Seminar L-functions:
The Zeta Function of a Quadratic Field

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The significance of the Riemann Zeta function comes from its Euler's product expression

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(\frac{1}{1-p^{-s}} \right), \tag{1}$$

which is a consequence, and in fact can be understood as an analytical formulation, of the fact that each natural number can be uniquely decomposed as a product of prime numbers. Since $\mathbb{Z}$ is a principal ideal ring, this can also be understood as the fact that each ideal has a unique factorisation in prime ideals, if we regard the Riemann Zeta function as the sum of $1/N(\mathfrak{a})^s$, over all proper ideals $\mathfrak{a} \subset \mathbb{Z}$, with $N(\mathfrak{a}) = |n|$ whenever $\mathfrak{a} = (n) := n\mathbb{Z}$.

The scope of this script is to generalise this construction and consider more general rings, in particular those of a quadratic field $K = \mathbb{Q}(\sqrt{d})$ (an algebraic number field of degree 2 over the rationals $\mathbb{Q}$), where $d \in \mathbb{Z}$ can be chosen square free. We recall that $\mathcal{D} \subset K$ is the ring of integers of K, that is, the set of rational roots of monic polynomials in $\mathbb{Z}[X]$, and that every element of K can be uniquely written as $\alpha + \beta\sqrt{\mathrm{d}}$, with $\alpha, \beta \in \mathbb{Q}$. Furthermore, the norm of $x = \alpha + \beta\sqrt{\mathrm{d}} \in K$ is defined to be $N(x) = xx'$, where $x' = \alpha - \beta\sqrt{\mathrm{d}}$ is the conjugate of x so that $N(x) = \alpha^2 - \beta^2 d$. Lastly, the trace of x is $\mathrm{tr}(x) = x + x' = 2\alpha$, so that $x = \alpha + \beta\sqrt{\mathrm{d}} \in K$ belongs to $\mathcal{D}$ if and only if both its trace and norm belong to $\mathbb{Z}$.

We also recall that, depending on the particular value of $d \mod 4$, we have different bases for $\mathcal{D}$:

$$\mathcal{D} = \begin{cases} \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \sqrt{\mathrm{d}} & \text{if } d \equiv 2, 3 \mod 4, \\ \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \frac{1+\sqrt{\mathrm{d}}}{2} & \text{if } d \equiv 1 \mod 4. \end{cases} \tag{2}$$

Borrowing last week's notation, we call $\omega = \sqrt{\mathrm{d}}$ if $d \equiv 2, 3 \mod 4$ and $\omega = \frac{1+\sqrt{\mathrm{d}}}{2}$ if $d \equiv 1 \mod 4$, so that $\{1, \omega\}$ is a basis of $\mathcal{D}$ as a $\mathbb{Z}$-module, denoted $\mathcal{D} = [1, \omega]$. Recalling that the discriminant D of K is defined as

$$D = \det \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix}^2,$$

where $\mathcal{D} = [\alpha, \beta]$ is any basis, we have that $D = 4d$ or $D = d$ for each of the corresponding cases above. For that reason, D is a fundamental discriminant.!!!?????

Before generalising the Riemann Zeta function we recall what the norm of an ideal $\mathfrak{a} \subset \mathcal{D}$ is $N(\mathfrak{a}) = [\mathcal{D} : \mathfrak{a}]$ (the order of the finite group $\mathcal{D} / \mathfrak{a}$). Likewise, its discriminant is

$$D(\mathfrak{a}) = \det \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix}^2$$

for any base such that $\mathfrak{a} = [\alpha, \beta]$. We recall that the norm is also multiplicative for ideals, $N(\mathfrak{a}\,\mathfrak{b}) = N(\mathfrak{a})N(\mathfrak{b})$, and that for any given $\xi \in \mathcal{D}$ we have $N((\xi)) = |N(\xi)|$ (where $(\xi) := \xi\,\mathcal{D}$ is the ideal generated by ξ). We obviously have that $D(\mathcal{D}) = D$, but also that $D(\mathfrak{a}) = N(\mathfrak{a})^2 D$ and that for an ideal $\mathfrak{a}$ and its conjugate ideal $\mathfrak{a}' = \{x' | x \in \mathfrak{a}\}$, we have $\mathfrak{a}\,\mathfrak{a}' = (N(\mathfrak{a}))$.

With all of this in mind it is therefore natural to define:

Definition 1. Given a quadratic field $K = \mathbb{Q}(\sqrt{\mathrm{d}})$ we define for $\mathrm{Re}(s) > 1$ formally its *Dedekind Zeta function*

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{D}} \frac{1}{N(\mathfrak{a})^s}, \tag{3}$$

where $\mathfrak{a}$ runs over all proper ideals of $\mathcal{D}$.

Remark 2. The Riemann Zeta function can be regarded as the Dedekind Zeta function for the particular case $K = \mathbb{Q}$, i.e. $D = 1$, since in that case it is well known that $\mathcal{D} = \mathbb{Z}$, that is, the roots of monic polynomials with rational coefficients are in fact the integers.

Dedekind Zeta functions share many of the properties of the Riemann Zeta function:

Proposition 3. *The Dedekind Zeta function ζ_K of a quadratic field $K = \mathbb{Q}(\sqrt{\mathrm{d}})$ has*

(i) abscissa of convergence $\sigma_0 = 1$,

(ii) a single pole at $s = 1$ as only singularity,

(iii) a functional equation under $s \to 1 - s$,

(iv) rational values at $s = 0, -1, -2, \ldots$.

We will only prove the first claim. To do that, we note the following. As we have said, ideals in $\mathcal{D}$ have unique factorisation in prime ideals, and since ideal norm is multiplicative, by exactly the same argument employed when expressing the Riemann Zeta function as the Euler's product 1 we obtain the corresponding Euler product for the Dedekind Zeta function of a quadratic field K:

Proposition 4. *Given a quadratic field $K = \mathbb{Q}(\sqrt{\mathrm{d}})$ we have for $Re(s) > 1$ the representation*

$$\zeta_K(s) = \prod_{\mathfrak{p} \subset \mathcal{D} \ \textit{prime ideal}} \left(\frac{1}{1 - N(\mathfrak{p})^{-s}} \right), \tag{4}$$

whenever any of both sides converges.

Proof of (i). We start by noticing that every prime ideal $\mathfrak{p} \subset \mathcal{D}$ must divide the ideal generated by a natural prime number $p \in \mathbb{N}$, $\mathfrak{p}\,|(p)$ (recall that divisibility among ideals is equivalent to inclusion, that is, $\mathfrak{a}\,|\,\mathfrak{b}$ if and only if $\mathfrak{b} \subset \mathfrak{a}$). To see this, we notice that $\mathfrak{p}\,|\,\mathfrak{p}\,\mathfrak{p}' = (N(\mathfrak{p}))$, so that $\mathfrak{p}$, being a prime ideal, must divide one the prime numbers that represent the prime factorisation of $N(\mathfrak{p})$ in $\mathbb{Z}$. This can alternatively be seen as a consequence of the fact that $\mathfrak{p} \cap \mathbb{Z} \neq \emptyset$, since for any $x \in \mathfrak{p}$, then $xx' = N(x) \in \mathbb{Z}$ while also $xx' \in x\,\mathcal{D} \subset \mathfrak{p}$. It follows that

$$N(\mathfrak{p})|N((p)) = N(p) = pp' = p^2.$$

Since p is prime, this can only mean that either $N(\mathfrak{p}) = p$ or $N(\mathfrak{p}) = p^2$ ($N(\mathfrak{p}) \neq 1$ for any prime ideal since $\mathfrak{p} \neq \mathcal{D}$, for it is a proper ideal). We now consider the prime ideal factorisation of (p) in $\mathcal{D}$,

$$(p) = \mathfrak{p}_1 \ldots \mathfrak{p}_r\,.$$

Taking norms in both sides and using the multiplicativity of the norm, we have

$$p^2 = N((p)) = N(\mathfrak{p}_1) \ldots N(\mathfrak{p}_r).$$

Since we are back on $\mathbb{Z}$, this implies that $r \leq 2$ and that we have only two possibilities:

(i) either $r = 1$ and (p) is a prime ideal of $\mathcal{D}$, so that $\mathfrak{p} = (p)$ and $N(\mathfrak{p}) = p^2$,

(ii) or $r = 2$ and $(p) = \mathfrak{p}_1\,\mathfrak{p}_2$ with $N(\mathfrak{p}_i) = p$ for $i = 1, 2$ (notice again we cannot have $N(\mathfrak{p}_i) = 1$ for any prime ideal $\mathfrak{p}_i$). We can of course take $\mathfrak{p}_1 = \mathfrak{p}$ and write $(p) = \mathfrak{p}\,\mathfrak{p}_2$.

In the first case we have

$$\prod_{\mathfrak{p}|(p),\ \mathfrak{p}\ \text{prime}} \left(\frac{1}{1 - N(\mathfrak{p})^{-s}}\right) = \left(\frac{1}{1 - p^{-2s}}\right);$$

while on the second case, if $\mathfrak{p}_1 \neq \mathfrak{p}_2$ we have

$$\prod_{\mathfrak{p}|(p),\ \mathfrak{p}\ \text{prime}} \left(\frac{1}{1 - N(\mathfrak{p})^{-s}}\right) = \left(\frac{1}{1 - p^{-s}}\right)^2;$$

but if $\mathfrak{p}_1 = \mathfrak{p}_2$ we have

$$\prod_{\mathfrak{p}|(p),\ \mathfrak{p}\ \text{prime}} \left(\frac{1}{1 - N(\mathfrak{p})^{-s}}\right) = \left(\frac{1}{1 - p^{-s}}\right).$$

Either way, we obtain the estimation:

$$\prod_{\mathfrak{p}|(p),\ \mathfrak{p}\ \text{prime}} \left|\frac{1}{1 - N(\mathfrak{p})^{-s}}\right| \leq \left|\frac{1}{1 - p^{-\sigma}}\right|^2.$$

Since every prime ideal divides the ideal generated by some prime number, $\zeta(s)^2$ is an upper bound for $\zeta_K(s)$. Hence, we have absolute convergence for the product, and with it for the Dedekind Zeta function, for $\sigma = \mathrm{Re}(s) > 1$. □

We now notice that we can express the Dedekind Zeta function in the following manner:

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{F(n)}{n^s}, \tag{5}$$

where we define

$$F(n) := \#\{\mathfrak{a} \subset \mathcal{D} \mid \text{proper ideal}, \; N(\mathfrak{a}) = n\}, \tag{6}$$

that is, as the number of occurrences of the number n as the norm of an ideal from $\mathcal{D}$.

Remark 5. In particular, this shows that $\zeta_K(s)$ is a Dirichlet series in the sense we have previously discussed.

We now obtain an alternative characterisation of the Dirichlet coefficients of the series.

Proposition 6. *The number $F(n)$ is equal to the number $R(n)$, defined as the number non-equivalent representations of the number n by quadratic forms of discriminant D.*

Proof. We start by recalling that a fractional ideal $\mathfrak{a}$ is an additive subgroup of K that is finitely generated and satisfies $\mathcal{D}\,\mathfrak{a} \subset \mathfrak{a}$. The norm of such $\mathfrak{a}$ is defined as $N(\mathfrak{a}) = \frac{1}{n^2} N(n\,\mathfrak{a}) \in \mathbb{Q}$ for any $n \in \mathbb{N}$ such that $n\,\mathfrak{a}$ is an ideal of $\mathcal{D}$. Discriminant, conjugate and product of fractional ideals are defined just as for usual ideals. We also recall that the fractional ideals $\mathfrak{a}, \mathfrak{b}$ are said to be equivalent whenever there exists $\xi \in K$ such that $\mathfrak{a} = \xi\,\mathfrak{b}$, and furthermore they are equivalent in the strict sense when we also have $N(\xi) > 0$. Let $h = h(D)$ be the number of distinct equivalence classes of fractional ideals in the strict sense, and $\{A_1, \ldots, A_h\}$ be such equivalence classes.

We recall the result stating that if $D \neq 1$ is a fundamental discriminant, then there is a bijective correspondence between the equivalence classes of binary quadratic forms with discriminant D (positive definite forms, if $D < 0$), and the strict equivalence classes of fractional ideals of K (this in particular implies that h is finite). The correspondence is given simply by assigning, to the fractional ideal $\mathfrak{a} = [\alpha, \beta]$, the quadratic function:

$$\xi \in \mathcal{D} \mapsto \phi(\xi) = \frac{N(\xi)}{N(\mathfrak{a})} \in \mathbb{Z}, \tag{7}$$

which can be regarded as a function over $\mathbb{Z}^2$ that takes on integer values

$$(x, y) \in \mathbb{Z}^2 \mapsto f(x, y) = \frac{N(x\alpha + y\beta)}{N(\mathfrak{a})} \in \mathbb{Z}\,. \tag{8}$$

Then, this quadratic function does not depend on the basis of $\mathfrak{a}$ nor on the representative $\mathfrak{a}$ of the strict equivalence class. Let f_i be the quadratic forms associated in this way to each class A_i.

It is clear that we have

$$\zeta_K(s) = \sum_{i=1}^{h} \zeta(A_i, s), \tag{9}$$

if for a given strict equivalence class A we denote

$$\zeta(A, s) = \sum_{\mathfrak{a} \in A, \mathfrak{a} \subset \mathcal{D}} \frac{1}{N(\mathfrak{a})^s} = \sum_{n=1}^{\infty} \frac{F(A, n)}{n^s}, \tag{10}$$

(with $\mathfrak{a} \subset \mathcal{D}$ we remind that here only ideals from $\mathcal{D}$ are considered, not fractional ones) where we define

$$F(A,n) := \#\{\mathfrak{a} \in A | N(\mathfrak{a}) = n\}. \tag{11}$$

Therefore, it is enough to show that $F_i(n) := F(A_i, n) = R(n, f_i)$, where $R(n, f)$ is defined to be the number of distinct representations of n via the quadratic form f (the cardinal of $f^{-1}(n)$), since obviously $R(n) = \sum_i^h R(n, f_i)$.

To show that $F_i(n) = R(n, f_i)$, we start noticing that the inverse class $A^{-1} = \{\mathfrak{a}^{-1} \,|\, \mathfrak{a} \in A\}$ of the inverse ideals of a given class A is equal to the conjugate class $A' = \{\mathfrak{a}' \,|\, \mathfrak{a} \in A\}$, $A^{-1} = A'$, so that:

$$\zeta(A^{-1}, s) = \zeta(A', s) = \sum_{\mathfrak{a} \in A, \mathfrak{a} \subset \mathcal{D}} \frac{1}{N(\mathfrak{a}')^s} = \sum_{\mathfrak{a} \in A, \mathfrak{a} \subset \mathcal{D}} \frac{1}{N(\mathfrak{a})^s} = \zeta(A, s),$$

since $N(\mathfrak{a}') = N(\mathfrak{a})$.

Let us now consider an ideal $\mathfrak{a}$, A its strict equivalence class and the associated quadratic form f. For any $\mathfrak{b} \in A^{-1}$, $\mathfrak{a}\,\mathfrak{b} = (\xi)$ is a principal ideal for some ξ with positive norm $N(\xi) > 0$. Conversely, given $\xi \in K$ with $N(\xi) > 0$ we have the fractional ideal $\mathfrak{b} = \xi\,\mathfrak{a}^{-1}$ belonging to A^{-1}, so that $\mathfrak{b}$ is an ideal from $\mathcal{D}$ if and only if $\mathfrak{a}\,|(\xi)$, i.e. $\xi \in \mathfrak{a}$. Therefore, the correspondence

$$\{\xi \in \mathfrak{a}\,| N(\xi) > 0\} \to \{\mathfrak{b} \in A^{-1}, \mathfrak{b} \subset \mathcal{D}\},$$
$$\xi \mapsto \xi\,\mathfrak{a}^{-1},$$

is well defined and surjective. Furthermore, two elements ξ_1, ξ_2 have the same image only if $\xi_1 | \xi_2$ and $\xi_2 | \xi_1$, so that $\xi_1 = \varepsilon \xi_2$ for some unit $\varepsilon \in \mathcal{D}$ which must also have positive norm $N(\varepsilon) > 0$. Hence, we have the bijection:

$$\{\xi \in \mathfrak{a}\,| N(\xi) > 0\}/U_+ \cong \{\mathfrak{b} \in A^{-1}, \mathfrak{b} \subset \mathcal{D}\},$$

where U_+ are the units with positive norm

$$U_+ = \{\varepsilon \in \mathcal{D}\,|\varepsilon^{-1} \in \mathcal{D}, N(\varepsilon) > 0\}.$$

Under this correspondence, we in particular have that

$$N(\mathfrak{b}) = N(\xi\,\mathfrak{a}^{-1}) = N(\xi) N(\mathfrak{a})^{-1}. \tag{12}$$

We therefore have

$$\zeta(A, s) = \zeta(A^{-1}, s) = \sum_{\mathfrak{b} \in A^{-1}, \mathfrak{b} \subset \mathcal{D}} \frac{1}{N(\mathfrak{b})^s} = \sum_{\xi \in \mathfrak{a}\,/U_+, N(\xi) > 0} \frac{N(\mathfrak{a})^s}{N(\xi)^s}.$$

Recalling the definition of the quadratic form 7, we finally see that

$$\zeta(A, s) = \sum_{(x,y) \in \mathbb{Z}^2\,/U_+, f(x,y) > 0} \frac{1}{f(x,y)^s}, \tag{13}$$

where we have identified $\mathfrak{a}$ with $\mathbb{Z}^2$ via an oriented basis (i.e. a basis such that $\mathfrak{a} = [\alpha, \beta]$ with $(\alpha'\beta - \alpha\beta')/\sqrt{D} > 0$).

But 13 can be now directly factored as

$$\zeta(A,s) = \sum_{n=1}^{\infty} \frac{R(n,f)}{n^s}, \tag{14}$$

so that finally we arrive at $F(A,n) = R(n,f)$, and in particular $F_i(n) = R(n,f_i)$. □

We will now give an elementary expression for the Dedekind Zeta function making use of its Euler's product expression 1. For that reason, we want to understand with precision how the ideal generated by a prime number (p) decomposes in prime ideals in $\mathcal{D}$, so as to apply it in each factor $(1-p^{-s})^{-1}$. We have already showed that there are only two possible cases, but we now further subdivide the second of these cases and give them separate names:

(i) $(p) = \mathfrak{p}$ is a prime ideal with $N(p) = p^2$. We call p an *inert* prime number.

(ii) $(p) = \mathfrak{p}_1\,\mathfrak{p}_2$ with $\mathfrak{p}_1 \neq \mathfrak{p}_2$ prime ideals and $N(\mathfrak{p}_1) = N(\mathfrak{p}_2) = p$. We call p a *totally split* prime number.

(iii) $(p) = \mathfrak{p}^2$ with $\mathfrak{p}$ a prime ideal and $N(\mathfrak{p}) = p$. We call p a *ramified* prime number.

Furthermore, since $\mathfrak{p}\,\mathfrak{p}' = (N(\mathfrak{p}))$, in the last two cases we can always take $\mathfrak{p}_1 = \mathfrak{p}$ and $\mathfrak{p}_2 = \mathfrak{p}'$, so that it is a totally split prime number if $\mathfrak{p} \neq \mathfrak{p}'$ and it is a ramified prime number if $\mathfrak{p} = \mathfrak{p}'$.

In the next result we show that which of the three cases we have depends exclusively on the value of $\chi_D(p)$, where D is the discriminant of K and χ_D the associated primitive character.

Theorem 7. *Let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic field and let $p \in \mathbb{Z}_{>0}$ be a prime. Letting (p) be the principal ideal in $\mathcal{D}$ generated by p we obtain*

$$(p) = \mathfrak{p}\mathfrak{p}',\ \ \mathfrak{p} \neq \mathfrak{p}' \iff \chi_D(p) = 1 \tag{15}$$

$$(p) = \mathfrak{p}^2 \iff \chi_D(p) = 0 \tag{16}$$

$$(p) = \mathfrak{p} \iff \chi_D(p) = -1, \tag{17}$$

where here $\mathfrak{p}$ and $\mathfrak{p}'$ are prime ideals in $\mathcal{D}$.

Proof. First assume that $p \neq 2$. We will proof (16) and (15) from which (17) is a consequence. Assume that $(p) = \mathfrak{p}^2$. Then we have $N(\mathfrak{p}) = p$. Because $\mathfrak{p} \neq \mathfrak{p}^2$ there exist some $a, b \in \mathbb{Z}$ such that $x = \frac{a+b\sqrt{D}}{2} \in \mathfrak{p}$ but $x \notin \mathfrak{p}^2$. Let $x' = \frac{a-b\sqrt{D}}{2} \in \mathfrak{p}$ be the conjugate of x. Then we may write $a = x + x'$ and $b\sqrt{D} = x - x'$ and see that $a, b\sqrt{D} \in \mathfrak{p}$. Hence,

$$a \in \mathfrak{p} \Rightarrow a^2 \in \mathfrak{p}^2 = (p) \Rightarrow p|a^2 \Rightarrow p|a$$

and

$$b\sqrt{D} \in \mathfrak{p} \Rightarrow (b\sqrt{D})^2 \in \mathfrak{p}^2 = (p) \Rightarrow p|b^2D \Rightarrow p|b \text{ or } p|D$$

Because by assumption $x \notin \mathfrak{p}^2 = (p)$ we can not have $a \in (p)$ and $b \in (p)$. We conclude that $p|D$. That is $\chi_D(p) = 0$.

Now assume that $\chi_D(p) = 0$, which means $p|D$. Because D or $\frac{D}{4}$ is squarefree we have $p^2 \not| D$. Therefor we can write $D = pD_1$ with $p \not| D_1$. From

$$(\sqrt{D})^2 = (D) = (p)(D_1)$$

it follows that (p) and (D_1) are square ideals. We obtain $(p) = \mathfrak{p}^2$ as desired and (16) follows.

Now assume that $(p) = \mathfrak{p}\,\mathfrak{p}'$ with $\mathfrak{p} \neq \mathfrak{p}'$. By (16) we have that $p \nmid D$. Let $R = \mathcal{D} / \mathfrak{p}$. Then $|R| = p$, and hence R is a field of order p. For $x \in R^\times$ we have $x^{p-1} = 1$, i.e.

$$x \in \mathcal{D},\ \mathfrak{p} \nmid x \Rightarrow x^{p-1} = 1\ (mod\ \ \mathfrak{p}).$$

Because $p \nmid D$ and hence $p \nmid \sqrt{D}$, we obtain choosing $x = \sqrt{D}$ that

$$D^{\frac{p-1}{2}} = 1\ (mod\ \ \mathfrak{p}).$$

Note that for any $a \in \mathbb{Z}$ we have $\mathfrak{p}\,|(a)$ if and only if $p|a$. In particular we get

$$D^{\frac{p-1}{2}} = 1\ (mod\ p).$$

We conclude $\chi_D(p) = 1$.

Finally, assume that p is a prime with $\chi_D(p) = 1$. Then there exists some $x \in \mathbb{Z}$ such that $x^2 = D\ mod\ p$. For a contradiction assume that $p|(x - \sqrt{D})$, then $p|(x + \sqrt{D})$. Taking the difference of these two it follows that $p|2\sqrt{D}$ which is a contradiction to $p \neq 2$ and $p \nmid D$. Hence,

$$(x - \sqrt{D})(x + \sqrt{D}) = x^2 - D \in (p)$$

is the product of two elements which are not in (p). Therefor $(p) = \mathfrak{p}\,\mathfrak{p}'$. By (16) we can only have $(p) = \mathfrak{p}^2$. The case of $p = 2$ is analog. □

This characterization of χ_D now allows us to prove the product decomposition of the Zeta function over a quadratic field K.

Corollary 8. *Let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic field and let $F(n) = |\{\mathfrak{a}; \mathfrak{a} \subset \mathcal{D}\ ideal,\ N(\mathfrak{a}) = n\}|$. Then*

$$F(n) = \sum_{m|n} \chi_D(m) \tag{18}$$

and obtain the decomposition

$$\zeta_K(s) = \zeta(s)L(s, \chi_D). \tag{19}$$

Proof. We first prove (18). Observe that since $F(n)$ and $\sum_{m|n} \chi_D(m)$ are both multiplicative it is enough to show equality for powers of primes $n = p^k$ for some k. With Theorem 7 we can now do a case distinction.
(i) If $\chi_D(p) = 1$, then $(p) = \mathfrak{p}\,\mathfrak{p}'$, $\mathfrak{p} \neq \mathfrak{p}'$, $N(\mathfrak{p}) = N(\mathfrak{p}') = p$ and

$$p^k = N(\mathfrak{p}^k) = N(\mathfrak{p}^{k-1}\,\mathfrak{p}') = \ldots = N(\mathfrak{p}'^{\,k}).$$

Hence, there are $k+1$ ideals with norm p^k and we get

$$F(p^k) = k+1 = \sum_{i=0}^{k} \chi_D(p^i).$$

(ii) If $\chi_D(p) = 0$, then $(p) = \mathfrak{p}^2$, $N(\mathfrak{p}) = p$ and there is only one ideal with norm p^k, namely $\mathfrak{p}^k$. Hence,

$$F(p^k) = 1 = \sum_{i=0}^{k} \chi_D(p^i).$$

(iii) If $\chi_D(p) = -1$, then $(p) = \mathfrak{p}$ and $N(\mathfrak{p}) = p^2$. If $k = 2l+1$, then there are no ideals with norm $p^k = p^{2l+1}$. We obtain

$$F(p^k) = 0 = \sum_{i=0}^{k} (-1)^i = \sum_{i=0}^{k} \chi_D(p^i).$$

Else if $k = 2l$, then there is only 1 ideal $(\mathfrak{p}^l)$ with norm $p^k = p^{2l}$. We obtain

$$F(p^k) = 1 = \sum_{i=0}^{k} (-1)^i = \sum_{i=0}^{k} \chi_D(p^i).$$

This yields (18). Finally, using the multiplication formula for Dirichlet series we conclude,

$$\begin{aligned}
\zeta(s)L(s,\chi_D) &= \left(\sum_{k=1}^{\infty} \frac{1}{k^s}\right)\left(\sum_{k=1}^{\infty} \frac{\chi_D(k)}{k^s}\right) \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{k^s}\left(\sum_{m|k} \chi_D(m)\cdot 1\right)\right) \\
&= \sum_{k=1}^{\infty} \frac{F(n)}{k^s} \\
&= \zeta_K(s).
\end{aligned}$$

□

Theorem 9. *Let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic field and $\sim$ be the equivalence relation in the strict sense on the set of fractional ideals of K. Let $A = [\mathfrak{a}]_\sim$ be an equivalence class and for $Re(s) > 1$ define*

$$\zeta(A,s) = \sum_{\mathfrak{a}\in A, \mathfrak{a}\subset\mathcal{D}} \frac{1}{N(\mathfrak{a})^s}.$$

Then $\zeta(A,s)$ has a meromorphic extension on the halfplane $\{Re(s) > \frac{1}{2}\}$ with a single pole at $s = 1$ with residue

$$res_{s=1}\zeta(A,s) = \kappa = \begin{cases} \frac{2\pi}{w\sqrt{|D|}}, & \text{if } D < 0 \\ \frac{\log \epsilon_0}{\sqrt{D}}, & \text{if } D > 0, \end{cases}$$

where w is the order of the unit group U and ϵ_0 is the Grundeinheit.

The proof of this statement is omitted, since it is essentially the same as the proof of the classformula in Chapter 8. We now turn our focus to more general characters and their related L-Series.

Definition 10. Let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic field and define

$$C = \{\text{fractional ideals of K}\}/\{\text{Principal ideals of K}\}.$$

An idealclass character is a character on C, i.e. $\chi : \{\text{fractional ideals of K}\} \to \mathbb{C}$ with

$$\chi(\mathfrak{a}\mathfrak{b}) = \chi(\mathfrak{a})\chi(\mathfrak{b}) \text{ for fractional ideals } \mathfrak{a}, \mathfrak{b}$$
$$\chi((\alpha)) = 1 \text{ for } \alpha \in K, N(\alpha) > 0.$$

We define its L-series for $\mathrm{Re}(s) > 1$ as

$$L_K(s, \chi) = \sum_{0 \neq \mathfrak{a} \subset \mathcal{D}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}.$$

Analogue to the Euler product representation of an L-series of a character we obtain the Euler product representation of an L-series of an idealclass character of a quadratic field.

Theorem 11. *Let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic field. Then we have the product representation*

$$L_K(s, \chi) = \prod_{\mathfrak{p} \subset \mathcal{D} \text{ prime ideal}} \left(1 - \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1}.$$

Proof. Because of the multiplicity of the idealclass character $\chi(\mathfrak{a}\,\mathfrak{b}) = \chi(\mathfrak{a})\chi(\mathfrak{b})$ we can invoke our theory on Dirichlet series and obtain the product formula. □

We have seen that for a character $\chi \neq \chi_0$ we have $L(1, \chi) \neq 0$. We can now generalize this statement for idealclass characters.

Theorem 12. *For every non-trivial idealclass character χ, we have $L_K(1, \chi) \neq 0$*

The proof of the Theorem is the same as the one seen in the case $K = \mathbb{Q}$. As a corollary of the theory developed in this chapter we obtain the following.

Corollary 13. *Let D be a fundamental discriminant. Then any quadratic form of D represents infinitely many primes, i.e. if $f(x, y) = ax^2 + bxy + cy^2$ is a quadratic form of the discriminant $D = b^2 - 4ac$, then the set*

$$\{p \in \mathbb{Z}; \exists x, y : f(x, y) = p,\ p \text{ prime}\}$$

is infinite.

References

[Zag81] Don Bernard Zagier. *Quadratische Körper und ihre Zetafunktionen.* Springer, 1981. Chap. 11, pp. 96–108.