Seminar Modular Forms

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General informations

The seminar takes place **Fridays** from **12-14** via Zoom, starting on **26.02.** until **04.06.** (13 talks; there are **no talks on 02.04. and 09.04.**). The Zoom link will be sent by email.

The talks should take about 100 minutes. Two students share a talk. For your presentation you can use slides (e.g. Beamer LaTex) or write on a tablet, for example. It will not be required to prepare extended lecture notes of your talk, but your slides or your handwritten notes should be made available to the participants on the website of the seminar after your talk, so keep this in mind during preparation.

Contact

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Topics

1. The modular group and modular forms (S. Horvath and T. Knötzsch)

Introduce the modular group $\mathrm{SL}_2(\mathbb{Z})$ and its action on the upper half-plane by fractional linear transformations (Moebius transformations); prove that T and S generate $\mathrm{SL}_2(\mathbb{Z})$; sketch the fundamental domain and explain its properties; if time permits, explain elliptic points; define the factor of automorphy and modular forms; explain what Fourier expansions are and give the integral formula for their coefficients; prove the Hecke bound and that there are no modular forms of negative weight.

References: [4], Section 2.1 – 2.2

2. Eisenstein series and the Delta function (I. Krstic and E. Rossi)

Introduce the Eisenstein series G_k: show convergence, modularity and the computation of the Fourier expansion; define the normalized Eisenstein series E_k and give its alternative definition using the slash operator; explain that one can write every modular form as an Eisenstein series plus a cusp form; give the evaluation of the Riemann zeta function at even natural numbers in terms of Bernoulli numbers, and use this to show that E_k has rational Fourier coefficients; introduce Ramanujan's Delta function and show that it is a cusp form of weight 12; mention the following things without proof: product expansion, multiplicative coefficients, some congruences, Ramanujan conjecture (Deligne bound), Lehmer's conjecture.

References: [4], Section 2.3 – 2.4

3. The valence formula and the structure of M_k (A. Imparato and D. Rusch)

Explain the order of a modular form at a point in the upper half-plane and at ∞; state, explain, and prove the valence formula; show that multiplication with Δ yields isomorphism $M_k \cong S_{k+12}$

and use this to prove the structure theorem for M_k for small k; prove the dimension formula for M_k and that it has a basis of products of Eisenstein series.

References: [4], Section 2.5

4. The j-invariant, the Eisenstein series of weight 2, and the Dedekind eta function (J. Hauenstein and J. Kock)

Define the j-invariant and determine its orders; define modular functions and show that every modular function is a rational function in j; mention that $j : \Gamma\backslash\mathbb{H} \to \mathbb{C}$ is a bijection; define the holomorphic (but non-modular) Eisenstein series G_2 and compute its Fourier expansion; prove its modular transformation law; define the non-holomorphic (but modular) Eisenstein series G_2^* and the normalized versions E_2 and E_2^*; define the Dedekind eta function η as an infinite product, and show that $\frac{\eta'}{\eta} = \frac{i}{4\pi}G_2$; deduce the modular transformation properties of η; show that $\eta^{24} = \Delta$ and thereby prove the product expansion of Δ.

References: [4], Section 2.6 – 2.8

5. Modular forms for congruence subgroups and the four-squares-theorem (E. Dubno and S. Zbinden)

Define congruence subgroups, in particular $\Gamma(N), \Gamma_1(N), \Gamma_0(N)$ and show that they have finite index in $\mathrm{SL}_2(\mathbb{Z})$; write down the explicit formula for their indices; introduce cusps; define modular forms for congruence subgroups, and explain what their expansion at different cusps are; introduce Eisenstein series for $\Gamma_0(N)$, in particular of weight 2; investigate the trace $\mathrm{tr}(f)$ and product $\pi(f)$; prove Sturm's bound on the dimension of spaces of modular forms for congruence subgroups; introduce the Jacobi theta function and prove its transformation under $\tau \mapsto -\frac{1}{4\tau}$ using Poisson summation; show that $\vartheta^4 \in M_2(\Gamma_0(4))$, but omit some details of the proof if necessary; state and prove the four-squares-theorem (omit some technical details, but make it clear how modular forms, in particular the theta function and Eisenstein series, come into play).

References: [4], Section 2.9– 2.10

6. The Petersson inner product and Poincaré series; Hecke operators I (Y. Ammann and C. Barcia)

Introduce the hyperbolic volume element and the Petersson inner product of two modular forms (one of which is a cusp form); prove its basic properties; show that the inner product is independent of the choice of fundamental domain; prove that Eisenstein series are orthogonal to cusp forms; define Poincaré series and show that they are cusp forms; prove that the inner product of a cusp form f with the m-th Poincaré series gives (essentially) the m-th coefficient of f; show that the Poincaré series span the space of cusp forms; introduce the set $\mathcal{M}_n$ of integral matrices with determinant n, and give a system of representatives for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{M}_n$; define Hecke operators and prove its action on the Fourier expansion; infer that Hecke operatos define endomorphisms of M_k and S_k.

References: [4], Section 3.1–3.3

7. Hecke operators II (L. Herren and M. Salerno)

Recall the definition and basic properties of Hecke operators from the last talk; explain what a simultaneous Hecke eigenform is and show what this means on the level of Fourier coefficients; show that Δ is a simultaneous eigenform and prove that its coefficients are multiplicative; show that the

Eisenstein series E_k is an eigenform, and state the formula for the action of Hecke operators on Poincaré series; prove that the algebra of Hecke operators is commutative and generated by the T_p for primes p, and give the precise composition laws (omit some details of the proof if necessary); show that the Hecke operators are self-adjoint with respect to the Petersson inner product; use this to show that S_k has an orthonormal basis of simultaneous Hecke eigenforms.

References: [4], Section 3.3–3.5

8. L-functions of modular forms (R. Pfitscher and A. Isakovic)

Define Dirichlet series and their convergence properties; define the Mellin transform and its inverse, and prove their relation; introduce the L-function $L_f(s)$ of a modular form f, and its completion $\Lambda_f(s)$; prove its meromorphic continuation (including the location of its poles and the residues) and functional equation; deduce that the L-function of a cusp form is entire; prove Hecke's converse theorem; prove the Euler product representation of the L-function of a simultaneous Hecke eigenform; give the L-function of E_k in terms of the Riemann zeta function.

References: [4], Section 4.1–4.2

9. Rankin L-functions and the Kronecker limit formula (M. Gröbner and M. Reho)

Define the non-holomorphic Eisenstein series $G(\tau, s)$ and compute its Fourier expansion (sketch the proof of the Lipschitz formula it time permits); mention the connection to the Epstein zeta function; introduce the normalized non-holomorphic Eisenstein series $G^*(\tau, s)$, and prove its meromorphic continuation and functional equation; prove the Kronecker limit formula; define the Rankin L-function of two cusp forms, and show its relation to the non-holomorphic Eisenstein series $G^*(\tau, s)$ via the Petersson inner product; use this to prove the meromorphic continuation and functional equation of the Rankin L-function; if time permits, state the Euler product representation of the Rankin L-function of two simultaneous Hecke eigenforms.

References: [4], Section 4.3

10. The singular values of the j-function (S. Bär and K. Lucca)

Extend the action of Hecke operators to modular functions; prove existence and uniqueness of the modular polynomial F_n; give F_1 and F_2 as examples; show that $F_j(X, j)$ is irreducible over the field of modular functions, and that $F_n(X, Y) = F_n(Y, X)$; prove that $j(\tau)$ is algebraic if τ lies in an imaginary quadratic field; introduce binary quadratic forms and traces of modular functions; state the modularity of the generating series of traces of singular moduli due to Zagier, and briefly explain what weight $\frac{3}{2}$ means; if time permits, give a very brief sketch of the proof of Zagier's result.

References: [4], Section 5.1– 5.2

11. Differential operators on modular forms (K. Andritsch and T. Haupt)

Introduce the derivative $Df = f'$ and prove the transformation behaviour (52) of f'; introduce the Serre derivative ϑ_k given in (53); define the ring of quasimodular forms for $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$; prove Proposition 15; introduce the 'raising operator' ∂_k in (55) and show that it raises the weight by 2; give the general relations (56) and (57) between the iterated raising operator and the differential D without proof; define Rankin-Cohen brackets as in (59) and prove Proposition 18; describe the application to identities for sums of powers of divisors; give the general definition of almost holomorphic modular forms and quasimodular forms and prove Proposition 20; if time permits, you can present one of the applications given in Chapter 5, e.g. from the Section 'Modular forms satisfy non-linear differential equations' or 'Exotic multiplication of modular forms'.

References: [7], Chapter 5, Section 5.1–5.3

12. Periods of modular forms (E. Eichelberg and H. Gui)

Define the periods $r_s(f)$ of a cusp form f; explain the period relations under the action of S and ST, and illustrate how they look like (see equations ES1–ES3 in [3]); state the Eichler-Shimura theorem without proof; compute the action of Hecke operators on periods and prove Manin's theorem (Theorem 4.1 in Chapter V of [3]) in as much detail as time permits.

References: [3], Chapter V. See also Section 1.1 in [2] for an overview.

13. The sphere packing problem in dimension 8 (A. Franklín and J. Rossier)

Explain the sphere packing problem, including a bit of its history, and describe the recent breakthrough solution of the problem in dimension 8 due to Maryna Viazovska [5]. Sketch the outline of the proof, and try to make it clear how modular forms come into play. Also mention the corresponding result in dimension 24 [6].

References: [5]

References

[1] Koecher, Krieg, *Elliptische Funktionen und Modulformen*

[2] Kohnen, Zagier, *Modular forms with rational periods*

[3] Lang, *Introduction to modular forms*

[4] Schwagenscheidt, *Modular forms*, lecture notes, available online

[5] Viazovska, *The sphere packing problem in dimension* 8

[6] Cohn, Kumar, Miller, Radchenko, Viazovska, *The sphere packing problem in dimension* 24

[7] Zagier's part of the book *The 1-2-3 of modular forms*