have:*

$$\sum_{n=1}^{\infty}\frac{e^{in\theta}}{n} = -\log\left(2\sin\frac{\theta}{2}\right) + i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$

Proof:

The series $\sum_{n=1}^{\infty} z^n/n$ converges for all $|z| \leq 1$ with $z \neq 1$ to $-\log(1-z)$ (where we take the branch of the complex logarithm defined on $\mathbb{C}\setminus\mathbb{R}^-$), so we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{e^{in\theta}}{n} &= -\log(1-e^{i\theta}) \\
&= -\log(-e^{i\theta/2}(e^{i\theta/2}+e^{-i\theta/2})) \\
&= -\log\left(-e^{i\theta/2}\cdot 2i\sin\frac{\theta}{2}\right) \\
&= -\log\left(e^{-\frac{i\pi}{2}+\frac{i\theta}{2}}\cdot 2\sin\frac{\theta}{2}\right) \\
&= -\log\left(2\sin\frac{\theta}{2}\right) + i\cdot\left(\frac{\pi}{2}-\frac{\theta}{2}\right)
\end{aligned}$$

as desired, where we take advantage of $-i = e^{-\pi/2}$ in the 4th line. □

Now we can finally compute $L(1,\chi)$ and obtain

Theorem 1.1. *Let N be an integer, $N > 1$, and χ a Dirichlet character* $\pmod N$*. We have:*

$$L(1,\chi) = -\frac{1}{\overline{G(\chi)}}\sum_{n=1}^{N-1}\overline{\chi(n)}\left(\log\left(2\sin\frac{\pi n}{N}\right) + i\cdot\left(\frac{\pi}{2} - \frac{\pi n}{N}\right)\right)$$

Proof:

We compute

$$\begin{aligned} L(1,\chi) &= \sum_{k=1}^{\infty}\frac{\chi(k)}{k} = \frac{1}{\overline{G(\chi)}}\sum_{k=1}^{\infty}\frac{1}{k}\sum_{n=1}^{N-1}\overline{\chi(n)}e^{-2\pi ink/N} \\ &= \frac{1}{\overline{G(\chi)}}\sum_{n=1}^{N-1}\overline{\chi(n)}\sum_{k=1}^{\infty}\frac{e^{-2\pi ikn/N}}{k} \\ &= -\frac{1}{\overline{G(\chi)}}\sum_{n=1}^{N-1}\overline{\chi(n)}\left(\log\left(2\sin\frac{\pi n}{N}\right) + i\cdot\left(\frac{\pi}{2} - \frac{\pi n}{N}\right)\right) \end{aligned}$$

where we use (4) in the first line and the conjugated form of Lemma 1.2 in the 3rd line. □

In the case where χ is primitive we have:

Theorem 1.2. *Let N be an integer, $N > 1$, and χ a primitive Dirichlet character* $\pmod N$*. We have:*

$$L(1,\chi) = -\frac{1}{\overline{G}}\sum_{n=1}^{N-1}\overline{\chi(n)}\log\sin\left(\frac{\pi n}{N}\right) + \frac{i\pi}{N\overline{G(\chi)}}\sum_{n=1}^{N-1}\chi(n)n$$

Proof:

We can write

$$\log\left(2\sin\frac{\pi n}{N}\right) = \log(2) + \log\sin\frac{\pi n}{N}$$

and as χ is primitive we have $\sum_{n=1}^{N-1}\overline{\chi(n)} = 0$ so we can write

$$\begin{aligned} L(1,\chi) &= -\frac{1}{\overline{G(\chi)}}\sum_{n=1}^{N-1}\overline{\chi(n)}\left(\log\left(2\sin\frac{\pi n}{N}\right) + i\cdot\left(\frac{\pi}{2} - \frac{\pi n}{N}\right)\right) \\ &= -\frac{1}{\overline{G}}\sum_{n=1}^{N-1}\overline{\chi(n)}\log\sin\left(\frac{\pi n}{N}\right) + \frac{i\pi}{N\overline{G(\chi)}}\sum_{n=1}^{N-1}\chi(n)n + \sum_{n=1}^{N-1}\overline{\chi(n)}\left(\log(2) + \frac{i\pi}{2}\right) \\ &= -\frac{1}{\overline{G}}\sum_{n=1}^{N-1}\overline{\chi(n)}\log\sin\left(\frac{\pi n}{N}\right) + \frac{i\pi}{N\overline{G(\chi)}}\sum_{n=1}^{N-1}\chi(n)n \end{aligned}$$

as desired. □

2 Computation of class numbers

Definition 2.1. *A fundamental discriminant D is an integer such that $D \equiv 1 \ (4)$ and D is square-free, or $D \equiv 0 \ (4)$, $\frac{D}{4} \equiv 2$ or $3 \ (4)$.*

Now, recall from [1] chapter 8 that the class number $h(D)$ for $D = b^2 - 4ac$ is given by the number of $SL_2(\mathbb{Z})$ equivalence classes of primitive quadratic forms of discriminant D (if $D < 0$, we take the quadratic forms to be positive-definite).
We have seen that

$$h(D) = \begin{cases} \frac{w\sqrt{D}}{2\pi}L(1,\chi_D), & D < 0 \\ \frac{\sqrt{D}}{\log\epsilon_0}L(1,\chi_D), & D > 0. \end{cases}$$

where ϵ_0 is the fundamental unit of f, and w is the order of the group of automorphisms of a quadratic form f, which was defined as follows:

$$\mathcal{U}(f) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{cases} a\alpha^2 + b\alpha\gamma + c\gamma^2 = a \\ 2a\alpha\beta + b(\beta\gamma + \alpha\delta) + 2c\alpha\delta = b \\ \alpha\beta^2 + b\beta\delta + c\delta^2 = c \end{cases} \right\}.$$

Let χ be the real character χ_D and $N = |D|$.
One can use Lemma 2 to obtain:

$$\begin{aligned} \overline{G} &= \sum_{n\ (N)} \overline{\chi(n)} e^{-2\pi i \frac{n}{N}} \\ &= \sum_{n\ (N)} \chi(n) e\left(\frac{-n}{N}\right) \\ &= \sum_{n\ (N)} \chi(-n) e\left(\frac{n}{N}\right) = \chi(-1)G, \end{aligned} \tag{1}$$

with $e(x) := e^{2\pi i x}$. Since

$$\chi_D(-1) = \begin{cases} 1, & D > 0 \\ -1, & D < 0 \end{cases} \tag{2}$$

G is either real or purely imaginary. One obtains from (1) that

$$G = \begin{cases} \pm\sqrt{D}, & D > 0 \\ \pm i\sqrt{D}, & D < 0. \end{cases} \tag{3}$$

Gauss has proven that the signs are as follows:

Theorem 2.1. *Let D be a fundamental discriminant. Then*

$$G = \begin{cases} \sqrt{D}, & D > 0 \\ i\sqrt{D}, & D < 0. \end{cases} \tag{4}$$

Now, consider again the sum:

$$L(1,\chi) = \frac{-1}{\overline{G}} \sum_{n=1}^{N-1} \overline{\chi(n)} \log\sin\frac{\pi n}{N} + i\frac{\pi}{N\overline{G}} \sum_{n=1}^{N-1} \overline{\chi}(n)n. \tag{5}$$

If χ is a real character, then $L(1,\chi)$ a real number, but since by (4) G could be either real or imaginary, we have that one of the two sums in (5) is equal to 0. If $\chi(-1) = -1$ then by (2) the first one equals 0, while for $\chi(-1) = 1$ the second one equals 0. Therefore:

Theorem 2.2. *Let D be a fundamental discriminant. Then*

$$L(1,\chi_D) = \begin{cases} \frac{-\pi}{|D|^{\frac{3}{2}}} \sum_{n=1}^{|D|-1} \chi_D(n)n, & D < 0 \\ \frac{-1}{\sqrt{D}} \sum_{n=1}^{D-1} \chi_D(n) \log\sin\frac{\pi n}{D}, & D > 0. \end{cases}$$

Using the formulas for the class number $h(D)$ that we recalled at the start of chapter 2 and Theorem 2.2 one obtains:

Theorem 2.3. *Let D be a fundamental discriminant. Then*

$$h(D) = \begin{cases} \frac{-w/2}{|D|} \sum_{n=1}^{|D|-1} \chi_D(n)n, & D < 0 \\ \frac{-1}{\log\epsilon_0} \sum_{n=1}^{D-1} \chi_D \log\sin\frac{\pi n}{D}, & D > 0. \end{cases}$$

Theorem 2.3 is useful for calculating the class numbers, because now to obtain the class number $h(D)$ we have to only compute certain finite sums instead of looking at the whole function $L(1, \chi_D)$.

Example 1 Let $D = -3$. We have seen that $w = 6$. Then

$$h(-3) = \frac{-6/2}{3} \sum_{n=1}^{2} \chi_{-3}(n)n = -(1-2) = 1.$$

Next, let $D = -4$, then $w = 4$ and

$$h(-4) = \frac{-4/2}{4} \sum_{n=1}^{3} \chi_{-4}(n)n = -\frac{1}{2}(1+0-3) = 1.$$

Similarly, one obtains:

$$\begin{aligned} h(-7) &= 1 \\ h(-8) &= 1 \\ h(-11) &= 1 \\ h(-15) &= 2 \\ h(-19) &= 1 \\ h(-23) &= 3. \end{aligned}$$

One observes that $h(D)$ is not always equal to 1. Later we will see that $h(D)$ is always even when D is divisible by at least 2 different prime numbers.
Next, we will take a look at some positive discriminants. One can rewrite

$$\begin{aligned} h(D) &= \frac{-1}{\log \epsilon_0} \sum_{n=1}^{D-1} \chi_D(n) \log \sin \frac{\pi n}{D} \\ &= \frac{1}{\log \epsilon_0} \sum_{n=1}^{D-1} \log \left((\sin \frac{\pi n}{D})^{-\chi_D(n)} \right). \end{aligned}$$

Hence,

$$\begin{aligned} \epsilon_0^{h(D)} &= \prod_{n=1}^{D-1} \left(\sin \frac{\pi n}{D} \right)^{-\chi_D(n)} \\ &= \prod_{\substack{0<n<D \\ \chi_D(n)=-1}} \sin \frac{\pi n}{D} \prod_{\substack{0<n<D \\ \chi_D(n)=1}} \left(\sin \frac{\pi n}{D} \right)^{-1}. \end{aligned}$$

Example 2 Let $D = 5$. Then $\epsilon_0 = \frac{3+\sqrt{5}}{2}$ (since $t = 3, u = 1$ are the smallest solutions of Pells equation $t^2 - 5u^2 = 4$ in positive reals). The right hand side is equal to

$$\frac{\sin \frac{2\pi}{5} \sin \frac{3\pi}{5}}{\sin \frac{\pi}{5} \sin \frac{4\pi}{5}} = (2 \cos \frac{\pi}{5})^2 = \frac{3+\sqrt{5}}{2},$$

hence $h(5) = 1$.
From the formula for $L(1, \chi_D)$ for $D > 0$ from Theorem 2.2 one obtains

$$L(1, \chi) = \frac{-1}{G} \sum_{n=1}^{N-1} \overline{\chi}(n) \log \left(1 - e\left(\frac{n}{N} \right) \right)$$

one obtains for $D > 0$:

$$h(D) = \frac{-1}{\log \epsilon_0} \sum_{n=1}^{N-1} \chi_D(n) \log \left(1 - e\left(\frac{n}{N} \right) \right).$$

Hence

$$\epsilon_0^{h(D)} = \prod_{n=1}^{D-1} \left(1 - e\left(\frac{n}{N}\right)\right)^{-\chi_D(n)}$$
$$= \prod_{\substack{0<n<D \\ \chi_D(n)=-1}} \left(1 - e\left(\frac{n}{N}\right)\right) \prod_{\substack{0<n<D \\ \chi_D(n)=1}} \left(1 - e\left(\frac{n}{N}\right)\right)^{-1}.$$

The last formula can also be useful for computing the class number, for example let $D = 8$. Then $\epsilon_0 = 3 + \sqrt{8}$ and

$$(3+\sqrt{8})^{h(8)} = \left(1 - e\left(\frac{3}{8}\right)\right)\left(1 - e\left(\frac{5}{8}\right)\right)\left(1 - e\left(\frac{1}{8}\right)\right)^{-1}\left(1 - e\left(\frac{7}{8}\right)\right)^{-1} = 3 + \sqrt{8}.$$

Therefore $h(8) = 1$.
One can show that for $D > 0$ the right hand side of the equation above has the form $\frac{t+u\sqrt{D}}{2}$, with $t^2 - u^2D = 4$ and hence is a proper power of ϵ_0.

If $D < 0$, then one can also show that

$$h(D) = \frac{-w/2}{|D|} \sum_{n=1}^{|D|-1} \chi_D(n)n$$

is an integer.
Proof for $D = -p < -3$, p a prime: Consider the following partition of the sum

$$\frac{1}{|D|} \sum_{n=1}^{|D|-1} \chi_D(n)n = \frac{1}{p}\sum_{n=1}^{p-1} \chi_{-p}(n)n = \frac{1}{p}\left(\sum_{\substack{n \not\equiv x^2\ (p) \\ n=1}}^{p-1} n - \sum_{\substack{n \equiv x^2\ (p) \\ n=1}}^{p-1} n \right). \tag{6}$$

We have that

$$\sum_{\substack{n \not\equiv x^2\ (p) \\ n=1}}^{p-1} n + \sum_{\substack{n \equiv x^2\ (p) \\ n=1}}^{p-1} n = \sum_{n=1}^{p-1} n = \frac{p(p-1)}{2} \equiv 0 \ \ (p)$$

and

$$2 \sum_{\substack{n \equiv x^2\ (p) \\ n=1}}^{p-1} n \equiv \sum_{n=1}^{p-1} n^2 = \frac{p(p-1)(2p-1)}{6} \equiv 0 \ \ (p).$$

The last two formulas both yield positive integers which are divisible by p. Then their difference will also be a an integer divisible by p, which shows that (6) is an integer. □

Theorem 2.4. *Let $D < -4$ be a fundamental discriminant. Then*

$$h(D) = \frac{1}{2 - \chi_D(2)} \sum_{0<k<\frac{|D|}{2}} \chi_D(k). \tag{7}$$

Since $h(D) > 0$ this means that there is more numbers in the interval $[0, \frac{|D|}{2}]$ with $\chi_D(k) = 1$, than with $\chi_D(k) = -1$, and the 'difference' is equal to $h(D), 2h(D)$ or $3h(D)$, depending on whether D is equal to $1 \ \ (8)$, $0 \ \ (4)$ or $5 \ \ (8)$.
Proof: Assume that D is odd. Let $Q := \sum_{n=1}^{|D|-1} \chi_D(n)n$. If n is even, then write $n = 2k$ with

$0 < k < \frac{|D|}{2}$, and if n is odd, write $n = 2k - |D|$ with $\frac{|D|}{2} < k < |D|$. Then

$$\begin{aligned} Q &= \sum_{0<k<\frac{|D|}{2}} \chi_D(2k) \cdot 2k + \sum_{\frac{|D|}{2}<k<|D|} \chi_D(2k - |D|)(2k - |D|) \\ &= \sum_{0<k<\frac{|D|}{2}} \chi_D(2k)2k + \sum_{\frac{|D|}{2}<k<|D|} \chi_D(2k)(2k - D) \\ &= 2 \sum_{0<k<|D|} \chi_D(2k)k - |D| \sum_{\frac{|D|}{2}<k<|D|} \chi_D(2k) \\ &= 2\chi_D(2)Q - |D|\chi_D(2) \sum_{\frac{|D|}{2}<k<|D|} \chi_D(k). \end{aligned}$$

Since $\chi_D(2) = \pm 1$ and $\sum_{0<k<\frac{|D|}{2}} \chi_D(k) = 0$, one has that

$$Q = \frac{-|D|}{2 - \chi_D(2)} \sum_{0<k<\frac{|D|}{2}} \chi_D(k).$$

Since $h(D)$ for $D < 0$ is equal to $\frac{-1}{|D|}Q$, this finishes the proof. □

Example 3 Let $D = -19$. Then applying Theorem 2.4 one obtains that for $p \in [0, 9]$ the integers smaller than 9 which are not equivalent to a square modulo -19 are $2, 3$ and 8, and the integers in $[0, 9]$ equivalent to a square modulo -19 are $1, 4, 5, 6, 7, 9$. Therefore $h(-19) = \frac{1}{3}(6 - 3) = 1$. Similarly one obtains $h(-23) = 3$.
We have seen that $h(D)$ attains different integer values, among those also $h(D) = 1$. One can ask, does $h(D)$ attain the value 1 infinitely many times?
For $D < 0$ the answer is: no, and the only integers $D < 0$ for which $h(D) = 1$ are

$$-3, -4, -7, -8, -11, -19, -43, -163.$$

This was showed by Heegner in [2] and we only give a reference here.
Another interesting question is about the growth of $h(D)$. Heilbronn showed that $h(D) \longrightarrow \infty$ as $D \longrightarrow -\infty$. Siegel showed afterwards that this result can be sharpened. He proved that for $\epsilon > 0$ there exists a $C > 0$ such that

$$h(D) > C|D|^{\frac{1}{2}-\epsilon}, \text{ for } D < 0.$$

Since there are finitely many equivalence classes of quadratic forms with a square-free discriminant, there exists a constant $C' > 0$ such that

$$h(D) < C'|D|^{\frac{1}{2}+\epsilon}, \text{ for } D < 0.$$

Altogether, one obtains

$$\lim_{D \longrightarrow \infty} \frac{\log h(D)}{\log |D|} = \frac{1}{2}.$$

Now, lets turn our attention to $D > 0$. Siegel showed (similarly to above) that

$$C|D|^{\frac{1}{2}-\epsilon} < h(D) \log \epsilon_0 < C'|D|^{\frac{1}{2}+\epsilon}, \text{ as } D \longrightarrow \infty$$

or equivalently

$$\lim_{D \longrightarrow \infty} \frac{\log(h(D) \log \epsilon_0)}{\log D} = \frac{1}{2}.$$

This result does not imply that $h(D) \longrightarrow \infty$, as $D \longrightarrow \infty$, since ϵ_0 can grow very large, comparing to D (for example for $D = 97$, one has $\epsilon_0 = 628096233 + 6377352\sqrt{97}$). But looking at 'rough' calculated table values, we only assume that $h(D)$ attains the value 1 infinitely many times for a fundamental discriminant $D \longrightarrow +\infty$. If we remove the restriction for D to be a fundamental discriminant and allow any discriminant, then there are infinitely many $D > 0$ with $h(D) = 1$.

In particular, for every $i > 0$ one has $h(5^{2i+1}) = 1$.

Proof: Let D be be a nonzero arbitrary discriminant which is equivalent to 0 or 1 modulo 4. One can write $D = D_0 \cdot r^2$, for D_0 a fundamental discriminant and $r \in \mathbb{N}$. One has that

$$\chi_D(m) = \begin{cases} \chi_{D_0}(m) & (m,r)=1 \\ 0 & otherwise. \end{cases}.$$

Then rewrite

$$h(D) = \frac{\sqrt{D}}{\log \epsilon_D} L(1,\chi_D) = \frac{r\sqrt{D_0}}{\log \epsilon_D} L(1,\chi_D).$$

Let $\mathcal{U}_D$ be the group generated by the solutions of Pells equation $t^2 - Du^2 = 4$. Analogously define $\mathcal{U}_{D_0}$. Since both $\mathcal{U}_D$ and $\mathcal{U}_{D_0}$ are cyclic groups and $D = D_0 \cdot r^2$, we have that $\mathcal{U}_D$ is a subgroup of $\mathcal{U}_{D_0}$. Then if the fundamental unit in $\mathcal{U}_D$ is given by ϵ, we have that the fundamental unit in $\mathcal{U}_D$ is given by

$$\epsilon_0^{[\mathcal{U}_{D_0} : \mathcal{U}_D]}.$$

Let $\upsilon_r := [\mathcal{U}_{D_0} : \mathcal{U}_D]$. Then

$$h(D_0) = \frac{\sqrt{D_0}}{\upsilon_r^{-1} \log \epsilon} L(1,\chi_{D_0})$$

and since one has that

$$L(1,\chi_D) = \prod_p \left(1 - \frac{\chi_D(p)}{p}\right)^{-1}$$
$$= \prod_p \left(1 - \frac{\chi_{D_0}(p)}{p}\right)^{-1} \prod_{q|r} \left(\left(1 - \frac{\chi_{D_0}(q)}{q}\right)^{-1}\right)^{-1}.$$

Altogether

$$h(D_0) = \frac{\sqrt{D_0}}{\upsilon_r^{-1}\log\epsilon} \prod_p \left(1 - \frac{\chi_{D_0}(p)}{p}\right)^{-1}$$
$$= \frac{\upsilon_r}{r} \prod_{p|r} \left(1 - \frac{\chi_{D_0}(p)}{p}\right)^{-1} h(D).$$

Now, let $D = 5 \cdot 5^{2i}$. We have seen before that $h(5) = 1$. Since $p = 5$ is the only prime dividing 5^{2i}, we have that

$$\prod_{p|5^{2i}} \left(1 - \frac{\chi_{D_0}(p)}{p}\right)^{-1} = 1.$$

Moreover $\upsilon_r = 5^i$, since the smallest solution of Pells equation $t^2 - 5^{2i+1}u^2 = 4$ can be obtained by looking at $t^2 - 5\tilde{u}^2 = 4$, where $\tilde{u} = 5^i u$. Therefore $h(5^{2i+1}) = \frac{\upsilon_r}{r} = \frac{5^i}{5^i} = 1$. Since i was arbitrary we have found infinitely many positive discriminants $D > 0$ for which $h(D) = 1$.

There is not much more known for $D > 0$, but there are results about asymptotic mean values of $h(D)$ and sums of $h(D)$, for D between 0 and $\pm N$. One has for $N \longrightarrow +\infty$ that

$$\sum_{\substack{-N<D<0 \\ D\equiv 0\ (4)}} h(D) \sim \frac{\pi}{42\zeta(3)} N^{\frac{3}{2}},$$

$$\sum_{\substack{0<D<N \\ D\equiv 0\ (4)}} h(D)\log\epsilon_0 \sim \frac{\pi^2}{42\zeta(3)} N^{\frac{3}{2}}.$$

One can also look at those sums for all D between 0 and $\pm N$ (not only $\equiv 0 \pmod 4$) and obtain similar asymptotic formulas as above, with 42 replaced by 18.

References

[1] D. Zagier, *Zetafunktionen und quadratische Korper*

[2] Kurt Heegner, *Diophantische Analysis und Modulfunktionen* Math. Z., 56:227–253, 1952.