Large financial markets, discounting, and no asymptotic arbitrage

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Abstract

For a large financial market (which is a sequence of usual, "small" financial markets), we introduce and study a concept of no asymptotic arbitrage (of the first kind) which is invariant under discounting. We give two dual characterisations of this property in terms of (1) martingale-like properties for each small market plus (2) a contiguity property, along the sequence of small markets, of suitably chosen "generalised martingale measures". Our results extend the work of Rokhlin and of Klein/Schachermayer and Kabanov/Kramkov to a discounting-invariant framework. We also show how a market on $[0, \infty)$ can be viewed as a large financial market and how no asymptotic arbitrage, both classic and in our new sense, then relates to no-arbitrage properties directly on $[0, \infty)$.

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1 Introduction

A large financial market is a sequence of usual (small) financial markets. This structure naturally comes up when one considers markets with (countably) infinitely many assets and studies their behaviour along approximating finite markets. One early motivation came from arbitrage pricing theory (APT) and factor models; see Ross [32], Huberman [16], Chamberlain/Rothschild [5]. Another is to view infinite-horizon models as limits of finite-horizon models.

The existing literature on large financial markets has several strands. Some recent papers have studied aspects of superreplication and utility maximisation (Baran [4], De Donno et al. [10], Rásonyi [28, 29], Roch [30]), transaction costs (Klein et al. [25]) or insider trading (Chau et al. [7]). A larger and more established strand studies absence-of-arbitrage (AOA) properties, and this is where our paper fits in.

The earliest AOA notion in a large financial market framework is *no asymptotic arbitrage of the first kind* (NAA or NAA1); it is due to Kabanov/Kramkov [19] who also gave a dual characterisation for the case of a sequence of complete markets. This was generalised to incomplete markets by Klein/Schachermayer [26] and Kabanov/Kramkov [20]; see also Klein/Schachermayer [27]. The stronger AOA condition of no asymptotic free lunch (NAFL) was introduced by Klein [23] and subsequently studied in more detail for continuous processes in Klein [24]. Exploiting the work of Karatzas/Kardaras [21], Rokhlin [31] managed to reduce the assumptions imposed on each small market and obtained a more general dual characterisation of NAA1. In a very recent paper, Cuchiero et al. [9] provide a unified analysis of NAA1 and NAFL together with dual characterisations, in the framework of one fixed stochastic basis. Other directions include an FTAP with an equivalent martingale measure on a projective limit space (Balbás/Downarowicz [2]), explicit constructions of asymptotic arbitrage strategies in specific diffusion settings (Dokuchaev/ Savkin [14]), or markets with a stochastically changing dimension (Strong [33]).

A unifying disadvantage of all the existing literature on large financial markets is that its formulations and results depend very strongly on the choice of the asset used to discount prices. In fact, characterising AOA properties by dual descriptions typically yields some kind of martingale property under an equivalent measure, but for the discounted, not for the original prices. Moreover, whether or not a given market is judged to be arbitrage-free very often depends, via the chosen AOA concept, on the asset used for discounting. These issues already appear in the classic Black–Scholes model for a single small market (see[1] [3, Example 1.1]), and Example 2.6 below illustrates that they only become worse in a sequence of markets.

Our goal is to develop and study an AOA concept which does not suffer from these

[1] After the present paper was written and accepted for publication, we have revised our work for small markets. The present paper refers to the version of [3] which is dated June 22, 2019. Because the order as well as the precise formulation of the results on small markets has changed during the revision, it is important to refer to the correct version.

problems — it should be *discounting-invariant* in the sense that AOA, with one choice of discounting process, implies AOA with respect to any other discounting process. For the case of one single small market, this has been implemented by Bálint/Schweizer [3], who introduced the concept of *dynamic share viability (DSV)* as a discounting-invariant form of an AOA condition and provided several dual characterisations of this property. The present paper focuses on large financial markets; it introduces a similar asymptotic AOA concept and exploits the results in [3] to provide again dual characterisations, now of course expressed in terms of the large market.

The paper is structured as follows. Section 2 fixes notation, presents a motivating example and recalls or extends a number of small market results. In Section 3, we introduce our new concept of *asymptotic strong share maximality* for large market strategies, use it to define *asymptotic dynamic share viability (ADSV)* and provide some preliminary results. Section 4 contains our main results, which are two dual characterisations of ADSV for general large financial markets. Theorem 4.1, extending the work of Rokhlin [31], describes ADSV via supermartingale properties of the wealth processes in each small market; Theorem 4.5 generalises Klein/Schachermayer [26] and Kabanov/Kramkov [20] and obtains local martingale properties for the sequence of underlying assets themselves. In both cases, as in the classic works [19, 26, 20], one has in addition a contiguity property along the sequence of small markets. The general results are specialised in Section 5 to markets on $[0, \infty)$ viewed as large markets and applied to the Black–Scholes example from Section 2. Finally, Section 6 contains a longish counterexample which shows that even if a strategy on $[0, \infty)$ is not strongly share maximal in the small market on $[0, \infty)$, the sequence of its restrictions to $[0, n]$ can be asymptotically strongly share maximal in the corresponding large market.

2 Preliminaries

The best-known absence-of-arbitrage concept for large markets is NAA. It was introduced by Kabanov/Kramkov [19] and also used in Klein/Schachermayer [26], Rokhlin [31] and Cuchiero et al. [9], among others. NAA means that there is no sequence of strategies with

$$\lim_{n\to\infty} (\text{initial wealth in market } n) = 0,$$
$$\limsup_{n\to\infty} P^n[(\text{final wealth in market } n) \geq 1] > 0.$$

In addition, one imposes for each small market an AOA property — the existence of an equivalent local martingale measure (ELMM) in [19, 20], and the existence of a supermartingale deflator in [31].

Like its small market counterpart NA1 = NUPBR, the concept NAA lacks stability with respect to discounting, even in very simple cases. This is illustrated in Section 2.2. Moreover, to the best of our knowledge, the time horizon in each small market is restricted to be finite in the existing large financial market literature. Finally, while NAA provides an asymptotic (in n) AOA property, it does not ensure any AOA property for the small markets; this must be assumed separately. All this provides ample scope for generalisation.

2.1 Framework

A *small market* is a triple $(\mathbb{B}, S, \zeta)$ consisting of a stochastic basis $\mathbb{B}$, a price process S and a time horizon ζ. Here, $\mathbb{B} = (\Omega, \mathcal{F}, \mathbb{F}, P)$ with a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions of right-continuity and P-completeness. We set $\mathcal{F}_\infty := \bigvee_{t\geq 0} \mathcal{F}_t = \sigma(\bigcup_{t\geq 0} \mathcal{F}_t)$ and assume that $\mathcal{F}_0$ is P-trivial and $\mathcal{F} = \mathcal{F}_\zeta$. The time horizon ζ is a general stopping time which can as usual take the value $+\infty$; we might even have $\zeta \equiv +\infty$. The price process S is an $\mathbb{R}^N$-valued semimartingale (chosen RCLL as usual) with $N \geq 2$ and defined on the stochastic interval

$$[\![0, \zeta]\!] = \{(\omega, t) \in \Omega \times [0, \infty) : 0 \leq t \leq \zeta(\omega)\}.$$

If $(\mathbb{B}, S, \zeta)$ is a small market, $\Theta^{\mathrm{sf}}(S)$ denotes the space of all $\mathbb{R}^N$-valued integrands $\vartheta \in L(S)$ satisfying $V(\vartheta, S) := \vartheta \cdot S = \vartheta_0 \cdot S_0 + \int \vartheta \,\mathrm{d}S =: \vartheta_0 \cdot S_0 + \vartheta \bullet S$ P-a.s., and $V(\vartheta, S)$ is the *value process* of the *self-financing strategy* ϑ, in the same units as S. If in addition $V(\vartheta, S) \geq 0$ P-a.s., we write $\vartheta \in \Theta^{\mathrm{sf}}_+(S)$. Here, $x \cdot y$ is the scalar product of $x, y \in \mathbb{R}^N$.

We extend all stochastic processes to $[\![0, \infty]\!] = [\![0, \infty[\![= \Omega \times [0, \infty)$, almost always by keeping them constant on $[\![\zeta, \infty]\!]$, with one important exception. To concatenate two strategies $\vartheta^1, \vartheta^2 \in \Theta^{\mathrm{sf}}(S)$ at some stopping time τ, we sometimes define, for a mapping F, a new strategy of the form $I_{[\![0,\tau]\!]}\vartheta^1 + I_{]\!]\tau,\infty]\!]}F(\vartheta^1, \vartheta^2)$. On the set $\{\tau = \zeta < \infty\}$, this is then constant for $t > \zeta(\omega)$, but not necessarily for $t \geq \zeta(\omega)$.

From now on, we assume that all processes are defined on $[\![0, \infty]\!]$ (but not necessarily on $\Omega \times [0, \infty]$). If a process Y is constant on $[\![\zeta, \infty]\!]$, we then have

$$\inf_{t\geq 0} Y_t(\omega) = I_{\{\zeta(\omega)=\infty\}} \inf_{0\leq t<\infty} Y_t(\omega) + I_{\{\zeta(\omega)<\infty\}} \inf_{0\leq t\leq \zeta(\omega)} Y_t(\omega),$$

$$\liminf_{t\to\infty} Y_t(\omega) = I_{\{\zeta(\omega)=\infty\}} \liminf_{t\to\infty} Y_t(\omega) + I_{\{\zeta(\omega)<\infty\}} Y_\zeta(\omega), \tag{2.1}$$

etc. Of course, if we write $\lim_{t\to\infty} Y_t$, we must make sure that this limit exists on $\{\zeta = \infty\}$.

Many of our results involve *discounting*, i.e., dividing prices by strictly positive processes. We define $\mathcal{S} := \{$all real-valued semimartingales$\}$ and set $\mathcal{S}_+ := \{D \in \mathcal{S} : D \geq 0\}$ and $\mathcal{S}_{++} := \{D \in \mathcal{S} : D > 0,\ D_- > 0\}$. (For any RCLL process Y, we set $Y_{0-} := Y_0$.) Elements $D \in \mathcal{S}_{++}$ are called *discounters*, and we note that $1/D \in \mathcal{S}_{++}$ if $D \in \mathcal{S}_{++}$. For $D \in \mathcal{S}_{++}$, we call S/D the *D-discounted prices*.

For D-discounted prices $\tilde{S} = S/D$, we have $V(\vartheta, \tilde{S}) = \vartheta \cdot \tilde{S} = V(\vartheta, S)/D$, the value process of ϑ in the currency units of $\tilde{S}$. It is shown in [15, Lemma 2.9] that if $\vartheta \in \Theta^{\mathrm{sf}}(S)$, then both $\vartheta \in L(\tilde{S})$ and $V(\vartheta, \tilde{S}) = \vartheta_0 \cdot \tilde{S}_0 + \vartheta \bullet \tilde{S}$ hold. Thus $\Theta^{\mathrm{sf}}(S) = \Theta^{\mathrm{sf}}(\tilde{S})$ does not depend on currency units even if value processes do. However, we still keep the argument S or $\tilde{S}$ because we use different markets in the sequel. We also need the spaces $\Theta^{\mathrm{sf}}_+(S) := \{\vartheta \in \Theta^{\mathrm{sf}}(S) : V(\vartheta, S) \in \mathcal{S}_+\}$ and $\Theta^{\mathrm{sf}}_{++}(S) := \{\vartheta \in \Theta^{\mathrm{sf}}(S) : V(\vartheta, S) \in \mathcal{S}_{++}\}$;

they do not depend on currency units either. Finally, a process Y is called *S-tradable* if it is the value process of some self-financing strategy, i.e., $Y = V(\vartheta, S)$ for some $\vartheta \in \Theta^{\mathrm{sf}}(S)$.

Definition 2.1. Fix a small market $(\mathbb{B}, S, \zeta)$. A *reference strategy* for S is an $\eta \in \Theta^{\mathrm{sf}}_{++}(S)$ with $\eta \geq 0$ (η is long-only) and[2] such that the η-discounted price process $S^\eta := S/(\eta \cdot S)$ is bounded uniformly in $t \geq 0$, P-a.s.

In the sequel, we usually work under the assumption that there exists a reference strategy η. Because $V(\eta, S) \in \mathcal{S}_{++}$ by definition, a reference strategy is a *desirable investment*, and it is expressed in numbers of shares. Note that if we pass from S to discounted prices $\tilde{S} = S/D$ with any $D \in \mathcal{S}_{++}$, we get $\tilde{S}^\eta := \tilde{S}/(\eta \cdot \tilde{S}) = S^\eta$; hence the notion of a reference strategy is discounting-invariant. In particular, $(S^\xi)^\eta = S^\eta$ for any $\xi, \eta \in \Theta^{\mathrm{sf}}_{++}(S)$.

Remark 2.2. The existence of a reference strategy η is a very weak condition on the price process S. Indeed, consider the *market portfolio*, i.e. the strategy $\mathbb{1} := (1, \dots, 1) \in \mathbb{R}^N$ of holding one share of each asset. If we have nonnegative prices $S \geq 0$, then $\mathbb{1} \in \Theta^{\mathrm{sf}}_{+}(S)$ and all components of the $\mathbb{1}$-discounted price process $S^{\mathbb{1}} = S/\sum_{i=1}^N S^{(i)}$ have values in $[0, 1]$. If $S \geq 0$ and the sum $\mathbb{1} \cdot S = \sum_{i=1}^N S^{(i)}$ of all prices is strictly positive and has strictly positive left limits, we even have $\mathbb{1} \in \Theta^{\mathrm{sf}}_{++}(S)$ so that the market portfolio is then a reference strategy. (Note that this always holds if $S = (1, X)$ for a d-dimensional semimartingale $X \geq 0$.) However, it is useful to work with a general reference strategy η because this gives a clearer view on a number of aspects.

Definition 2.3. Fix a strategy $\eta \in \Theta^{\mathrm{sf}}(S)$. A strategy $\vartheta \in \Theta^{\mathrm{sf}}(S)$ is called an *η-buy-and-hold strategy* if it is of the form $\vartheta = c\eta$, where $c \in L^\infty(\mathcal{F}_0; \mathbb{R}^N)$ and the multiplication is componentwise.

Because $\mathcal{F}_0$ is trivial, ϑ is η-buy-and-hold if and only if it is a coordinatewise nonrandom multiple of η. If $\eta \equiv \mathbb{1}$ is the market portfolio, this reduces to the classic concept of buying and holding a fixed number of shares of each asset, with $\vartheta \equiv \vartheta_0 \in \mathbb{R}^N$. More generally, if η is a reference strategy, it is desirable to have $\eta^{(i)}_t$ shares of asset i at time t, and the above buy-and-hold concept is then a natural generalisation from the classic case of the market portfolio. Note that η itself is always an η-buy-and-hold strategy.

A *large market* is a sequence $(\mathbb{B}^n, S^n, \zeta^n)_{n \in \mathbb{N}}$ of small markets. In particular, each small market only contains a finite number N^n of assets. For compact notation, we write $(\mathbb{B}, S, \zeta)$ for a generic small market and $(\mathbb{B}^n, S^n, \zeta^n)_{n \in \mathbb{N}}$ for a large market. A *(large market) strategy* is a sequence $\vec{\vartheta} = (\vartheta^n)_{n \in \mathbb{N}}$ where each ϑ^n is in $\Theta^{\mathrm{sf}}_{+}(S^n)$, and we write $\vec{0} := (0^n)_{n \in \mathbb{N}}$ for the large market zero strategy, with $0^n := 0^{N^n} := (0, \dots, 0) \in \mathbb{R}^{N^n}$. We denote by $\mathbb{1}^n := \mathbb{1}^{N^n} := (1, \dots, 1) \in \mathbb{R}^{N^n}$ the *market portfolio* in the n-th small market. Sometimes, we use $\mathbb{1}^d := (1, \dots, 1) \in \mathbb{R}^d$, and we write just $\mathbb{1}$ if the dimension is clear

[2] When revising the paper [3], we have been able to remove the assumption of P-a.s. boundedness of S^η. So this can be omitted without loss. See, however, footnote 1.

from the context. Concerning indices, we use S^n for the n-th process from a sequence, $S^{(i)}$ for the i-th coordinate of a process S, and $S_{\cdot\wedge\tau}$ for a process S stopped at some time τ. We use e^i for the i-th unit vector. For $g \in L^0_+(P)$, we denote as in [31] by $g \cdot P$ the measure defined by $(g \cdot P)[A] := E^P[gI_A]$ for $A \in \mathcal{F}$.

The following definition is due to Kabanov/Kramkov [19]; see also Rokhlin [31].

Definition 2.4. A large market $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$ with $\zeta^n < \infty$ P^n-a.s. for each n *admits no asymptotic arbitrage* or *satisfies NAA* if $\limsup_{n\to\infty} P^n[V_{\zeta^n}(\vartheta^n, S^n) \geq 1] = 0$ for any large market strategy $\vec{\vartheta}$ with $\lim_{n\to\infty} V_0(\vartheta^n, S^n) = 0$.

Remark 2.5. 1) Typical bond markets with an uncountable number of maturities do not fit into the above framework and need a different approach.

2) In the spirit of Kabanov [17], some papers start directly from an abstract set of processes satisfying some structural properties and designed to describe the wealth processes one can obtain (in some underlying market) from self-financing trading; see e.g. Kardaras [22]. It has also been suggested that this could represent a description of a large financial market. While directly working with (abstract) wealth processes allows elegant proofs and gives a clear view on some underlying mathematical structures, it is a coarser approach because it no longer allows to disentangle the underlying basic assets from the trading activities in the market. Therefore it yields in general less precise results.

2.2 A motivating example

Example 2.6. Consider the classic Black–Scholes (BS) model of geometric Brownian motion. This is given, for constants $r \in \mathbb{R}$, $m \in \mathbb{R}$, $\sigma > 0$ and for $t \geq 0$, by

$$Y^{(1)}_t = e^{rt}, \qquad Y^{(2)}_t = e^{mt+\sigma W_t - \frac{1}{2}\sigma^2 t},$$

where $W = (W_t)_{t\geq 0}$ is a one-dimensional Brownian motion. Take any $(\Omega, \mathcal{F}, P)$ supporting W and let $\mathbb{F}$ be generated by Y (or W) and P-augmented; we set $\mathbb{B} := (\Omega, \mathcal{F}, \mathbb{F}, P)$.

In this example, it is usual to discount all prices by the bank account $Y^{(1)}$ and hence look at the process $Y/Y^{(1)} = (1, X)$ with

$$X_t = e^{(m-r)t+\sigma W_t - \frac{1}{2}\sigma^2 t}. \tag{2.2}$$

But one can also discount by the stock $Y^{(2)}$ and look at $Y/Y^{(2)} = (X', 1)$ with $X' := 1/X$.

For the large market, we take $\mathbb{B}^n \equiv \mathbb{B}$, $N^n \equiv 2$ and $\zeta^n := n$ for all n. Discounted asset prices are either $S^n := (I_{[\![0,n]\!]}, X_{\cdot\wedge n})$ or $(S')^n := (X'_{\cdot\wedge n}, I_{[\![0,n]\!]})$, and we introduce on $[\![0,\infty]\!]$ the process $S = (1, X)$ respectively $S' = (X', 1)$. One naturally hopes that any reasonable AOA property holds for one kind of discounting if and only if it holds for the other.

It is well known that for both choices of discounting, every small market admits an (even unique) ELMM; hence the assumptions in [19, 31] are satisfied. Moreover, Proposition 5.1 below proves that in this special setup, NAA along $(S^n)_{n\in\mathbb{N}}$ or $((S')^n)_{n\in\mathbb{N}}$ is equivalent to NUPBR for S respectively S' on $[\![0,\infty]\!]$. Now choose $m = r$. Then in the first discounting scenario, the value process of any strategy $\vartheta \in \Theta^{\mathrm{sf}}_{+}(S)$ is a supermartingale and so NUPBR holds for S. However, in the second discounting scenario, X'_t converges to $+\infty$ P-a.s. as $t \to \infty$, and hence NUPBR does not hold for S'. So we see that NAA can hold or fail, depending on the choice of discounting.

Example 2.6 shows that NAA crucially depends on how the units of price denomination evolve over time; so it is not discounting-invariant (see later after Definition 3.2 for a precise definition). This only becomes visible if we discount dynamically over time; just rescaling each small market at time 0 (maybe differently for each n) does not affect NAA. The next result makes this precise; its easy proof is left to the reader.

Lemma 2.7. *Take $c^n \in \mathbb{R}^{N^n}_{++}$ for each n. Then a large market $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$ satisfies NAA if and only if $(\mathbb{B}^n, S^n/c^n, \zeta^n)_{n\in\mathbb{N}}$ satisfies NAA, where the division is componentwise.*

2.3 Small market results

Small market notions naturally have a certain importance in a large market framework as well. In this section, we recall some small market terminology and results.

Definition 2.8. Let $(\mathbb{B}, S, \zeta)$ be a small market and define, for $a \geq 0$,

$$\begin{aligned}\mathcal{X}(S) &:= \{V(\vartheta, S) : \vartheta \in \Theta^{\mathrm{sf}}_{+}(S)\},\\ \mathcal{X}^a(S) &:= \{X \in \mathcal{X}(S) : X_0 = a\},\\ \mathcal{X}^1_\infty(S) &:= \Big\{\lim_{t\to\infty} X_t : X \in \mathcal{X}^1(S) \text{ and } \lim_{t\to\infty} X_t \text{ exists}\Big\}.\end{aligned}$$

Then S or $(\mathbb{B}, S, \zeta)$ satisfies *NUPBR* if $\mathcal{X}^1_\infty(S)$ is bounded in L^0.

Remark 2.9. 1) If $S = (1, X)$ for an $\mathbb{R}^d$-valued semimartingale, then

$$\mathcal{X}^a(S) = a + \{H \bullet X = \textstyle\int H \,\mathrm{d}X : H \in L(X) \text{ and } \int H\,\mathrm{d}X \geq -a\}.$$

As a consequence, our definition of NUPBR coincides with the classic concept from the literature if we consider the classic framework $S = (1, X)$; see e.g. [21, Definition 4.1].

2) In the classic framework, NUPBR has been studied both for models X indexed by $[0, T]$ with $0 < T < \infty$ and by $[0, \infty)$. Our approach with $[\![0, \zeta]\!]$ contains both as special cases. See however Remark 5.6 for a difference between finite and infinite horizons.

The next result shows that if we enlarge a market S by the value process of a self-financing strategy, we obtain the same set of wealth processes. This is used later.

Lemma 2.10. *If $\eta \in \Theta^{\mathrm{sf}}(S)$, then $\mathcal{X}(S) = \mathcal{X}(S, \eta \cdot S)$.*

Proof. See Appendix. □

Definition 2.11. Fix a small market $(\mathbb{B}, S, \zeta)$ and a discounter $D \in \mathcal{S}_{++}$. If $D_0 = 1$ and X/D is for each $X \in \mathcal{X}(S)$ a local/σ-/supermartingale, we call D a *local/σ-/supermartingale discounter (LMD/σMD/SMD) for $\mathcal{X}(S)$*. If these properties hold for S instead of all $X \in \mathcal{X}(S)$, D is called an *LMD/σMD/SMD for S*. For $\mathcal{E} \in \{\mathrm{L}, \sigma, \mathrm{S}\}$ and $\eta \in \Theta^{\mathrm{sf}}_{++}(S)$, an $\mathcal{E}\mathrm{MD}^{\eta+}$ is an $\mathcal{E}$MD D with the extra property $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s. (Note that an $\mathcal{E}\mathrm{MD}^{\eta+}$ can only differ from an $\mathcal{E}$MD if $P[\zeta = \infty] > 0$.)

Remark 2.12. As both $\eta \cdot S = V(\eta, S)$ and D are in $\mathcal{S}_{++}$ in Definition 2.11, so is $\eta \cdot (S/D)$. Thus $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s. is equivalent to $\liminf_{t\to\infty}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s.

The following result clarifies the connections between the different concepts just introduced. This is essentially known and easy to argue, but we include it for completeness. We also point out that our discounters are almost, but not exactly the reciprocals of the *deflators* in [21] (we have no local martingale property for $1/D$).

Lemma 2.13. *Let $(\mathbb{B}, S, \zeta)$ be a small market and D a discounter with $D_0 = 1$.*

1) *In general, we have*

$$\begin{array}{ccccc} D \text{ is LMD for } S & \Rightarrow & D \text{ is } \sigma\text{MD for } S & & D \text{ is SMD for } S \\ \Downarrow & & \Downarrow & & \\ D \text{ is LMD for } \mathcal{X}(S) & \Leftrightarrow & D \text{ is } \sigma\text{MD for } \mathcal{X}(S) & \Rightarrow & D \text{ is SMD for } \mathcal{X}(S) \end{array}$$

2) *If $S \geq 0$, then*

$$\begin{array}{ccccc} D \text{ is LMD for } S & \Leftrightarrow & D \text{ is } \sigma\text{MD for } S & \Rightarrow & D \text{ is SMD for } S \\ \Updownarrow & & \Updownarrow & & \Uparrow \\ D \text{ is LMD for } \mathcal{X}(S) & \Leftrightarrow & D \text{ is } \sigma\text{MD for } \mathcal{X}(S) & \Rightarrow & D \text{ is SMD for } \mathcal{X}(S) \end{array}$$

Proof. See Appendix. □

Remark 2.14. 1) We get the same statement in Lemma 2.13 if we replace the set $\mathcal{X}(S)$ by

$$\mathcal{X}_{\mathrm{adm}}(S) := \{Y = V(\vartheta, S) : \vartheta \in \Theta^{\mathrm{sf}}(S) \text{ and } Y \geq -a \text{ for some } a \geq 0\}.$$

In the classic setup $S = (1, X)$ and for $\mathcal{F}_0$ trivial, any $Y \in \mathcal{X}_{\mathrm{adm}}(S)$ is the sum of a constant and a stochastic integral $H \bullet X$ of some admissible $\mathbb{R}^d$-valued integrand H.

2) A missing arrow in Lemma 2.13 indicates that the corresponding implication does not hold in general. It is not hard to find counterexamples, and we leave this to the reader.

3) If $S \geq 0$ is continuous and D is S-tradable, one can show that D is an LMD for S if and only if it is an SMD for $\mathcal{X}(S)$. But there is no such result for general D.

We next recall from Bálint/Schweizer [3] maximality and AOA notions for small markets. The corresponding interpretations are also given in [3].

Definition 2.15. Fix a small market $(\mathbb{B}, S, \zeta)$ and a strategy $\eta \in \Theta^{\mathrm{sf}}(S)$. A strategy $\vartheta \in \Theta^{\mathrm{sf}}_{+}(S)$ is called *strongly share maximal (ssm) for* η if there is no $[0,1]$-valued adapted process $\psi = (\psi_t)_{t\geq 0}$ converging P-a.s. as $t \to \infty$ to some $\psi_\infty \in L^\infty_+ \setminus \{0\}$ and such that for every $\varepsilon > 0$, there exists some $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_{+}(S)$ with $V_0(\hat{\vartheta}^\varepsilon, S) \leq V_0(\vartheta, S) + \varepsilon$ and

$$\liminf_{t\to\infty}(\hat{\vartheta}^\varepsilon_t - \vartheta_t - \psi_t\eta_t) \geq 0 \qquad P\text{-a.s.}$$

A strategy $\vartheta \in \Theta^{\mathrm{sf}}_{+}(S)$ is called *terminally strongly share maximal (tssm) for* η if there is no $\psi_\infty \in L^\infty_+ \setminus \{0\}$ such that for every $\varepsilon > 0$, there exists some $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_{+}(S)$ with $V_0(\hat{\vartheta}^\varepsilon, S) \leq V_0(\vartheta, S) + \varepsilon$ and

$$\liminf_{t\to\infty}(\hat{\vartheta}^\varepsilon_t - \vartheta_t - \psi_\infty\eta_t) \geq 0 \qquad P\text{-a.s.}$$

Lemma 2.16. *Fix a small market* $(\mathbb{B}, S, \zeta)$ *and a strategy* $\eta \in \Theta^{\mathrm{sf}}(S)$*. If* η *is bounded uniformly in* $t \geq 0$*,* P*-a.s., then any* $\vartheta \in \Theta^{\mathrm{sf}}_{+}(S)$ *is ssm for* η *if and only if it is tssm for* η*.*

Proof. If we only have ψ_∞, we define an adapted process ψ by $\psi_t := E[\psi_\infty|\mathcal{F}_t]$ for $t \geq 0$. For both implications, we then get $\lim_{t\to\infty}\psi_t = \psi_\infty$ P-a.s. (either by assumption or by martingale convergence) and hence $\lim_{t\to\infty}(\psi_\infty - \psi_t)\eta_t = 0$ because η is bounded, P-a.s. So $\hat{\vartheta}^\varepsilon - \vartheta - \psi\eta$ and $\hat{\vartheta}^\varepsilon - \vartheta - \psi_\infty\eta$ have the same lim inf as $t \to \infty$, and the result follows. □

Remark 2.17. By using [3, Theorem 3.4], one can show that tssm for η and ssm for η are also equivalent if $S \geq 0$ and η is a reference strategy (but not necessarily bounded). However, the proof needs considerably more work and we do not give it here.

Definition 2.18. Fix a small market $(\mathbb{B}, S, \zeta)$ and a strategy $\eta \in \Theta^{\mathrm{sf}}(S)$. We say that S satisfies *dynamic share viability (DSV) for* η if the zero strategy $0 \in \Theta^{\mathrm{sf}}_{+}(S)$ is strongly share maximal for η, and *DSE (DSE) for* η if every η-buy-and-hold strategy $\vartheta \in \Theta^{\mathrm{sf}}_{+}(S)$ is strongly share maximal for η.

We first obtain from [3, Theorem 2.11] a dual characterisation of dynamic share viability for small markets.

Proposition 2.19. *Fix a small market* $(\mathbb{B}, S, \zeta)$*. If* $S \geq 0$ *and there exists a reference strategy* $\eta \in \Theta^{\mathrm{sf}}_{++}(S)$*, then* S *satisfies DSV for* η *if and only if there exists an LMD*$^{\eta+}$ D *for* S*.*

Proof. Because $S \geq 0$, [3, Theorem 2.11] and Lemma 2.13, 2) imply that dynamic share viability for η holds if and only if there exists an LMD D for S with $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s. This gives the result. □

Our next result uses and extends [21, Theorem 4.12] to give another dual characterisation of dynamic share viability for a small market.

Proposition 2.20. *Fix a small market* $(\mathbb{B}, S, \zeta)$. *If* $S \geq 0$ *and there exists a reference strategy* $\eta \in \Theta^{\mathrm{sf}}_{++}(S)$, *then the following are equivalent:*

(a) *S satisfies DSV for η.*

(b) *There exists an SMD$^{\eta+}$ D for $\mathcal{X}(S)$.*

(c) *There exists an S-tradable SMD$^{\eta+}$ $\bar{D}$ for $\mathcal{X}(S)$.*

Moreover, $\bar{D}$ is unique if it exists.

Proof. By [3, Theorem 2.14], S satisfies DSV for η if and only if the η-discounted price process $S^\eta = S/(\eta \cdot S)$ satisfies NUPBR, i.e., $\mathcal{X}^1_\infty(S^\eta)$ is bounded in L^0. By (the proof of) Lemma 2.10 and due to $\eta \cdot S^\eta \equiv 1$, we have $\mathcal{X}^a(S^\eta) = \mathcal{X}^a(1, S^\eta)$, and this has several consequences. First, S^η satisfies NUPBR if and only if $(1, S^\eta)$ satisfies NUPBR. Second, a discounter $G \in \mathcal{S}_{++}$ is an SMD for $\mathcal{X}(1, S^\eta)$ if and only if it is an SMD for $\mathcal{X}(S^\eta)$. Third, G is $(1, S^\eta)$-tradable if and only if it is S^η-tradable.

Now we want to use [21, Theorem 4.12] for $(1, S^\eta)$. As pointed out in [21, Section 4.8], that result holds even if S respectively S^η are not necessarily strictly positive semimartingales (i.e., in $\mathcal{S}^N_{++}$). More precisely, define for any semimartingale $\tilde{S} \in \mathcal{S}^N$ the sets

$$\mathcal{Y}^a(\tilde{S}) := \{X \in \mathcal{X}^a(\tilde{S}) : X > 0 \text{ and } X_- > 0\} = \mathcal{X}^a(\tilde{S}) \cap \mathcal{S}_{++},$$
$$\mathcal{Y}^1_\infty(\tilde{S}) := \Big\{ \lim_{t\to\infty} X_t : X \in \mathcal{Y}^1(\tilde{S}) \text{ and } \lim_{t\to\infty} X_t \text{ exists} \Big\}.$$

Then [21, Theorem 4.12] in conjunction with the comment in [21, Section 4.8] implies that the following are equivalent:

(A) $\mathcal{Y}^1_\infty(1, S^\eta)$ is bounded in L^0.

(B) There exists $Z \geq 0$ with $Z_0 = 1$, $\lim_{t\to\infty} Z_t > 0$ P-a.s. and such that ZY is a supermartingale for all $Y \in \mathcal{Y}^1(1, S^\eta)$.

(C) There exists $\bar{Z}$ with the same properties as Z in (B) and in addition $1/\bar{Z} \in \mathcal{Y}^1(1, S^\eta)$.

Because $\mathcal{Y}^1(1, S^\eta)$ contains $Y \equiv 1$, Z in (B) is a supermartingale and hence $Z > 0$ and $Z_- > 0$ by the minimum principle for supermartingales (see Dellacherie/Meyer [13, Theorem VI.17]). So Z is in $\mathcal{S}_{++}$ and so is then $G := 1/Z$; in particular, $\lim_{t\to\infty} Z_t > 0$ P-a.s. is equivalent to $\inf_{t\geq 0} 1/G_t > 0$ P-a.s. The same applies to $\bar{Z}$ in (C).

For any $\varepsilon > 0$ and $X \in \mathcal{X}^1(1, S^\eta)$, the process $Y := (1-\varepsilon)X + \varepsilon$ is in $\mathcal{Y}^1(1, S^\eta)$. This first implies $\mathcal{Y}^1(1, S^\eta) \subseteq \mathcal{X}^1(1, S^\eta) \subseteq (\mathcal{Y}^1(1, S^\eta) - \varepsilon)/(1-\varepsilon)$ so that $\mathcal{Y}^1_\infty(1, S^\eta)$ is bounded in L^0 if and only if $\mathcal{X}^1_\infty(1, S^\eta)$ is. In consequence, (a) is equivalent to (A). Second, for any $m \in \mathbb{N}$ and $X \in \mathcal{X}^1(1, S^\eta)$, we have $Y^m := (1 - \frac{1}{m})X + \frac{1}{m} \in \mathcal{Y}^1(1, S^\eta)$ and $\frac{m}{(m-1)} Y^m = X + \frac{1}{m-1}$. So if ZY is a supermartingale for all $Y \in \mathcal{Y}^1(1, S^\eta)$, monotone convergence gives $E[Z_t X_t \,|\, \mathcal{F}_s] = \lim_{m\to\infty} E[Z_t(X_t + \frac{1}{m}) \,|\, \mathcal{F}_s] \leq \lim_{m\to\infty} Z_s(X_s + \frac{1}{m}) = Z_s X_s$ for $s \leq t$ so that ZX is also a supermartingale for all $X \in \mathcal{X}^1(1, S^\eta)$. The converse is clear because $\mathcal{Y}^1(1, S^\eta) \subseteq \mathcal{X}^1(1, S^\eta)$. If Z is as in (B), then $G := 1/Z$ is an SMD for $\mathcal{X}(1, S^\eta)$ with $\inf_{t\geq 0} 1/G_t > 0$ P-a.s. Because $\eta \cdot S^\eta \equiv 1 > 0$, $1/G$ is thus an SMD$^{\eta+}$ for $\mathcal{X}(S^\eta) = \mathcal{X}(1, S^\eta)$. Thus (B) is equivalent to

(B′) There exists an SMD$^{\eta+}$ G for $\mathcal{X}^1(S^\eta)$.

Because $(1, S^\eta)$-tradability is the same as S^η-tradability, (C) is analogously equivalent to

(C′) There exists an S^η-tradable SMD$^{\eta+}$ $\bar{G}$ for $\mathcal{X}^1(S^\eta)$.

So up to here, we have shown that (a), (B′) and (C′) are all equivalent.

It remains to pass from S^η to S. Take an SMD$^{\eta+}$ G for $\mathcal{X}(S^\eta)$ and define $D := (\eta \cdot S)G$. Then $V(\vartheta, S)/D = V(\vartheta, S^\eta)/G$ is a supermartingale for any $\vartheta \in \Theta^{\mathrm{sf}}_+(S) = \Theta^{\mathrm{sf}}_+(S^\eta)$, and so D is an SMD for $\mathcal{X}(S)$. Moreover, $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) = \inf_{t \geq 0} 1/G_t > 0$ P-a.s. shows that D is an SMD$^{\eta+}$ for $\mathcal{X}(S)$. Finally, if $\bar{G}$ is S^η-tradable, then $\bar{D}$ is S-tradable because $\bar{D} = (\eta \cdot S)\bar{G} = (\eta \cdot S)V(\vartheta, S^\eta) = V(\vartheta, S)$ for some $\vartheta \in \Theta^{\mathrm{sf}}_+(S^\eta) = \Theta^{\mathrm{sf}}_+(S)$. Analogously, if D is an (S-tradable) SMD$^{\eta+}$ for $\mathcal{X}(S)$, then $G := D/(\eta \cdot S)$ is an (S^η-tradable) SMD$^{\eta+}$ for $\mathcal{X}(S^\eta)$. So (B′) and (C′) are equivalent to (b) and (c), respectively, and this proves the equivalence statement.

The proof of uniqueness is standard. Take two S-tradable SMDs $\bar{D}, \bar{D}'$ for $\mathcal{X}(S)$ and recall that $\bar{D}_0 = 1 = \bar{D}'_0$. Due to $\bar{D}, \bar{D}' \in \mathcal{X}(S)$, both $X := \bar{D}/\bar{D}'$ and $1/X$ are supermartingales and hence $E[X_t] \leq E[X_0] = 1$ and $1/E[X_t] \leq E[1/X_t] \leq E[1/X_0] = 1$ for any t by Jensen's inequality. Thus $E[X_t] \equiv 1$ and so by Jensen again, $X \equiv X_0$ and $\bar{D} = \bar{D}'$. □

Proposition 2.20 can be viewed as a generalisation of [21, Theorem 4.12] to our setting with DSV instead of NUPBR. Analogously, the next result extends [18, Theorem 2.1].

Proposition 2.21. *Fix a small market $(\mathbb{B}, S, \zeta)$. If $S \geq 0$ and there exists a reference strategy $\eta \in \Theta^{\mathrm{sf}}_{++}(S)$, then we have:*

1) *$(\mathbb{B}, S, \zeta)$ satisfies DSV for η if and only if for any $\varepsilon > 0$, there exists $Q \approx P$ on $\mathcal{F}$ with $\sup_{A \in \mathcal{F}} |Q[A] - P[A]| < \varepsilon$ and such that there exists an S-tradable Q-LMD$^{\eta+}$ $\bar{D}$ for S. (More precisely, this means that $\bar{D} \in \mathcal{S}_{++}$ is S-tradable, has $\bar{D}_0 = 1$, $S/\bar{D}$ is a Q-local martingale and $\inf_{t \geq 0}(\eta_t \cdot (S_t/\bar{D}_t)) > 0$ P-a.s.)*
2) *Suppose $S = (1, X)$ for some $\mathbb{R}^d_+$-valued semimartingale $X \geq 0$. If $(\mathbb{B}, S, \zeta)$ satisfies NUPBR, there exists for any $\varepsilon > 0$ a pair $(Q, \bar{D})$ as above. If in addition X is bounded, the converse holds as well.*

Proof. 1) If $(\mathbb{B}, S, \zeta)$ satisfies DSV for η, then $(\mathbb{B}, S^\eta, \zeta)$ satisfies NUPBR = NA1 by [3, Theorem 2.14], and $S^\eta_\infty := \lim_{t\to\infty} S^\eta_t$ exists P-a.s. due to [3, Theorem 3.7]. (In more detail, 0 is strongly value maximal for S^η by [3, Theorem 2.14], and $\xi := \eta$ has $(S^\eta)^\xi = S^\eta$ and $V(\xi, S^\eta) = \eta \cdot S^\eta \equiv 1$. Because $S \geq 0$, we also have $S^\eta \geq 0$ and hence $\mathrm{e}^i \in \Theta^{\mathrm{sf}}_+(S^\eta)$ implies by [3, Theorem 3.7] the P-a.s. existence of $V_\infty(\mathrm{e}^i, S^\eta) = \lim_{t\to\infty}(S^\eta_t)^{(i)}$ for $i = 1, \dots, N$.) Setting now $\tilde{X}_t := S^\eta_{t\wedge\zeta}$ and $\tilde{S} := (1, \tilde{X})$ thus gives an $\mathbb{R}^{1+N}_+$-valued semimartingale $\tilde{S}$ defined on the closed interval $[0, \infty]$, and $\tilde{S}$ satisfies NA1 as well because $\tilde{X}$ does and $\mathcal{X}(\tilde{S}) = \mathcal{X}(\eta \cdot \tilde{X}, \tilde{X}) = \mathcal{X}(\tilde{X})$ due to $\eta \cdot S^\eta \equiv 1$ and Lemma 2.10. Applying [18, Theorem 2.1] to $(\mathbb{B}, \tilde{S}, \infty)$ yields for any $\varepsilon > 0$ a $Q \approx P$ on $\mathcal{F}$ with $\sup_{A\in\mathcal{F}} |Q[A] - P[A]| < \varepsilon$ and an $\tilde{S}$-tradable Q-LMD $\bar{D}'$ for $\mathcal{X}(\tilde{S})$. Due to $\tilde{X} \geq 0$, $\bar{D}'$ is also a Q-LMD for $\tilde{S}$ by

Lemma 2.13, and even a Q-LMD$^{\eta+}$ for $\tilde{S}$ as both $\tilde{S}$ and $\bar{D}'$ are defined on the closed interval $[0,\infty]$. Finally, $\bar{D}'$ is S^η-tradable because $\mathcal{X}(S^\eta)=\mathcal{X}(\tilde{X})=\mathcal{X}(\tilde{S})$ as seen above. But now $\bar{D}:=(\eta\cdot S)\bar{D}'$ on $[\![0,\zeta]\!]$ clearly defines an S-tradable Q-LMD$^{\eta+}$ $\bar{D}$ for S, and so we get the "only if" part. Conversely, if we have $Q\approx P$ and a Q-LMD$^{\eta+}$ $\bar{D}$ for S and denote by Z the density process of Q with respect to P, then $D:=\bar{D}/Z$ is a P-LMD$^{\eta+}$ for S by the Bayes rule. (Note that D, unlike $\bar{D}$, is not S-tradable in general.) Then the "if" part follows from Proposition 2.19.

2) If $S=(1,X)$ with $X\geq 0$, then $\eta\equiv\mathbb{1}$ is a reference strategy and NUPBR implies DSV for $\mathbb{1}$ by [3, Proposition 5.6]. So the first part of 2) follows from 1). If we have for any $\varepsilon>0$ the existence of a pair $(Q,\bar{D})$ as in 1), S satisfies DSV for $\mathbb{1}$ by 1), and $S^{\mathbb{1}}=(1/(1+\sum_i X^{(i)}),X/(1+\sum_i X^{(i)}))$ satisfies NUPBR due to [3, Theorem 2.14]. By [3, Proposition 3.6] for $S^{\mathbb{1}}$ and $\xi\equiv\mathbb{1}$, 0 is thus strongly value maximal for $S^{\mathbb{1}}$. If in addition $X\geq 0$ is bounded, $1+\sum_i X^{(i)}$ is P-a.s. bounded away from 0 and ∞, and so 0 is strongly value maximal for S as well by [3, Lemma 3.1] with $D:=1+\sum_i X^{(i)}$. Using again [3, Proposition 3.6], for $S=(1,X)$ and $\xi\equiv\mathrm{e}^1$, shows that NUPBR holds for $(1,X)=S$. □

Before proving an auxiliary result for later use, we recall from [3, Equation (3.1)] an operation on strategies. Fix $\xi\in\Theta^{\mathrm{sf}}_{++}(S)$ and a stopping time τ. The ξ-*concatenation* at time τ of $\vartheta^1,\vartheta^2\in\Theta^{\mathrm{sf}}_{+}(S)$ is defined by

$$\vartheta^1 \circledS^\xi_\tau \vartheta^2 := I_{[\![0,\tau]\!]}\vartheta^1 + I_{]\!]\tau,\infty]\!]}\Big(I_\Gamma\vartheta^1 + I_{\Gamma^c}\big(\vartheta^2+\xi V_\tau(\vartheta^1-\vartheta^2,S^\xi)\big)\Big) \quad (2.3)$$
$$\text{with } \Gamma:=\{V_\tau(\vartheta^1,S)<V_\tau(\vartheta^2,S)\}.$$

By [3, Lemma 3.3], $\vartheta^1 \circledS^\xi_\tau \vartheta^2$ is in $\Theta^{\mathrm{sf}}_{+}(S)$. Note that $\vartheta^1 \circledS^\xi_\tau \vartheta^2=\vartheta^1$ on $\{\tau=\infty\}\cap\{\tau=\zeta\}$.

Lemma 2.22. *Fix a small market* $(\mathbb{B},S,\zeta)$ *and assume that* $S\geq 0$ *and that there exists a reference strategy* $\eta\in\Theta^{\mathrm{sf}}_{++}(S)$. *Suppose that* $\vartheta\in\Theta^{\mathrm{sf}}_{+}(S)$ *is ssm for* η *and* $\hat{\vartheta}\in\Theta^{\mathrm{sf}}_{+}(S)$ *is such that* $\liminf_{t\to\infty}(\hat{\vartheta}_t-\vartheta_t)\geq 0$ *P-a.s. Then* $\hat{\vartheta}-\vartheta\in\Theta^{\mathrm{sf}}_{+}(S)$.

Proof. See Appendix. □

3 Asymptotic strong share maximality

In this section, we introduce a new concept of maximal strategies for large markets and use this to define AOA concepts which are discounting-invariant in a sense we make precise. In analogy to the small market case, we could introduce the notion of a large market reference strategy $\vec{\eta}$ and then define asymptotic strong share maximality with respect to $\vec{\eta}$. But to reduce technicalities and in order to facilitate comparisons with NAA and NUPBR, we opt for the choice $\vec{\eta}=\vec{\mathbb{1}}=(1,1,1,\dots)$, the large market analogue of the market portfolio. In view of Lemma 2.16, we can then equivalently use either ssm or tssm, and the latter concept gives the crispest formulations. Note how Definitions 3.1 and 3.2 parallel Definitions 2.15 and 2.18, respectively.

Definition 3.1. Fix a large market $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$. A large market strategy $\vec{\vartheta} = (\vartheta^n)_{n\in\mathbb{N}}$ is *asymptotically strongly share maximal (assm) for* $\vec{\mathbb{1}}$ if there is no $p > 0$ such that for every $\varepsilon > 0$, there are some $n \in \mathbb{N}$, $A^n \in \mathcal{F}^n$ with $P^n[A^n] \geq p$ and a strategy $\vec{\hat{\vartheta}}$ with $V_0(\hat{\vartheta}^n, S^n) \leq V_0(\vartheta^n, S^n) + \varepsilon(\mathbb{1}^n \cdot S_0^n)$ and $\liminf_{t\to\infty}(\hat{\vartheta}_t^n - \vartheta_t^n) \geq pI_{A^n}\mathbb{1}^n$ P^n-a.s.

Note that n, A^n and $\vec{\hat{\vartheta}}$ above can of course depend on ε. We usually omit a corresponding index for ease of notation.

Definition 3.2. A large market $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$ satisfies *asymptotic dynamic share viability (ADSV) for* $\vec{\mathbb{1}}$ if $\vec{0} = (0^n)_{n\in\mathbb{N}}$ is asymptotically strongly share maximal for $\vec{\mathbb{1}}$, and *asymptotic DSE (ADSE) for* $\vec{\mathbb{1}}$ if every $\vec{\vartheta} = (\vartheta^n)_{n\in\mathbb{N}}$ with $\vartheta^n \in \Theta_+^{\mathrm{sf}}(S^n)$ being $\mathbb{1}^n$-buy-and-hold (i.e., $\vartheta^n \equiv \vartheta_0^n \in \mathbb{R}_+^{N^n}$) for each n is asymptotically strongly share maximal for $\vec{\mathbb{1}}$.

The definition directly implies that asymptotic strong share maximality is *discounting-invariant* in the sense that $\vec{\vartheta}$ is assm for $\vec{\mathbb{1}}$ in $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$ if and only if it is assm for $\vec{\mathbb{1}}$ in $(\mathbb{B}^n, S^n/D^n, \zeta^n)_{n\in\mathbb{N}}$ for any sequence $\vec{D} = (D^n)_{n\in\mathbb{N}}$, where each D^n is a discounter in the market $(\mathbb{B}^n, S^n, \zeta^n)$. The reason is that maximality is formulated not in terms of wealth, but of holdings in assets, and these do not change if we change the numéraire. Because we want invariance under discounting not only for each small market, but along the entire sequence, it is important that we do not normalise discounters to $D_0^n = 1$ (as can be done if one only works in a fixed small market), but allow D_0^n to depend on n. In turn, this makes it necessary that the allowed small extra initial wealth in Definition 3.1 depends on the n-th market's size via the term $\mathbb{1}^n \cdot S_0^n$. This is economically natural; if for instance prices in model n are simply a c_n-multiple of prices in model 1 with a sequence $(c_n)_{n\in\mathbb{N}}$ going to 0, a fixed initial wealth amount ε becomes more and more valuable along the sequence of models, and in the absence of the term $\mathbb{1}^n \cdot S_0^n$, this might in itself asymptotically generate some arbitrage opportunities. On the other hand, if $\lim_{n\to\infty}(\mathbb{1}^n \cdot S_0^n) = +\infty$, having the term $\mathbb{1}^n \cdot S_0^n$ allows more strategies for trying to generate arbitrage, and hence forbidding them gives a more restrictive AOA concept than if the term is absent.

Remark 3.3. In Bálint/Schweizer [3] and in Section 2, we have defined and used strong share maximality and the derived concepts DSV and DSE with respect to a reference strategy η in the small market $(\mathbb{B}, S, \zeta)$. As mentioned at the beginning of this section, the natural extension to a large market would be to define asymptotic strong share maximality with respect to some $\vec{\eta} = (\eta^n)_{n\in\mathbb{N}}$, where each η^n is a reference strategy in the n-th small market $(\mathbb{B}^n, S^n, \zeta^n)$. One obvious question is then how the results depend on the choice of $\vec{\eta}$. For a small market, we have shown in [3, Lemma 5.1] that if $S \geq 0$ and η, η' are reference strategies satisfying $0 < \inf_{t\geq 0}(\eta_t \cdot S_t/\eta_t' \cdot S_t) \leq \sup_{t\geq 0}(\eta_t \cdot S_t/\eta_t' \cdot S_t) < \infty$ P-a.s., then any strategy $\vartheta \in \Theta_+^{\mathrm{sf}}(S)$ is ssm for η if and only if it is ssm for η', which implies that DSV for η and DSV for η' are equivalent. For a large market, one would probably not only need to control each pair $(\eta^n, \eta'^{,n})$, but in addition also the relative behaviour of the *sequences* $\vec{\eta}$ and $\vec{\eta}'$. We leave this question for future research.

We first show that if a large market strategy is asymptotically strongly share maximal for $\vec{\mathbb{1}}$, its coordinates are strongly share maximal for $\mathbb{1}$ in their small market. This contrasts NAA which does not imply any AOA property along the sequence of small markets. Recall from Remark 2.2 that $S \geq 0$ plus $\mathbb{1} \cdot S \in \mathcal{S}_{++}$ implies that $\mathbb{1}$ is a reference strategy for S.

Lemma 3.4. *Suppose for each small market* $(\mathbb{B}^n, S^n, \zeta^n)$ *that* $S^n \geq 0$ *and* $\mathbb{1}^n \cdot S^n \in \mathcal{S}_{++}$*. If* $\vec{\vartheta}$ *is assm for* $\vec{\mathbb{1}}$*, then* ϑ^n *is ssm for* $\mathbb{1}^n$ *in* $(\mathbb{B}^n, S^n, \zeta^n)$ *for each* n*. In particular, ADSV or ADSE for* $\vec{\mathbb{1}}$ *implies that each small market satisfies DSV or DSE for* $\mathbb{1}^n$*, respectively.*

Proof. Suppose that ϑ^n is not ssm for $\mathbb{1}^n$ in $(\mathbb{B}^n, S^n, \zeta^n)$ for some n. Then Lemma 2.16 yields a $\psi^n_\infty \in L^\infty_+(\mathcal{F}^n) \setminus \{0\}$ such that for any $\varepsilon > 0$, there is a $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+(S^n)$ with $V_0(\hat{\vartheta}^\varepsilon, S^n) \leq V_0(\vartheta^n, S^n) + \varepsilon(\mathbb{1}^n \cdot S^n_0)$ and $\liminf_{t\to\infty}(\hat{\vartheta}^\varepsilon_t - \vartheta^n_t) \geq \psi^n_\infty \mathbb{1}^n$ P^n-a.s. Choose $p > 0$ and $A^n \in \mathcal{F}^n$, not depending on ε, such that $P^n[A^n] \geq p$ and $\psi^n_\infty \geq pI_{A^n}$ P^n-a.s. Then $\vec{\hat{\vartheta}} := (\vartheta^1, \dots, \vartheta^{n-1}, \hat{\vartheta}^\varepsilon, \vartheta^{n+1}, \dots)$ is a large market strategy, and for the above p and given any ε, the triple $(n, A^n, \vec{\hat{\vartheta}})$ satisfies the requirements in Definition 3.1. Because $\varepsilon > 0$ was arbitrary, this shows that $\vec{\vartheta}$ is not assm for $\vec{\mathbb{1}}$. □

We next show a consistency result: If the sequence $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$ is constant in n, i.e., we morally have a small market, then strong share maximality and asymptotic strong share maximality are equivalent.

Proposition 3.5. *Let the sequence* $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$ *of small markets be constant in* n*, assume* $S^1 \geq 0$ *and* $\mathbb{1}^{N^1} \cdot S^1 \in \mathcal{S}_{++}$*, and let* $\vec{\vartheta}$ *be a large market strategy with* $\vartheta^n \equiv \vartheta \in \Theta^{\mathrm{sf}}_+(S^1)$ *for all* n*. Then* $\vec{\vartheta}$ *is assm for* $\vec{\mathbb{1}}$ *if and only if* ϑ *is ssm for* $\mathbb{1}^{N^1}$ *in* $(\mathbb{B}^1, S^1, \zeta^1)$*.*

Proof. The "only if" direction is a direct consequence of Lemma 3.4. For the "if" direction, we have $\mathbb{1}^n \cdot S^n_0 \equiv \mathbb{1}^{N^1} \cdot S^1_0$ because $(\mathbb{B}^n, S^n, \zeta^n) \equiv (\mathbb{B}^1, S^1, \zeta^1)$, and as asymptotic strong share maximality for $\vec{\mathbb{1}}$ is discounting-invariant, we can assume without loss of generality that $\mathbb{1}^{N^1} \cdot S^1_0 = 1$. Suppose ϑ is ssm for $\mathbb{1}^{N^1}$, but $\vec{\vartheta}$ is not assm for $\vec{\mathbb{1}}$. Consider the set

$$\mathcal{G}^{\vartheta,\varepsilon} := \Big\{ g \in L^0_+ : \exists \varphi \in \Theta^{\mathrm{sf}}_+(S^1) \text{ with } V_0(\varphi - \vartheta, S^1) \leq \varepsilon \\ \text{and } \liminf_{t\to\infty}(\varphi_t - \vartheta_t) \geq g \mathbb{1}^{N^1} \ P^1\text{-a.s.} \Big\}$$

and define $\mathcal{G}^\vartheta := \bigcup_{\varepsilon>0} \mathcal{G}^{\vartheta,\varepsilon}/\varepsilon$. We first show that $\mathcal{G}^\vartheta$ is not bounded in L^0. Because $\vec{\vartheta}$ is not assm for $\vec{\mathbb{1}}$, there exists $p > 0$ such that for each $\varepsilon > 0$, there are $n \in \mathbb{N}$, $A^{n,\varepsilon} \in \mathcal{F}^1$ with $P^1[A^{n,\varepsilon}] \geq p$ and a strategy $\vec{\hat{\vartheta}}$ with $V_0(\hat{\vartheta}^n, S^1) \leq V_0(\vartheta^n, S^1) + \varepsilon(\mathbb{1}^{N^1} \cdot S^1_0) = V_0(\vartheta, S^1) + \varepsilon$ and

$$pI_{A^{n,\varepsilon}} \mathbb{1}^{N^1} \leq \liminf_{t\to\infty}(\hat{\vartheta}^n_t - \vartheta^n_t) = \liminf_{t\to\infty}(\hat{\vartheta}^n_t - \vartheta_t) \qquad P^1\text{-a.s.}$$

It follows that $pI_{A^{n,\varepsilon}} \in \mathcal{G}^{\vartheta,\varepsilon}$ and hence $pI_{A^{n,\varepsilon}}/\varepsilon \in \mathcal{G}^\vartheta$. As $P^1[A^{n,\varepsilon}] \geq p$ and $\varepsilon > 0$ is arbitrary, $\mathcal{G}^\vartheta$ is not bounded in L^0.

In a second step, we show that $\mathcal{G}^\vartheta$ is convex. For $a, b \in \mathcal{G}^\vartheta$, let $\varepsilon^a > 0$, $\varphi^a \in \Theta_+^{\mathrm{sf}}(S^1)$ and $\varepsilon^b > 0$, $\varphi^b \in \Theta_+^{\mathrm{sf}}(S^1)$ be the corresponding objects as in the definition of $\mathcal{G}^\vartheta$. We need to show that $\lambda a + (1-\lambda)b \in \mathcal{G}^\vartheta$ for any $\lambda \in (0,1)$. Fix a parameter $x \in (0,1)$ and consider the strategy $\varphi^x := x\varphi^a + (1-x)\varphi^b \in \Theta_+^{\mathrm{sf}}(S^1)$. Then $V_0(\varphi^x - \vartheta, S^1) \le x\varepsilon^a + (1-x)\varepsilon^b$ and

$$\begin{aligned}\liminf_{t\to\infty}(\varphi_t^x - \vartheta_t) &= \liminf_{t\to\infty}\Big(x(\varphi_t^a - \vartheta_t) + (1-x)(\varphi_t^b - \vartheta_t)\Big)\\ &\ge x\liminf_{t\to\infty}(\varphi_t^a - \vartheta_t) + (1-x)\liminf_{t\to\infty}(\varphi_t^b - \vartheta_t)\\ &\ge x\varepsilon^a a\mathbb{1}^{N^1} + (1-x)\varepsilon^b b\mathbb{1}^{N^1} \qquad P^1\text{-a.s.}\end{aligned}$$

so that $c := (x\varepsilon^a a + (1-x)\varepsilon^b b)/(x\varepsilon^a + (1-x)\varepsilon^b) \in \mathcal{G}^\vartheta$. Choose $x := (\lambda\varepsilon^b)/((1-\lambda)\varepsilon^a + \lambda\varepsilon^b)$, which is in $(0,1)$, and calculate to get $(x\varepsilon^a)/(x\varepsilon^a + (1-x)\varepsilon^b) = \lambda$ and then of course $((1-x)\varepsilon^b)/(x\varepsilon^a + (1-x)\varepsilon^b) = 1-\lambda$. This means that $\lambda a + (1-\lambda)b = c \in \mathcal{G}^\vartheta$.

As $\mathcal{G}^\vartheta$ is convex and unbounded in L^0, [3, Lemma A.2] yields $g \in L_+^0 \setminus \{0\}$ and a sequence $(U^n)_{n\in\mathbb{N}} \subseteq \mathcal{G}^\vartheta$ with $U^n \ge ng$ for each n. Take $\varepsilon > 0$ with $\varepsilon U^n \in \mathcal{G}^{\vartheta,\varepsilon}$ and the corresponding strategy $\varphi' \in \Theta_+^{\mathrm{sf}}(S^1)$. Writing $\varphi' = \vartheta + (\varphi' - \vartheta) =: \vartheta + \psi$, note that because ϑ is ssm and in view of the definition of $\mathcal{G}^\vartheta$, ψ is in $\Theta_+^{\mathrm{sf}}(S^1)$ by Lemma 2.22 and hence also $\varphi := \vartheta + \psi/(n\varepsilon) \in \Theta_+^{\mathrm{sf}}(S^1)$. But now the properties of φ' and ψ yield $V_0(\varphi - \vartheta, S^1) \le \varepsilon/(n\varepsilon) = 1/n$ and

$$\liminf_{t\to\infty}(\varphi_t - \vartheta_t) = \liminf_{t\to\infty}\psi_t/(n\varepsilon) = \liminf_{t\to\infty}(\varphi_t' - \vartheta_t)/(n\varepsilon) \ge (\varepsilon U^n)/(n\varepsilon)\mathbb{1}^{N^1} \ge g\mathbb{1}^{N^1},$$

which is by Lemma 2.16 a contradiction to the strong share maximality of ϑ for $\mathbb{1}^{N^1}$. ☐

Choosing $\vec{\vartheta} = \vec{0}$ and $\vartheta = 0$ immediately yields

Corollary 3.6. *Let the sequence $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$ of small markets be constant in n and assume $S^1 \ge 0$ and $\mathbb{1}^{N^1}\cdot S^1 \in \mathcal{S}_{++}$. Then the large market $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$ satisfies ADSV for $\vec{\mathbb{1}}$ if and only if the small market $(\mathbb{B}^1, S^1, \zeta^1)$ satisfies DSV for $\mathbb{1}^{N^1}$.*

With a bit more work, we get an analogous result also for efficiency.

Corollary 3.7. *Let the sequence $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$ of small markets be constant in n and assume $S^1 \ge 0$ and $\mathbb{1}^{N^1}\cdot S^1 \in \mathcal{S}_{++}$. Then the large market $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$ satisfies ADSE for $\vec{\mathbb{1}}$ if and only if the small market $(\mathbb{B}^1, S^1, \zeta^1)$ satisfies DSE for $\mathbb{1}^{N^1}$.*

Proof. The "only if" part is clear from Lemma 3.4. For the "if" part, suppose each $\vartheta \in \mathbb{R}_+^{N^1}$ is ssm for $\mathbb{1}^{N^1}$ in $(\mathbb{B}^1, S^1, \zeta^1)$, but there is a $\vec{\vartheta} = (\vartheta^n)_{n\in\mathbb{N}}$ with $\vartheta^n \equiv \vartheta_0^n \in \mathbb{R}_+^{N^1}$ for each n which is not assm for $\vec{\mathbb{1}}$. Then there exist $p > 0$ and a subsequence $(\vartheta^{n_k})_{k\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$, there are $A^k \in \mathcal{F}^1$ with $P^1[A^k] \ge p$ and $\hat{\vartheta}^k \in \Theta_+^{\mathrm{sf}}(S^1)$ with $V_0(\hat{\vartheta}^k, S^1) \le V_0(\vartheta^{n_k}, S^1) + (\mathbb{1}^{N^1}\cdot S_0^1)/k$ and $\liminf_{t\to\infty}(\hat{\vartheta}_t^k - \vartheta_0^{n_k}) \ge pI_{A^k}\mathbb{1}^{N^1}$ P^1-a.s. By Lemma 2.22, each $\tilde{\vartheta}^k := \hat{\vartheta}^k - \vartheta^{n_k}$ is then in $\Theta_+^{\mathrm{sf}}(S^1)$, and so $\vec{0}$ is not assm for $\vec{\mathbb{1}}$ in the large market $(\mathbb{B}^{n_k}, S^{n_k}, \zeta^{n_k})_{k\in\mathbb{N}}$. By Proposition 3.5, the zero strategy 0 is then not ssm for $\mathbb{1}^{N^1}$ in $(\mathbb{B}^1, S^1, \zeta^1)$, which is a contradiction. ☐

4 Absence of arbitrage and dual characterisations

This section considers general large markets and provides two dual characterisations of our ADSV concept in terms of martingale properties in each small market plus some contiguity property. Recall that two sequences $(Q^n)_{n\in\mathbb{N}}, (\tilde{Q}^n)_{n\in\mathbb{N}}$ of sub-probability measures with each $Q^n, \tilde{Q}^n$ on $\mathcal{F}^n$ are *contiguous*, written $(Q^n)_{n\in\mathbb{N}} \lhd (\tilde{Q}^n)_{n\in\mathbb{N}}$, if for any sequence $(A^n)_{n\in\mathbb{N}}$ of sets $A^n \in \mathcal{F}^n$ with $\lim_{n\to\infty} \tilde{Q}^n[A^n] = 0$, we have $\lim_{n\to\infty} Q^n[A^n] = 0$.

The classic characterisation of absence of arbitrage in a small market (in the sense of NFLVR) is the existence of an ELMM Q for the underlying process $S = (1, X)$ (assuming $S \geq 0$, to avoid σ-martingales); see Delbaen/Schachermayer [11, 12]. This was complemented by Karatzas/Kardaras [21] who proved that NUPBR in a small market is equivalent to the existence of a supermartingale discounter for $\mathcal{X}(S)$ with a positivity property at $+\infty$. Conceptually, this generalises the existence of an equivalent separating measure, which gives information about all stochastic integrals of S, but not necessarily about S itself. (If $S \geq 0$, an SMD for $\mathcal{X}(S)$ is also an SMD for S, by Lemma 2.13; so then we intuitively get for S a supermartingale, but maybe not a local martingale property.)

For large markets, the classic characterisation of NAA in Klein/Schachermayer [26] and Kabanov/Kramkov [20], assuming that each small market admits an ELMM Q^n, is that there exists a sequence $(Q^n)_{n\in\mathbb{N}}$ of ELMMs with $(P^n)_{n\in\mathbb{N}} \lhd (Q^n)_{n\in\mathbb{N}}$. This was generalised by Rokhlin [31] who showed that if each S^n only satisfies NUPBR, NAA is equivalent to the existence of a sequence of SMDs D^n for $\mathcal{X}(S^n)$ such that $(P^n)_{n\in\mathbb{N}} \lhd ((1/D^n_\zeta) \cdot P^n)_{n\in\mathbb{N}}$. Our first main result, Theorem 4.1 below, is in the same spirit, but uses the more general concepts of DSV for $\mathbb{1}$ and ADSV for $\vec{\mathbb{1}}$ instead of NUPBR and NAA. This similarity to the ideas and results in [31] also shows up in parts of the proofs.

In the spirit of the results in [11, 12] and [26, 20], we should also like to have a characterisation of ADSV with local martingale properties for the S^n instead of only supermartingale properties for the $\mathcal{X}(S^n)$. However, there is a problem. We shall see below in Lemma 4.2 and Theorem 4.1 that deriving a contiguity property crucially needs tradability for each discounter D^n. To work with an LMD for S^n (and not only an SMD for $\mathcal{X}(S^n)$), we thus must find an S^n-*tradable* local martingale deflator — and a counterexample in Takaoka/Schweizer [34] shows that this does not exist in general. Fortunately, Kabanov et al. [18] recently proved that in a small market under NUPBR, we can still find an S-tradable Q-LMD for $S = (1, X)$, under some $Q \approx P$ which can even be chosen arbitrarily close to P. Combining an extension of this result from NUPBR to DSV for $\mathbb{1}$ with the invariance of ADSV for $\vec{\mathbb{1}}$ under bi-contiguous measure changes (see Corollary 4.4 below) then allows us to derive in Theorem 4.5 our second main result. It is a dual characterisation of ADSV for $\vec{\mathbb{1}}$, now with local martingale properties for S^n itself, and as usual with a contiguity property.

If $S^n \geq 0$ and $\mathbb{1}^n \cdot S^n \in \mathcal{S}_{++}$, then $\mu^n := (S^n)^{\mathbb{1}^n} = S^n/(\mathbb{1}^n \cdot S^n)$ is well defined and an $\mathbb{R}^{N^n}$-valued semimartingale with values in $[0, 1]^{N^n}$. It describes the (price-based) *market* *weights* of the N^n assets in the small market $(\mathbb{B}^n, S^n, \zeta^n)$ and frequently appears later.

We start with our first main result.

Theorem 4.1. *Let every small market* $(\mathbb{B}^n, S^n, \zeta^n)$ *satisfy* $S^n \geq 0$ *and* $\mathbb{1}^n \cdot S^n \in \mathcal{S}_{++}$. *Then the following are equivalent:*

(a) *The large market* $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$ *satisfies ADSV for* $\vec{\mathbb{1}}$.

(b) *There exists for each n a (unique)* S^n*-tradable SMD*$^{\mathbb{1}^n+}$ $\bar{D}^n$ *for* $\mathcal{X}(S^n)$ *and we have the contiguity* $(P^n)_{n\in\mathbb{N}} \lhd ((1/\bar{g}^n)\cdot P^n)_{n\in\mathbb{N}}$, *where* $\bar{g}^n := ((\mathbb{1}^n \cdot S_0^n)\bar{D}^n_{\zeta^n})/(\mathbb{1}^n \cdot S^n_{\zeta^n})$.

(c) *There exists for each n an SMD*$^{\mathbb{1}^n+}$ D^n *for* $\mathcal{X}(S^n)$ *such that we have the contiguity relation* $(P^n)_{n\in\mathbb{N}} \lhd ((1/g^n)\cdot P^n)_{n\in\mathbb{N}}$, *where* $g^n := ((\mathbb{1}^n \cdot S_0^n)D^n_{\zeta^n})/(\mathbb{1}^n \cdot S^n_{\zeta^n})$.

For each n and $\vartheta \in \Theta^{\mathrm{sf}}_+(\mu^n) = \Theta^{\mathrm{sf}}_+(S^n)$, the ratio $\frac{V(\vartheta,\mu^n)}{D^n/(\mathbb{1}^n\cdot S^n)} = \frac{V(\vartheta,S^n)}{D^n}$ is a P^n-supermartingale. This means that each $(1/g^n)\cdot P^n$ is (by the Bayes rule) essentially a supermartingale measure for all wealth processes in $\mathcal{X}(\mu^n)$ and hence could be called a "generalised martingale measure".

The core of the proof for Theorem 4.1 is given by the following technical result, which is inspired by Rokhlin [31, Theorem 2.1] and the results of [3].

Lemma 4.2. *Let every small market* $(\mathbb{B}^n, S^n, \zeta^n)$ *satisfy* $S^n \geq 0$, $\mathbb{1}^n \cdot S^n \in \mathcal{S}_{++}$ *and DSV for* $\mathbb{1}^n$. *Denote by* $\bar{D}^n$ *(see Proposition 2.20) the (unique)* S^n*-tradable SMD*$^{\mathbb{1}^n+}$ *for* $\mathcal{X}(S^n)$ *in the n-th market. Set* $\bar{g}^n := ((\mathbb{1}^n \cdot S_0^n)\bar{D}^n_{\zeta^n})/(\mathbb{1}^n \cdot S^n_{\zeta^n})$. *Then the following are equivalent:*

(a) *The large market* $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$ *satisfies ADSV for* $\vec{\mathbb{1}}$.

(b) *For each large market strategy* $\vec{\vartheta}$ *such that* $V_0(\vartheta^n, S^n) = \mathbb{1}^n \cdot S_0^n$ *for all n, the sequence* $(P^n \circ (V_{\zeta^n}(\vartheta^n, S^n)/(\mathbb{1}^n \cdot S^n_{\zeta^n}))^{-1})_{n\in\mathbb{N}}$ *is tight.*

(c) *The sequence* $(P^n \circ (\bar{g}^n)^{-1})_{n\in\mathbb{N}}$ *is tight.*

(d) $(P^n)_{n\in\mathbb{N}} \lhd ((1/\bar{g}^n)\cdot P^n)_{n\in\mathbb{N}}$.

Proof. Recalling $\mu^n = S^n/(\mathbb{1}^n \cdot S^n)$, note first that even if $P^n[\zeta^n = \infty] > 0$, the quantity $V_{\zeta^n}(\vartheta^n, S^n)/(\mathbb{1}^n \cdot S^n_{\zeta^n}) = V_{\zeta^n}(\vartheta^n, \mu^n)$ is well defined and finite P^n-a.s. Indeed, by [3, Theorem 2.14], DSV for $\mathbb{1}^n$ in the n-th market implies that 0^n is strongly value maximal for μ^n, and hence due to [3, Theorem 3.7] applied for μ^n and $\xi = \mathbb{1}^n$, $\lim_{t\to\infty} V_t(\vartheta^n, \mu^n)$ exists on $\{\zeta^n = \infty\}$; see also (2.1). The existence of $\bar{D}^n$ comes from Proposition 2.20. Because $\mathbb{1}^n \cdot S^n \in \mathcal{S}_{++}$, every $(\mathbb{1}^n \cdot S^n)/\bar{D}^n$ is then a strictly positive P^n-supermartingale and hence converges P^n-a.s. on $\{\zeta^n = \infty\}$ to a finite limit, which is P^n-a.s. strictly positive because $\bar{D}^n$ is an SMD$^{\mathbb{1}^n+}$ for $\mathcal{X}(S^n)$. Thus all expressions in Lemma 4.2 are well defined even on $\{\zeta^n = \infty\}$, and $\bar{g}^n$ can be written as $\bar{g}^n = \lim_{t\to\infty}((\mathbb{1}^n \cdot S_0^n)\bar{D}^n_t)/(\mathbb{1}^n \cdot S^n_t)$ and has values in $(0,\infty)$ P^n-a.s. with $E^{P^n}[1/\bar{g}^n] \leq 1$. In particular, $1/\bar{g}^n$ is P-a.s. the limit of a function which is bounded in $t \geq 0$ (but not in ω or n). Moreover, $V(\vartheta, S^n)/\bar{D}^n$ is a P^n-supermartingale ≥ 0 for $\vartheta \in \Theta^{\mathrm{sf}}_+(S^n)$ and hence P^n-a.s. convergent on $\{\zeta^n = \infty\}$.

"(a) $\Rightarrow$ (b)": Write $V(\vartheta^n, S^n)/(\mathbb{1}^n \cdot S^n) = V(\vartheta^n, \mu^n)$ and assume there is a large market strategy $\vec{\vartheta}$ satisfying $V_0(\vartheta^n, S^n) = \mathbb{1}^n \cdot S_0^n$ for all n such that $(P^n \circ (V_{\zeta^n}(\vartheta^n, \mu^n))^{-1})_{n\in\mathbb{N}}$ is not tight. Then there exists $\delta > 0$ such that for any $M > 0$, there is $n = n(M) \in \mathbb{N}$ with

$$P^n[B^n] \geq 2\delta \qquad \text{for } B^n := \{V_{\zeta^n}(\vartheta^n, \mu^n) \geq 2M\}.$$

Define the stopping time $\tau^n := \inf\{t \geq 0 : V_t(\vartheta^n, \mu^n) \geq M\}$ and note that $\tau^n < \infty$ on B^n. So we can find N such that $A^n := B^n \cap \{\tau^n \leq N\} \in \mathcal{F}^n$ has $P^n[A^n] \geq \delta$, and then $\varrho^n := \tau^n \wedge N$ is finite and satisfies by right-continuity that

$$V_{\varrho^n}(\vartheta^n, \mu^n) = V_{\tau^n}(\vartheta^n, \mu^n) \geq M \qquad \text{on } A^n,\ P^n\text{-a.s.} \tag{4.1}$$

Set $p = \delta$, take $\varepsilon > 0$, choose $M = \frac{1}{\varepsilon}$ and then n, A^n as above for this M. By [3, Lemma 3.3], the concatenated strategy $\tilde{\vartheta}^n := (\vartheta^n/M) \oslash^{\mathbb{1}^n}_{\varrho^n} 0$ is then in $\Theta^{\mathrm{sf}}_+(S^n)$ so that $\vec{\tilde{\vartheta}} := (0, \dots, 0, \tilde{\vartheta}^n, 0, \dots)$ with $\tilde{\vartheta}^n$ at the n-th position is a large market strategy. Moreover, $(n, A^n, \vec{\tilde{\vartheta}})$ satisfies $P^n[A^n] \geq p$ and $V_0(\tilde{\vartheta}^n, S^n) = (\mathbb{1}^n \cdot S_0^n)/M = \varepsilon(\mathbb{1}^n \cdot S_0^n)$, and the definition of $\tilde{\vartheta}^n$, see (2.3), gives $\tilde{\vartheta}^n = I_{[\![0,\varrho^n]\!]}(\vartheta^n/M) + I_{]\!]\varrho^n,\infty[\![}\mathbb{1}^n(V_{\varrho^n}(\vartheta^n, \mu^n)/M)$. Using $\varrho^n < \infty$, $\tilde{\vartheta}^n \in \Theta^{\mathrm{sf}}_+(S^n)$ and (4.1) therefore yields

$$\liminf_{t\to\infty} \tilde{\vartheta}^n_t = \Big(V_{\varrho^n}(\vartheta^n, \mu^n)/M\Big)\mathbb{1}^n \geq I_{A^n}\mathbb{1}^n \qquad P^n\text{-a.s.}$$

This means that $\vec{0}$ is not assm for $\vec{\mathbb{1}}$ and ADSV for $\vec{\mathbb{1}}$ does not hold.

"(b) $\Rightarrow$ (c)": As each $\bar{D}^n$ is S^n-tradable, we have $\bar{D}^n = V(\bar{\vartheta}^n, S^n)$ for some $\bar{\vartheta}^n \in \Theta^{\mathrm{sf}}_+(S^n)$ with $V_0(\bar{\vartheta}^n, S^n) = 1$. So the large market strategy $\vec{\vartheta}$ given by $\vartheta^n := (\mathbb{1}^n \cdot S_0^n)\bar{\vartheta}^n$ for each n satisfies the conditions in (b), and (c) follows. This explicitly uses tradability of $\bar{D}^n$.

"(c) $\Rightarrow$ (d)": This is analogous to the proof of [31, Theorem 2.1, "(c) $\Rightarrow$ (d)"]; however, we do not use the precise properties of $\bar{D}^n$ or S^n, but only that $\bar{g}^n$ is well defined and $\bar{g}^n > 0$ P^n-a.s. In more detail, suppose that $\lim_{n\to\infty}((1/\bar{g}^n)\cdot P^n)[A^n] = 0$ for some sequence $(A^n)_{n\in\mathbb{N}}$ with $A^n \in \mathcal{F}^n$, so that $\lim_{n\to\infty} E^{P^n}[(1/\bar{g}^n)I_{A^n}] = 0$. Then writing

$$\begin{aligned} P^n[A^n] &= P^n[A^n \cap \{\bar{g}^n \geq M\}] + E^{P^n}[\bar{g}^n(1/\bar{g}^n)I_{A^n\cap\{\bar{g}^n<M\}}] \\ &\leq P^n[\bar{g}^n \geq M] + ME^{P^n}[(1/\bar{g}^n)I_{A^n}] \end{aligned}$$

shows that $\limsup_{n\to\infty} P^n[A^n] \leq \limsup_{n\to\infty} P^n[\bar{g}^n \geq M]$ for any fixed $M > 0$. Thus the tightness in (c) implies $\lim_{n\to\infty} P^n[A^n] = 0$, and we have the desired contiguity.

"(d) $\Rightarrow$ (a)": If ADSV for $\vec{\mathbb{1}}$ fails, there are $p > 0$ and for every $\varepsilon := \frac{1}{k}$ some $(n, A^n, \vec{\hat{\vartheta}})$ with $P^n[A^n] \geq p$, $V_0(\hat{\vartheta}^n, S^n) \leq (\mathbb{1}^n \cdot S_0^n)/k$ and $\liminf_{t\to\infty} \hat{\vartheta}^n_t \geq pI_{A^n}\mathbb{1}^n$ P^n-a.s. Note that $n = n^k$ and $A^{n^k} = A^{n,1/k}$ depend on k. Due to $0 \leq \mu^n \leq 1$, [3, Lemma A.1] yields

$$0 \leq \Big(\liminf_{t\to\infty}(\hat{\vartheta}^n_t - pI_{A^n}\mathbb{1}^n)\Big) \cdot \Big(\liminf_{t\to\infty} \mu^n_t\Big) \leq \liminf_{t\to\infty} \Big((\hat{\vartheta}^n_t - pI_{A^n}\mathbb{1}^n) \cdot \mu^n_t\Big) \qquad P^n\text{-a.s.},$$

and in view of $\mathbb{1}^n \cdot \mu^n \equiv 1$, this gives

$$(4.2) \qquad 0 \le E^{P^n}\Big[(1/\bar{g}^n) \liminf_{t\to\infty} \Big((\hat{\vartheta}^n_t - pI_{A^n}\mathbb{1}^n)\cdot \mu^n_t\Big)\Big] = E^{P^n}\Big[(1/\bar{g}^n) \liminf_{t\to\infty}(\hat{\vartheta}^n_t \cdot \mu^n_t)\Big] - E^{P^n}[(1/\bar{g}^n)pI_{A^n}].$$

For the first term on the RHS, we successively use the definition of $\bar{g}^n$, [3, Lemma A.1] and $\hat{\vartheta}^n \in \Theta^{\mathrm{sf}}_+(S^n)$, Fatou's lemma, the fact that $(\hat{\vartheta}^n \cdot S^n)/\bar{D}^n = V(\hat{\vartheta}^n, S^n)/\bar{D}^n$ is a P^n-supermartingale and finally the initial condition on $\hat{\vartheta}^n$ above to obtain

$$(4.3) \qquad E^{P^n}\Big[(1/\bar{g}^n) \liminf_{t\to\infty}(\hat{\vartheta}^n_t \cdot \mu^n_t)\Big] = E^{P^n}\Big[\lim_{t\to\infty} \frac{\mathbb{1}^n \cdot S^n_t}{(\mathbb{1}^n \cdot S^n_0)\bar{D}^n_t} \liminf_{t\to\infty} \frac{\hat{\vartheta}^n_t \cdot S^n_t}{\mathbb{1} \cdot S^n_t}\Big] \le E^{P^n}\Big[\liminf_{t\to\infty} \frac{\hat{\vartheta}^n_t \cdot S^n_t}{(\mathbb{1}^n \cdot S^n_0)\bar{D}^n_t}\Big] \le \liminf_{t\to\infty} E^{P^n}\Big[\frac{\hat{\vartheta}^n_t \cdot S^n_t}{(\mathbb{1}^n \cdot S^n_0)\bar{D}^n_t}\Big] \le E^{P^n}\Big[\frac{\hat{\vartheta}^n_0 \cdot S^n_0}{(\mathbb{1}^n \cdot S^n_0)\bar{D}^n_0}\Big] = \frac{V_0(\hat{\vartheta}^n, S^n)}{\mathbb{1}^n \cdot S^n_0} \le \frac{1}{k}.$$

For the last term in (4.2), consider $(n^k, A^{n^k})_{k\in\mathbb{N}}$. Then $\inf_{k\in\mathbb{N}} P^{n^k}[A^{n^k}] \ge p$ and so by (d), there is $\delta > 0$ (not depending on k) with $\liminf_{k\to\infty}((1/\bar{g}^{n^k}) \cdot P^{n^k})[A^{n^k}] \ge 2\delta$. In particular, for any $k \in \mathbb{N}$, there exist $j \ge k$ and $(n^j, A^{n^j}, \vec{\hat{\vartheta}})$ as above satisfying in addition $((1/\bar{g}^{n^j}) \cdot P^{n^j})[A^{n^j}] \ge \delta$. Passing to a subsequence gives $((1/\bar{g}^{n^k}) \cdot P^{n^k})[A^{n^k}] \ge \delta$ for any $k \in \mathbb{N}$, and therefore

$$(4.4) \qquad E^{P^{n^k}}[(1/\bar{g}^{n^k})pI_{A^{n^k}}] = p\Big((1/\bar{g}^{n^k}) \cdot P^{n^k}\Big)[A^{n^k}] \ge p\delta.$$

Now plugging (4.3) (with n^k instead of n) and (4.4) into (4.2) yields for any $k \in \mathbb{N}$ that $0 \le \frac{1}{k} - p\delta$, which is a contradiction as $p, \delta > 0$ do not depend on k. ☐

Remark 4.3. The only place where we need and exploit that each discounter $\bar{D}^n$ is S^n-tradable is the proof of "(b) ⇒ (c)". In particular, "(d) ⇒ (a)" does not use this.

Proof of Theorem 4.1. "(a) ⇒ (b)": Under ADSV for $\vec{\mathbb{1}}$, $\vec{0} = (0^n)_{n\in\mathbb{N}}$ is assm for $\vec{\mathbb{1}}$, each $0^n \in \Theta^{\mathrm{sf}}_+(S^n)$ is ssm for $\mathbb{1}^n$ in $(\mathbb{B}^n, S^n, \zeta^n)$ by Lemma 3.4, and so each $(\mathbb{B}^n, S^n, \zeta^n)$ satisfies DSV for $\mathbb{1}^n$. Proposition 2.20 thus implies for each n the existence of a (unique) S^n-tradable SMD$^{\mathbb{1}^n+}$ $\bar{D}^n$ for $\mathcal{X}(S^n)$, and so we can use Lemma 4.2 and conclude via "(a) ⇒ (d)". This needs S^n-tradability of $\bar{D}^n$.

"(b) ⇒ (c)" is trivial.

"(c) ⇒ (a)": If each $\mathcal{X}(S^n)$ admits an SMD$^{\mathbb{1}^n+}$ D^n, every small market satisfies DSV for $\mathbb{1}^n$ by Proposition 2.20. If in addition $(P^n)_{n\in\mathbb{N}} \lhd ((1/g^n) \cdot P^n)_{n\in\mathbb{N}}$, Lemma 4.2 via "(d) ⇒ (a)" yields ADSV for $\vec{\mathbb{1}}$. As pointed out in Remark 4.3, that argument does not need S^n-tradability for D^n. ☐

One consequence of Theorem 4.1 is that ADSV for $\vec{\mathbb{1}}$ is invariant under bi-contiguous measure changes. Recall that each basis $\mathbb{B}^n = (\Omega^n, \mathcal{F}^n, \mathbb{F}^n, P^n)$ includes a probability measure P^n and that $\mathcal{F}^n = \mathcal{F}^n_{\zeta^n}$ for all n.

Corollary 4.4. *Let every small market $(\mathbb{B}^n, S^n, \zeta^n)$ satisfy $S^n \geq 0$ and $\mathbb{1}^n \cdot S^n \in \mathcal{S}_{++}$, and suppose the large market $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$ satisfies ADSV for $\vec{\mathbb{1}}$. If we have probability measures $Q^n \approx P^n$ on $\mathcal{F}^n$, $n \in \mathbb{N}$, with $(Q^n)_{n\in\mathbb{N}} \lhd (P^n)_{n\in\mathbb{N}}$, then also the large market $((\Omega^n, \mathcal{F}^n, \mathbb{F}^n, Q^n), S^n, \zeta^n)_{n\in\mathbb{N}}$ satisfies ADSV for $\vec{\mathbb{1}}$.*

Proof. For brevity, write P-ADSV and Q-ADSV. Under P-ADSV for $\vec{\mathbb{1}}$, Theorem 4.1 (c) yields for each n a P^n-SMD$^{\mathbb{1}^n+}$ D^n for $\mathcal{X}(S^n)$ such that $(P^n)_{n\in\mathbb{N}} \lhd ((1/g^n) \cdot P^n)_{n\in\mathbb{N}}$, where $g^n := ((\mathbb{1}^n \cdot S^n_0) D^n_{\zeta^n})/(\mathbb{1}^n \cdot S^n_{\zeta^n})$. By Bayes' theorem, $\hat{D}^n := D^n Z^n$ is then a Q^n-SMD$^{\mathbb{1}^n+}$ for $\mathcal{X}(S^n)$, where Z^n denotes the density process of Q^n with respect to P^n. If we define $\hat{g}^n := ((\mathbb{1}^n \cdot S^n_0)\hat{D}^n_{\zeta^n})/(\mathbb{1}^n \cdot S^n_{\zeta^n}) = g^n Z^n_{\zeta^n}$, then for any $A^n \in \mathcal{F}^n = \mathcal{F}^n_{\zeta^n}$, we obtain

$$\Big((1/g^n) \cdot P^n\Big)[A^n] = E^{P^n}[(1/\hat{g}^n) Z^n_{\zeta^n} I_{A^n}] = \Big((1/\hat{g}^n) \cdot Q^n\Big)[A^n],$$

and so we get $(Q^n)_{n\in\mathbb{N}} \lhd (P^n)_{n\in\mathbb{N}} \lhd ((1/g^n)\cdot P^n)_{n\in\mathbb{N}} = ((1/\hat{g}^n)\cdot Q^n)_{n\in\mathbb{N}}$. Using the transitivity of contiguity and applying Theorem 4.1 again (now for the Q^n) yields Q-ADSV for $\vec{\mathbb{1}}$. □

We are now ready for our second main result.

Theorem 4.5. *Let every small market $(\mathbb{B}^n, S^n, \zeta^n)$ satisfy $S^n \geq 0$ and $\mathbb{1}^n \cdot S^n \in \mathcal{S}_{++}$. Then the following are equivalent:*

(a) *The large market $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$ satisfies ADSV for $\vec{\mathbb{1}}$.*

(b) *There exist for each n a $Q^n \approx P^n$ on $\mathcal{F}^n$ and an S^n-tradable Q^n-LMD$^{\mathbb{1}^n+}$ $\bar{D}^n$ for S^n, and we have the contiguity $(P^n)_{n\in\mathbb{N}} \lhd ((1/\bar{g}^n) \cdot Q^n)_{n\in\mathbb{N}}$, where $\bar{g}^n$ is defined by $\bar{g}^n := ((\mathbb{1}^n \cdot S^n_0)\bar{D}^n_{\zeta^n})/(\mathbb{1}^n \cdot S^n_{\zeta^n})$.*

(c) *There exists for each n a P^n-LMD$^{\mathbb{1}^n+}$ D^n for S^n such that we have the contiguity relation $(P^n)_{n\in\mathbb{N}} \lhd ((1/g^n) \cdot P^n)_{n\in\mathbb{N}}$, where $g^n := ((\mathbb{1}^n \cdot S^n_0) D^n_{\zeta^n})/(\mathbb{1}^n \cdot S^n_{\zeta^n})$.*

Proof. "(a) ⇒ (b)": Each small market satisfies DSV for $\mathbb{1}^n$ by Lemma 3.4. So by Proposition 2.21, 1), there exists for every n a $Q^n \approx P^n$ on $\mathcal{F}^n$ with $\sup_{B\in\mathcal{F}^n} |P^n[B] - Q^n[B]| < 2^{-n}$ and such that there exists an S^n-tradable Q^n-LMD$^{\mathbb{1}^n+}$ $\bar{D}^n$ for S^n. Note that we have $(P^n)_{n\in\mathbb{N}} \lhd\rhd (Q^n)_{n\in\mathbb{N}}$; indeed, for any sequence $(A^n)_{n\in\mathbb{N}}$ of sets $A^n \in \mathcal{F}^n$ with $Q^n[A^n] \to 0$,

$$P^n[A^n] \leq |P^n[A^n] - Q^n[A^n]| + Q^n[A^n] \leq 2^{-n} + Q^n[A^n] \longrightarrow 0$$

so that $(P^n)_{n\in\mathbb{N}} \lhd (Q^n)_{n\in\mathbb{N}}$, and $(Q^n)_{n\in\mathbb{N}} \lhd (P^n)_{n\in\mathbb{N}}$ follows by symmetry. By Corollary 4.4, $((\Omega^n, \mathcal{F}^n, \mathbb{F}^n, Q^n), S^n, \zeta^n)$ thus satisfies ADSV for $\vec{\mathbb{1}}$. Moreover, by uniqueness and Lemma 2.13, 1), the S^n-tradable Q^n-LMD$^{\mathbb{1}^n+}$ $\bar{D}^n$ for S^n coincides with the unique S^n-tradable Q^n-SMD$^{\mathbb{1}^n+}$ for $\mathcal{X}(S^n)$. Applying Theorem 4.1, (a) ⇒ (b), for

$((\Omega^n, \mathcal{F}^n, \mathbb{F}^n, Q^n), S^n, \zeta^n)$ therefore implies $(P^n)_{n\in\mathbb{N}} \triangleleft (Q^n)_{n\in\mathbb{N}} \triangleleft ((1/\bar{g}^n)\cdot Q^n)_{n\in\mathbb{N}}$ and hence the result by the transitivity of contiguity.

"(b) $\Rightarrow$ (c)": If Z^n is the density process of Q^n with respect to P^n, then $D^n := \bar{D}^n/Z^n$ is a P^n-LMD$^{\mathbb{1}^n+}$ for S^n by Bayes' theorem. As in the proof of Corollary 4.4, we obtain $(1/g^n)\cdot P^n = (1/\bar{g}^n)\cdot Q^n$, and therefore $(P^n)_{n\in\mathbb{N}} \triangleleft ((1/\bar{g}^n)\cdot Q^n)_{n\in\mathbb{N}} = ((1/g^n)\cdot P^n)_{n\in\mathbb{N}}$.

"(c) $\Rightarrow$ (a)": By Lemma 2.13, 2), an LMD$^{\mathbb{1}^n+}$ for S^n and an LMD$^{\mathbb{1}^n+}$ for $\mathcal{X}(S^n)$ are the same thing when $S^n \geq 0$. So we can just use Theorem 4.1, "(c) $\Rightarrow$ (a)". ☐

Analogously to Theorem 4.1, $\frac{\mu^n}{D^n/(\mathbb{1}^n\cdot S^n)} = \frac{S^n}{D^n}$ is a local P^n-martingale for each n, so that $(1/g^n)\cdot P^n$ could again be called a "generalised martingale measure".

Conceptually, an LMD$^{\mathbb{1}^n+}$ for S is a generalised form of an ELMM for S. So the equivalence of (a) and (c) in Theorem 4.5 corresponds precisely to the classic equivalence result going back to Klein/Schachermayer [26] and Kabanov/Kramkov [20]. It is remarkable that we are able to obtain such a result in the generality of our setup and ADSV. It is also noteworthy that while the sufficiency of (c) for ADSV for $\vec{\mathbb{1}}$ can be proved fairly easily, the necessity crucially involves a *tradable* LMD$^{\mathbb{1}+}$. This is different from the original result in [26, 20] because working directly with ELMMs eliminates this difficulty.

Theorem 4.5 (c) asserts under ADSV for $\vec{\mathbb{1}}$ the existence of a sequence of LMD$^{\mathbb{1}^n+}$s with a contiguity property. This contiguity does not hold for *every* sequence $(D^n)_{n\in\mathbb{N}}$.

Example 4.6. *ADSV for $\vec{\mathbb{1}}$ can hold even if there is a sequence $(D^n)_{n\in\mathbb{N}}$ of LMD$^{\mathbb{1}^n+}$s for S^n such that $(P^n\circ(g^n)^{-1})_{n\in\mathbb{N}}$ is not tight, where $g^n := ((\mathbb{1}^n\cdot S^n_0)D^n_{\zeta^n})/(\mathbb{1}^n\cdot S^n_{\zeta^n})$.* Let $(Y^{n,i})_{n\in\mathbb{N}}$, $i = 1,2$, be independent sequences of i.i.d. Bernoulli random variables (with values 0 and 1) with parameter $\frac{1}{2}$. Set $\zeta^n = 2$ for every n and define for $i = 1,2$ the single-jump processes

$$S^{n,i} := (S^n)^{(i)} := I_{[\![0,1[\![} + I_{[\![1,2]\!]}\Big(2^{-n} + 2Y^{n,i}(1-2^{-n})\Big).$$

Let $\mathbb{B}$ be a minimal stochastic basis supporting the above, set $\mathbb{B}^n := \mathbb{B}$ for each n and fix n. Then $(\mathbb{B}^n, S^n, \zeta^n)$ satisfies $S^n \geq 0$ and $\mathbb{1}^n\cdot S^n \in \mathcal{S}_{++}$, and S^n is a strictly positive UI martingale so that $D^n \equiv 1$ is an LMD$^{\mathbb{1}^n+}$ for S^n. By Proposition 2.19, every small market $(\mathbb{B}^n, S^n, \zeta^n)$ therefore satisfies DSV for $\mathbb{1}^n$.

The process D^n is not S^n-tradable, and the sequence $(P^n\circ(g^n)^{-1})_{n\in\mathbb{N}}$ is not tight because $g^n = 2/(S^{n,1}+S^{n,2})$ has $P[g^n = 2^n] = P[Y^{n,1} = Y^{n,2} = 0] = \frac{1}{4}$. But $\bar{D}^n := \mathbb{1}^n\cdot S^n$ is clearly S^n-tradable, and also an LMD$^{\mathbb{1}^n+}$ for S. Indeed, $S^{n,1}/\bar{D}^n = \mu^{n,1}$ is a single-jump process which starts at $\frac{1}{2}$, jumps at $t = 1$ to either $2^{-(n+1)}$ or $1-2^{-(n+1)}$ with probability $\frac{1}{4}$ each, and stays constant at $t=1$ with probability $\frac{1}{2}$. Thus $\mu^{n,1}$ is a (UI) martingale, so is then $\mu^{n,2} = 1-\mu^{n,1}$, and $\mathbb{1}^n\cdot(S^n/\bar{D}^n) = \mathbb{1}^n\cdot\mu^n \equiv 1$. Computing now $\bar{g}^n := (\mathbb{1}^n\cdot S^n_0)\bar{D}^n_{\zeta^n}/(\mathbb{1}^n\cdot S^n_{\zeta^n}) = \mathbb{1}^n\cdot S^n_0 \equiv 2$ shows that $(P^n\circ(\bar{g}^n)^{-1})_{n\in\mathbb{N}}$ is obviously tight, (c) in Lemma 4.2 is satisfied and ADSV for $\vec{\mathbb{1}}$ holds.

The preceding results are summarised in Figure 1. All one-sided implications are due to Lemma 2.13, 1) or trivial. The equivalences are due to Theorems 4.1 and 4.5. A counterexample for both invalid implications (crossed arrows) is given by [34, Remark 2.8].

$$
\begin{array}{ccccc}
\begin{array}{c}\forall n\ \exists\ \mathrm{LMD}^{\mathbb{1}^n+}\ \bar{D}^n \text{ for } S^n\\ \text{with } \bar{D}^n \text{ being } S^n\text{-tradable}\\ \text{and } (P^n)_{n\in\mathbb{N}} \lhd ((1/\bar{g}^n)\cdot P^n)_{n\in\mathbb{N}}\end{array} & \begin{array}{c}\Rightarrow\\ \not\Leftarrow\end{array} & \begin{array}{c}\forall n\ \exists\ \mathrm{SMD}^{\mathbb{1}^n+}\ \bar{D}^n \text{ for } \mathcal{X}(S^n)\\ \text{with } \bar{D}^n \text{ being } S^n\text{-tradable}\\ \text{and } (P^n)_{n\in\mathbb{N}} \lhd ((1/\bar{g}^n)\cdot P^n)_{n\in\mathbb{N}}\end{array} & & \\
\Downarrow \not\Uparrow & & \Updownarrow & & \\
\begin{array}{c}\forall n\ \exists\ \mathrm{LMD}^{\mathbb{1}^n+}\ D^n \text{ for } S^n\\ \text{with } (P^n)_{n\in\mathbb{N}} \lhd ((1/g^n)\cdot P^n)_{n\in\mathbb{N}}\end{array} & \Leftrightarrow & \begin{array}{c}\forall n\ \exists\ \mathrm{SMD}^{\mathbb{1}^n+}\ D^n \text{ for } \mathcal{X}(S^n)\\ \text{with } (P^n)_{n\in\mathbb{N}} \lhd ((1/g^n)\cdot P^n)_{n\in\mathbb{N}}\end{array} & \Leftrightarrow & \text{ADSV for } \vec{\mathbb{1}}
\end{array}
$$

Figure 1: Overview of results for Section 4. We assume $S^n \geq 0$ and $\mathbb{1}^n \cdot S^n \in \mathcal{S}_{++}$ as well as $\mathcal{F}^n = \mathcal{F}^n_{\zeta^n}$ for every small market. Notation is $g^n = ((\mathbb{1}^n \cdot S^n_0)D^n_{\zeta^n})/(\mathbb{1}^n \cdot S^n_{\zeta^n})$, $\bar{g}^n = ((\mathbb{1}^n \cdot S^n_0)\bar{D}^n_{\zeta^n})/(\mathbb{1}^n \cdot S^n_{\zeta^n})$.

The next result gives a sufficient condition for ADSV for $\vec{\mathbb{1}}$ in the classic setup.

Corollary 4.7. *Let every small market be of the form* $(\mathbb{B}^n, S^n, \zeta^n) = (\mathbb{B}^n, (1, X^n), \zeta^n)$ *with* $N^n = 1 + d^n$*, where each* $X^n \geq 0$ *is an* $\mathbb{R}^{d^n}_+$*-valued semimartingale and* $\zeta^n < \infty$*. Suppose also that* $\sup_{n\in\mathbb{N}}(\mathbb{1}^n \cdot S^n_0) < \infty$ *and that every small market satisfies NUPBR. Then if the large market satisfies NAA, it also satisfies ADSV for* $\vec{\mathbb{1}}$*.*

Proof. Any small market $(\mathbb{B}^n, (1, X^n), \zeta^n)$ with NUPBR satisfies DSV for $\mathbb{1}^n$ by [3, Proposition 5.6] and hence admits an S^n-tradable $\mathrm{SMD}^{\mathbb{1}^n+}$ $\bar{D}^n$ for $\mathcal{X}(S^n)$ by Proposition 2.20. Set $\bar{g}^n := ((\mathbb{1}^n \cdot S^n_0)\bar{D}^n_{\zeta^n})/(\mathbb{1}^n \cdot S^n_{\zeta^n})$. If we have the above for all n, ADSV for $\vec{\mathbb{1}}$ follows by Theorem 4.1 if $(P^n)_{n\in\mathbb{N}} \lhd ((1/\bar{g}^n) \cdot P^n)_{n\in\mathbb{N}}$. Take a sequence $(A^n)_{n\in\mathbb{N}}$ with $A^n \in \mathcal{F}^n$ and $\lim_{n\to\infty}((1/\bar{g}^n) \cdot P^n)[A^n] = 0$. Note that $\bar{g}^n \leq (\mathbb{1}^n \cdot S^n_0)\bar{D}^n_{\zeta^n} \leq C\bar{D}^n_{\zeta^n}$ because $X^n \geq 0$ gives $\mathbb{1}^n \cdot S^n \geq 1$. So $\limsup_{n\to\infty}((1/\bar{D}^n_{\zeta^n}) \cdot P^n)[A^n] \leq C \lim_{n\to\infty}((1/\bar{g}^n) \cdot P^n)[A^n] = 0$, all assumptions in [31] are satisfied, and we conclude from [31, Theorem 2.1] that NAA implies $(P^n)_{n\in\mathbb{N}} \lhd ((1/\bar{D}^n_{\zeta^n}) \cdot P^n)_{n\in\mathbb{N}}$. Hence we get $\lim_{n\to\infty} P^n[A^n] = 0$ and we are done. □

The conditions in Corollary 4.7 are sufficient for ADSV for $\vec{\mathbb{1}}$ in that setting, but not necessary. The converse can already fail when each X^n comes from stopping at n a fixed process X.

Example 4.8. *ADSV for* $\vec{\mathbb{1}}$ *does not imply NAA.* Fix a small market $(\mathbb{B}, (1, X), \infty)$ and consider the large market $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}} := (\mathbb{B}, (1, X_{\cdot\wedge n}), n)_{n\in\mathbb{N}}$. Then Corollary 5.4 below shows that NAA for the large market is equivalent to NUPBR for $(\mathbb{B}, (1, X), \infty)$, and that ADSV for $\vec{\mathbb{1}}$ in the large market is equivalent to DSV for $\mathbb{1}$ in $(\mathbb{B}, (1, X), \infty)$. Now [3, Example 6.7] provides a small market of the form $(\mathbb{B}, (1, X), \infty)$ with $X \geq 0$ which does not satisfy NUPBR, but satisfies DSV for $\mathbb{1}$. Hence the above large market does not satisfy NAA, but satisfies ADSV for $\vec{\mathbb{1}}$.

The last result in this section gives another necessary and sufficient condition for ADSV for $\vec{\mathbb{1}}$, linking this to the classic concept NAA.

Corollary 4.9. *Let every small market $(\mathbb{B}^n, S^n, \zeta^n)$ satisfy $S^n \geq 0$, $\mathbb{1}^n \cdot S^n \in \mathcal{S}_{++}$ and $\zeta^n < \infty$. Recall the market weight process $\mu^n = S^n/(\mathbb{1}^n \cdot S^n)$. Then the following are equivalent:*

(a) *The large market $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$ satisfies ADSV for $\vec{\mathbb{1}}$.*

(b) *Each small market $(\mathbb{B}^n, \mu^n, \zeta^n)$ satisfies NUPBR and the large market $(\mathbb{B}^n, \mu^n, \zeta^n)_{n\in\mathbb{N}}$ satisfies NAA.*

Proof. If $(\mathbb{B}^n, \mu^n, \zeta^n)$ satisfies NUPBR, it satisfies DSV for $\mathbb{1}^n$ by [3, Theorem 2.14] and hence by Proposition 2.20 admits a unique μ^n-tradable SMD$^{\mathbb{1}^n+}$ $\bar{D}^n$ for $\mathcal{X}(\mu^n)$. Using $\mathbb{1}^n \cdot \mu^n \equiv 1$ gives $\bar{g}^n := (\mathbb{1}^n \cdot \mu^n_0)\bar{D}^n_{\zeta^n}/(\mathbb{1}^n \cdot \mu^n_{\zeta^n}) = \bar{D}^n_{\zeta^n}$. If we have all this for each n and if also the large market $(\mathbb{B}^n, \mu^n, \zeta^n)_{n\in\mathbb{N}}$ satisfies NAA, we get $(P^n)_{n\in\mathbb{N}} \lhd ((1/\bar{g}^n) \cdot P^n)_{n\in\mathbb{N}}$ from [31, Theorem 2.1], and so Theorem 4.1 implies that $(\mathbb{B}^n, \mu^n, \zeta^n)_{n\in\mathbb{N}}$ satisfies ADSV for $\vec{\mathbb{1}}$. But then so does $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$ because asymptotic strong share maximality for $\vec{\mathbb{1}}$ is discounting-invariant. This proves "(b) $\Rightarrow$ (a)".

Conversely, if $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$ satisfies ADSV for $\vec{\mathbb{1}}$, every $(\mathbb{B}^n, S^n, \zeta^n)$ satisfies DSV for $\mathbb{1}^n$ due to Lemma 3.4, and [3, Theorem 2.14] implies that $(\mathbb{B}^n, \mu^n, \zeta^n)$ satisfies NUPBR for each n. As in the above proof of "(b) $\Rightarrow$ (a)", we therefore obtain a unique μ^n-tradable SMD$^{\mathbb{1}^n+}$ $\bar{D}^n$ for $\mathcal{X}(\mu^n)$ and $\bar{g}^n = \bar{D}^n_{\zeta^n}$. Because ADSV for $\vec{\mathbb{1}}$ is discounting-invariant, $(\mathbb{B}^n, \mu^n, \zeta^n)_{n\in\mathbb{N}}$ also satisfies ADSV for $\vec{\mathbb{1}}$, and applying Theorem 4.1 to $(\mathbb{B}^n, \mu^n, \zeta^n)_{n\in\mathbb{N}}$ gives $(P^n)_{n\in\mathbb{N}} \lhd ((1/\bar{g}^n) \cdot P^n)_{n\in\mathbb{N}}$. But then [31, Theorem 2.1] implies that $(\mathbb{B}^n, \mu^n, \zeta^n)_{n\in\mathbb{N}}$ satisfies NAA and so we get "(a) $\Rightarrow$ (b)". ☐

Remark 4.10. Corollaries 4.7 and 4.9 assume $\zeta^n < \infty$, because NAA up to now has only been defined for that setting. We believe that with a suitable generalisation of NAA for the case $P[\zeta^n = \infty] > 0$, these results still hold (i.e., one could omit the condition $\zeta^n < \infty$). But we do not pursue this here in more detail.

5 Models indexed by $[0, \infty)$ as large markets

In this section, we study the special case where the stochastic basis $\mathbb{B}^n = (\Omega^n, \mathcal{F}^n, \mathbb{F}^n, P^n)$ is constant in n, $\zeta^1 \leq \zeta^2 \leq \cdots < \infty$ with $\lim_{n\to\infty} \zeta^n = \infty$ and S^n is of the stopped form $S^n = S_{\cdot\wedge\zeta^n}$ for a fixed semimartingale $S = (S_t)_{t\geq 0}$ on the basis $\mathbb{B}^1 = \mathbb{B}$. This very specific large market framework morally coincides with the small market framework where $\zeta \equiv \infty$ so that we consider a model on $[\![0, \infty]\!] = \Omega \times [0, \infty)$. In particular, models indexed by the right-open time interval $[0, \infty)$ come up naturally in this way.

Proposition 5.1. *Let $\mathbb{B}^n \equiv \mathbb{B}$ and $\mathbb{N}^n \equiv 1 + d \geq 2$. Suppose the stopping times $\zeta^1 \leq \zeta^2 \leq \cdots < \infty$ satisfy $\lim_{n\to\infty} \zeta^n = \infty$ and each S^n is of the stopped form $S^n = S_{\cdot\wedge\zeta^n}$ with $S = (1, X)$ for a fixed $\mathbb{R}^d$-valued semimartingale $X = (X_t)_{t\geq 0}$ on the basis $\mathbb{B}$.*

Then NAA holds for $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$ *if and only if NUPBR holds for the small market* $(\mathbb{B}, (1, X), \infty)$.

Proof. Throughout the proof, $\mathrm{e}^1 := (1, 0, \dots, 0) \in \mathbb{R}^{1+d}$ is the buy-and-hold strategy for the riskless asset so that $V(\mathrm{e}^1, (1, X)) \equiv 1$. Moreover, probability always refers to $P = P^1$.

If NAA fails, there are $\delta > 0$ and a large market strategy $\vec{\vartheta}$ such that any $\varepsilon > 0$ admits $n \in \mathbb{N}$ with $V_0(\vartheta^n, (1, X)) = \varepsilon$ and $P[V_{\zeta^n}(\vartheta^n, (1, X)) \geq 1] \geq \delta$. Due to [3, Lemma 3.3], the concatenated strategy $\tilde{\vartheta}^n := (\vartheta^n/\varepsilon) \oslash^{\mathrm{e}^1}_{\zeta^n} 0$ is in $\Theta^{\mathrm{sf}}_+(1, X)$, and the definition of $\tilde{\vartheta}^n$, see (2.3), yields $V_0(\tilde{\vartheta}^n, (1, X)) = 1$ and that $U(\varepsilon) := \lim_{t\to\infty} V_t(\tilde{\vartheta}^n, (1, X))$ exists and is in $\mathcal{X}^1_\infty(1, X)$ with $P[U(\varepsilon) \geq \frac{1}{\varepsilon}] \geq \delta > 0$. As δ is fixed and $\varepsilon > 0$ is arbitrary, $\mathcal{X}^1_\infty(1, X)$ cannot be bounded in L^0 and NUPBR fails.

If NUPBR fails, there is $\delta > 0$ and for any $n \in \mathbb{N}$ a strategy $\vartheta^n \in \Theta^{\mathrm{sf}}_+(1, X)$ with $V_0(\vartheta^n, (1, X)) = 1$ and such that $U(n) := \lim_{t\to\infty} V_t(\vartheta^n, (1, X))$ exists and satisfies $P[U(n) \geq 2n] \geq 2\delta$. But then any n admits $k = k(n) > k(n-1)$ (with $k(0) := 0$) with $P[V_{\zeta^{k(n)}}(\vartheta^n, (1, X)) \geq n] > \delta$, and so each $\tilde{\vartheta}^n := (\vartheta^n/n) I_{[\![0,\zeta^{k(n)}]\!]} \in \Theta^{\mathrm{sf}}_+(S^{k(n)})$ has $V_0(\tilde{\vartheta}^n, (1, X)) \leq \frac{1}{n}$ and $P[V_{\zeta^{k(n)}}(\tilde{\vartheta}^n, (1, X)) \geq 1] > \delta$. Define $\vec{\vartheta}$ by $\vartheta^{k(n)} := \tilde{\vartheta}^n$ and $\vartheta^i := 0$ for $i \neq k(n)$, $n \in \mathbb{N}$. Then $\lim_{n\to\infty} V_0(\vartheta^n, S^n) = 0$ and NAA fails because

$$\limsup_{n\to\infty} P[V_{\zeta^n}(\vartheta^n, S^n) \geq 1] \geq \limsup_{n\to\infty} P[V_{\zeta^{k(n)}}(\vartheta^{k(n)}, S^{k(n)}) \geq 1] \geq \delta.$$

□

Remark 5.2. Results similar to Proposition 5.1 can already be found in the literature. The same equivalence is obtained for a very special situation (a two-dimensional discrete-time model with a kind of recurrent asset price dynamics) by combining [31, Theorem 6.2] with [21, Theorem 4.12]. For a large market framework with a finite horizon, an analogous equivalence result is proved in [9, Proposition 4.3].

Proposition 5.3. *Let* $\mathbb{B}^n \equiv \mathbb{B}$ *and* $N^n \equiv N \geq 2$ *for all* n, *assume that the stopping times* $\zeta^1 \leq \zeta^2 \leq \cdots < \infty$ *satisfy* $\lim_{n\to\infty} \zeta^n = \infty$, *and let* S^n *be of the stopped form* $S^n = S_{\cdot\wedge\zeta^n}$ *for a fixed* $\mathbb{R}^N$*-valued semimartingale* $S = (S_t)_{t\geq 0}$ *on the basis* $\mathbb{B}$. *Suppose that* $S \geq 0$ *and* $\mathbb{1} \cdot S \in \mathcal{S}_{++}$.

1) *Fix* $\vartheta \in \Theta^{\mathrm{sf}}_+(S)$ *and let* $\vec{\vartheta} = (\vartheta I_{[\![0,\zeta^n]\!]})_{n\in\mathbb{N}}$ *be the corresponding large market strategy. Then* $\vec{\vartheta}$ *is assm for* $\vec{\mathbb{1}}$ *if* ϑ *is ssm for* $\mathbb{1}$ *in the small market* $(\mathbb{B}, S, \infty)$.
2) *For* $\vartheta \equiv 0$, *the converse holds as well, i.e.* $\vec{0} = (0^n)_{n\in\mathbb{N}}$ *is assm for* $\vec{\mathbb{1}}$ *if and only if 0 is ssm for* $\mathbb{1}$ *in the small market* $(\mathbb{B}, S, \infty)$.
3) *In particular, the large market satisfies ADSV for* $\vec{\mathbb{1}}$ *if and only if* S *satisfies DSV for* $\mathbb{1}$ *on* $[0, \infty)$.

Proof. As all the S^n come from S, we have $\mathbb{1}^n \cdot S^n_0 \equiv \mathbb{1} \cdot S_0$ and can rescale simultaneously all the S^n with one single constant. This does not affect asymptotic strong share maximality for $\vec{\mathbb{1}}$, and so we can assume without loss of generality that $\mathbb{1} \cdot S_0 = 1$. We use this later to get $V_0(\,\cdot\,, S) = V_0(\,\cdot\,, \mu)$.

1) This is proved similarly to the "if" part of Proposition 3.5. Suppose ϑ is ssm but $\vec{\vartheta}$ is not assm for $\vec{\mathbb{1}}$, consider the set

$$\mathcal{G}^{\vartheta,\varepsilon} := \Big\{ g \in L^0_+ : \exists \varphi \in \Theta^{\mathrm{sf}}_+(S) \text{ with } V_0(\varphi - \vartheta, S) \le \varepsilon \text{ and } \liminf_{t\to\infty}(\varphi_t - \vartheta_t) \ge g\mathbb{1} \ P\text{-a.s.} \Big\}$$

and define $\mathcal{G}^\vartheta := \bigcup_{\varepsilon>0} \mathcal{G}^{\vartheta,\varepsilon}/\varepsilon$. We first show again that $\mathcal{G}^\vartheta$ is not bounded in L^0. Because $\vec{\vartheta}$ is not assm for $\vec{\mathbb{1}}$ and each ζ^n is finite, there exists a $p > 0$ such that for each $\varepsilon > 0$, there are some $n \in \mathbb{N}$, $A^{n,\varepsilon} \in \mathcal{F}$ with $P[A^{n,\varepsilon}] \ge p$ and a strategy $\vec{\hat{\vartheta}}$ with

$$V_0(\hat{\vartheta}^n, S^n) \le V_0(\vartheta^n, S^n) + \varepsilon(\mathbb{1}^n \cdot S^n_0) = V_0(\vartheta, S^n) + \varepsilon,$$
$$\liminf_{t\to\infty}(\hat{\vartheta}^n_t - \vartheta_t) = \hat{\vartheta}^n_{\zeta^n} - \vartheta_{\zeta^n} = \hat{\vartheta}^n_{\zeta^n} - \vartheta^n_{\zeta^n} \ge pI_{A^{n,\varepsilon}}\mathbb{1} \qquad P\text{-a.s.;}$$

see (2.1). This implies $V_{\zeta^n}(\hat{\vartheta}^n - \vartheta, \mu) \ge V_{\zeta^n}(pI_{A^{n,\varepsilon}}\mathbb{1}, \mu) = pI_{A^{n,\varepsilon}} \ge 0$, using $\mu \ge 0$ due to $S \ge 0$ and $V(\mathbb{1}, \mu) \equiv 1$. Then $\varphi := \hat{\vartheta}^n \circledS^{\mathbb{1}}_{\zeta^n} \vartheta$ defined as in (2.3) is in $\Theta^{\mathrm{sf}}_+(S)$ by [3, Lemma 3.3], and $V_0(\varphi, S) = V_0(\hat{\vartheta}^n, S) \le V_0(\vartheta, S) + \varepsilon$. Because $V_{\zeta^n}(\hat{\vartheta}^n, S) \ge V_{\zeta^n}(\vartheta, S)$, the definition of $\circledS^{\mathbb{1}}_{\zeta^n}$ gives $\varphi_t = \vartheta_t + V_{\zeta^n}(\hat{\vartheta}^n - \vartheta, \mu)\mathbb{1}$ for $t > \zeta^n(\omega)$ and therefore

$$\liminf_{t\to\infty}(\varphi_t - \vartheta_t) = V_{\zeta^n}(\hat{\vartheta}^n - \vartheta, \mu)\mathbb{1} \ge pI_{A^{n,\varepsilon}}\mathbb{1} \qquad P\text{-a.s.}$$

so that $pI_{A^{n,\varepsilon}}/\varepsilon \in \mathcal{G}^\vartheta$ for any $\varepsilon > 0$ and hence again, $\mathcal{G}^\vartheta$ is not bounded in L^0. From here, one can argue word by word as in the second and third paragraph in the proof of Proposition 3.5 to conclude that ϑ is not ssm for $\mathbb{1}$, which is a contradiction.

2) If $\vartheta \equiv 0$ is not ssm for $\mathbb{1}$ in $(\mathbb{B}, S, \infty)$, and hence also not in $(\mathbb{B}, \mu, \infty)$, by discounting-invariance, Lemma 2.16 yields a $\psi_\infty \in L^\infty_+ \setminus \{0\}$ such that for any $\varepsilon > 0$, there is a $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+(S)$ with $V_0(\hat{\vartheta}^\varepsilon, \mu) \le \varepsilon$ and $\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon, \mu) \ge \psi_\infty$ P-a.s. For this ψ_∞, we can find $p > 0$ and $B \in \mathcal{F}$ (not depending on ε) such that $P[B] \ge 2p$ and $2pI_B \le \psi_\infty$ P-a.s. Define the stopping time $\tau := \inf\{t \ge 0 : V_t(\hat{\vartheta}^\varepsilon, \mu) \ge p\}$ and note that $\tau < \infty$ P-a.s. on B. Because $\lim_{n\to\infty} \zeta^n = \infty$, we can thus find n such that $A^n := B \cap \{\zeta^n \ge \tau\}$ has $P[A^n] \ge p$, and clearly $\varrho^n := \tau \wedge \zeta^n < \infty$ satisfies $\varrho^n = \tau$ on A^n. Moreover, $V(\hat{\vartheta}^\varepsilon, \mu) \ge 0$ as $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+(S)$, and $V_\tau(\hat{\vartheta}^\varepsilon, \mu) \ge p$ on B P-a.s. by right-continuity, so that we obtain

$$V_{\varrho^n}(\hat{\vartheta}^\varepsilon, \mu) = V_\tau(\hat{\vartheta}^\varepsilon, \mu) \ge p \qquad \text{on } A^n,\ P\text{-a.s.} \tag{5.1}$$

Now define the strategy $\hat{\vartheta}^n := \hat{\vartheta}^\varepsilon \circledS^{\mathbb{1}}_{\varrho^n} 0$ as in (2.3). Then $\hat{\vartheta}^n$ is in $\Theta^{\mathrm{sf}}_+(S)$ by [3, Lemma 3.3], has $V_0(\hat{\vartheta}^n, S) = V_0(\hat{\vartheta}^n, \mu) = V_0(\hat{\vartheta}^\varepsilon, \mu) \le \varepsilon$, and by (2.3), using $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+(S)$,

$$\hat{\vartheta}^n = I_{[\![0,\varrho^n]\!]}\hat{\vartheta}^\varepsilon + I_{]\!]\varrho^n,\infty[\![}V_{\varrho^n}(\hat{\vartheta}^\varepsilon, \mu)\mathbb{1}.$$

Using $\varrho^n < \infty$, $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+(S)$ and (5.1) therefore yields

$$\liminf_{t\to\infty} \hat{\vartheta}^n_t = V_{\varrho^n}(\hat{\vartheta}^\varepsilon, \mu)\mathbb{1} \ge pI_{A^n}\mathbb{1} \qquad P\text{-a.s.}$$

So $(n, A^n, \hat{\vartheta}^n)$ satisfies the conditions in Definition 3.1 and $\vec{0} = (0^n)_{n\in\mathbb{N}}$ is not assm for $\vec{\mathbb{1}}$.

3) follows directly from 2) and the definitions. □

For $\vartheta \not\equiv 0$, the converse of Proposition 5.3, 1) is not true in general. Example 6.1 below constructs a large market strategy $\vec{\vartheta} = (\vartheta^n)_{n\in\mathbb{N}}$ of the form $\vartheta^n = \vartheta I_{[\![0,n]\!]}$ which is assm for $\vec{\mathbb{1}}$ even if ϑ is not ssm for $\mathbb{1}$.

Example 2.6. Let us now see what this section tells us about the Black–Scholes model $S = (1, X)$ from (2.2) when viewed as a large market as above. By Proposition 5.1, we have NAA if and only if X satisfies NUPBR on $[0,\infty)$, and this holds if and only if $m = r$. In that case, X is a martingale and we even have NFLVR. By Proposition 5.3, we have ADSV for $\vec{\mathbb{1}}$ if and only if S (or equivalently Y, in the notation of Example 2.6) satisfies DSV for $\mathbb{1}$, and by [3, Theorem 6.4, 1)], this holds if and only if $m = r$ or $m = r + \sigma^2$. In the former case, $Y/Y^{(1)} = (1, X)$ is a martingale while in the latter case, $Y/Y^{(2)} = (1/X, 1)$ is a martingale. Thus ADSV is more symmetric than NAA, as DSV is more symmetric than NUPBR.

The results for models indexed by $[0,\infty)$ viewed as large markets are summarised in the following result. We choose $S = (1, X)$ with $X \geq 0$ so that we can cover the intersection of the classic setup with the framework where $S \geq 0$ and $\mathbb{1} \cdot S \in \mathcal{S}_{++}$.

Corollary 5.4. *Let $\mathbb{B}^n \equiv \mathbb{B}$, $N^n \equiv N = 1 + d \geq 2$ and assume that the stopping times $\zeta^1 \leq \zeta^2 \leq \cdots < \infty$ satisfy $\lim_{n\to\infty} \zeta^n = \infty$. Let $S = (1, X)$ for a fixed $\mathbb{R}^d_+$-valued semimartingale $X = (X_t)_{t\geq 0}$ on $\mathbb{B}$ and for each n, let S^n be of the stopped form $S^n = S_{\cdot\wedge\zeta^n}$. Then the following relations hold:*

large market $(\mathbb{B}^n, S^n, \zeta^n)_{n\in\mathbb{N}}$		*small market* $(\mathbb{B}, S, \infty)$		*dual conditions for* $(\mathbb{B}, S, \infty)$
NAA	$\Longleftrightarrow$	*NUPBR is satisfied on* $[0,\infty)$	$\Longleftrightarrow$	$\exists$ *LMD* D *for* $\mathcal{X}(S)$ *with* $\lim_{t\to\infty} D_t < \infty$ *P-a.s.*
$\Downarrow \not\Uparrow$		$\Downarrow \not\Uparrow$		$\Downarrow \not\Uparrow$
ADSV for $\vec{\mathbb{1}}$	$\Longleftrightarrow$	*DSV for* $\mathbb{1}$	$\Longleftrightarrow$	$\exists$ *LMD*$^{\mathbb{1}+}$ *for* $\mathcal{X}(S)$
$\Downarrow \not\Uparrow$		$\Downarrow \not\Uparrow$		$\Downarrow \not\Uparrow$
NUPBR is satisfied in each small market	$\Longleftrightarrow$	*NUPBR is satisfied on* $[\![0,\zeta^n]\!]$ *for each* n	$\Longleftrightarrow$	$\exists$ *LMD for* $\mathcal{X}(S)$

Proof. In the first line, the first equivalence is from Proposition 5.1, and the second follows by combining [3, Proposition 5.8 (with $\eta \equiv \mathrm{e}^1$)] with Lemma 2.13, 2). In the second line, the first equivalence is from Proposition 5.3, and the second from [3, Theorem 2.11 (with $\eta \equiv \mathbb{1}$)]. In the third line, the first equivalence is a tautology; the second is from Chau et al. [6, Proposition 2.1]. The lower downward implication in the third column is obvious, and the upper follows by noting that due to $X \geq 0$, $\lim_{t\to\infty} D_t < \infty$ P-a.s. implies that

$$\liminf_{t\to\infty} (\mathbb{1} \cdot S_t)/D_t = \liminf_{t\to\infty} (1 + \sum_i X^{(i)}_t)/D_t \geq \lim_{t\to\infty} 1/D_t > 0 \qquad P\text{-a.s.}$$

Finally, the downward implications for the other two columns follow from the equivalences proved above. □

Remark 5.5. In the very special small market case $(\mathbb{B}, (1, X), \zeta)$ with a bounded horizon $\zeta \leq C < \infty$ and a semimartingale $X \geq 0$, we have automatically both $S \geq 0$ and $0 < \inf_{t\geq 0}(\mathbb{1} \cdot S_t) \leq \sup_{t\geq 0}(\mathbb{1} \cdot S_t) < \infty$ P-a.s. Then DSV for $\mathbb{1}$ and NUPBR are equivalent by [3, Corollary 3.5 (with $\xi \equiv \mathbb{1}$) and Proposition 3.6 (with $\xi \equiv \mathrm{e}^1$)]. Thus we could reformulate the third line in Corollary 5.4 by twice writing DSV for $\mathbb{1}$ instead of NUPBR.

Remark 5.6. In the literature, NUPBR is usually considered to be localisation-stable in the sense that NUPBR holds over a finite interval $[0, T]$ (with a deterministic $T < \infty$) if and only if it holds on each $[\![0, \tau_n]\!]$ for a sequence $(\tau_n)_{n\in\mathbb{N}}$ with $\tau_n \nearrow T$ stationarily. This follows from [34, Theorem 2.6], and was pointed out explicitly in [8, Section 3].

In contrast, *NUPBR on* $[0, \infty)$ *is not stable under localisation.* For a counterexample, take $S = (1, X)$, where $1/X$ is given by the stochastic exponential $\mathcal{E}(W)$ for a Brownian motion W. Then $X_t = e^{-W_t + \frac{1}{2}t}$ gives $\lim_{t\to\infty} X_t = +\infty$ P-a.s. so that S does not satisfy NUPBR on $[0, \infty)$. But for each $n \in \mathbb{N}$, the stopped process $S_{\cdot\wedge n}$ admits an equivalent martingale measure (we can simply remove its drift on $[0, n]$ by a Girsanov transformation), and so $S_{\cdot\wedge n}$ satisfies NFLVR and a fortiori NUPBR, on $[0, \infty)$. In other words, S satisfies NUPBR on $[0, n]$, for any $n \in \mathbb{N}$.

The symbols $\Downarrow\not\Uparrow$ in Corollary 5.4 indicate that the upward implications there are not true in general. As all columns are equivalent, we only need two counterexamples; see Example 4.8 for the upper implication and [3, Example 6.10] for the lower one. The latter example has an $S \geq 0$ so that an LMD is the same as a σMD by Lemma 2.13, 2).

6 A counterexample

Example 6.1. *A large market strategy* $\vec{\vartheta} = (\vartheta^n)_{n\in\mathbb{N}}$ *of the form* $\vartheta^n = \vartheta I_{[\![0,n]\!]}$ *can be asymptotically strongly share maximal (assm) for* $\vec{\mathbb{1}}$ *even if* ϑ *is not strongly share maximal (ssm) for* $\mathbb{1}$. Let $(\Omega, \mathcal{F}, P)$ be a probability space on which we have an Exp(1)-distributed random variable U and define the $\mathbb{R}^2_{++}$-valued process $S = (1, X)$ by

$$X_t = e^{-t} I_{\{t<U\}} + 2 I_{\{t\geq U\}}, \qquad t \geq 0. \tag{6.1}$$

The filtration $\mathbb{F}$ is generated by S (or X) and made right-continuous, and we consider the small market $(\mathbb{B}, S, \infty)$. For each $n \in \mathbb{N}$, we set $\mathbb{B}^n := \mathbb{B}$, $\zeta^n := n$ and $S^n := S_{\cdot\wedge\zeta^n}$ to obtain a large market $(\mathbb{B}, S^n, n)_{n\in\mathbb{N}}$.

Intuitively, it is clear that the buy-and-hold strategy e^2 for X dominates e^1 on $[0, \infty)$; indeed, both start with initial value 1, but while $V(\mathrm{e}^1, S)$ stays at 1, $V(\mathrm{e}^2, S) = X$ eventually jumps to 2 (at the jump time $U < \infty$ P-a.s.). However, on any finite interval $[0, n]$, e^2 cannot dominate e^1 — if we go long e^2, we continuously lose wealth relative to e^1 until the jump at time U, which need not occur on $[0,n]$, and if we go short e^2, we risk a big loss relative to e^1 when X jumps at time U, which may well occur on $[0,n]$. Making this intuition precise in the sequel involves a careful handling of nullsets in several places.

Let us first prove (the easy part) that

$$\text{the strategy } \vartheta \equiv \mathrm{e}^1 \in \Theta^{\mathrm{sf}}_+(S) \text{ is not ssm for } \mathbb{1} \text{ (and } S\text{)},$$

by showing that $\bar\vartheta := \mathrm{e}^2 \circledS^{\mathbb{1}}_U \mathrm{e}^1$ improves ϑ. Indeed, $\bar\vartheta \in \Theta^{\mathrm{sf}}_+(S)$ by [3, Lemma 3.3], we have $V_0(\bar\vartheta, S) = V_0(\vartheta, S) = 1$, and using that $V_U(\mathrm{e}^1, S) = 1$, $V_U(\mathrm{e}^2, S) = X_U = 2$ yields $V_U(\mathrm{e}^2 - \mathrm{e}^1, \mu) = (2-1)/(\mathbb{1}\cdot S_U) = \frac{1}{3}$ so that (2.3) gives $\bar\vartheta = I_{[\![0,U]\!]}\mathrm{e}^2 + I_{]\!]U,\infty[\![}(\mathrm{e}^1 + \frac{1}{3}\mathbb{1})$. As $U<\infty$ P-a.s., we thus get $\liminf_{t\to\infty}(\bar\vartheta_t - \vartheta_t) = \frac{1}{3}\mathbb{1}$ which shows that ϑ is not ssm for $\mathbb{1}$.

Now define $\vec\vartheta = (\vartheta^n)_{n\in\mathbb{N}}$ by setting $\vartheta^n := \vartheta I_{[\![0,n]\!]} = \mathrm{e}^1 I_{[\![0,n]\!]}$ for all n. We claim that

$$\text{the large market strategy } \vec\vartheta \text{ is assm for } \vec{\mathbb{1}}, \tag{6.2}$$

and this needs substantially more work.

Because $S=(1,X)$ and $\mathcal{F}_0$ is trivial, every $\vartheta \in \Theta^{\mathrm{sf}}(S)$ can be identified with a pair $(v_0, H) \in \mathbb{R}\times L(X)$ via $v_0 = V_0(\vartheta, S)$, $H = \vartheta^{(2)}$ and $\vartheta^{(1)} = v_0 + \int H\,\mathrm{d}X - H\cdot X$, $\vartheta^{(2)} = H$, and then $V(\vartheta,S) = v_0 + \int H\,\mathrm{d}X$. We write $\vartheta \mathrel{\widehat{=}} (v_0, H)$. Because X (and S) is constant on $[\![U,\infty[\![$, so is $\int H\,\mathrm{d}X$ as well as $V(\vartheta,S)$, and so we only need $HI_{[\![0,U]\!]}$. But due to $\mathbb{F} = \mathbb{F}^S$ and the structure of S which is deterministic on $[\![0,U[\![$ and has one single jump at U, a standard monotone class argument shows that every $\mathbb{F}$-predictable H can be written as

$$HI_{[\![0,U]\!]} = h(U\wedge\,\cdot\,)I_{[\![0,U]\!]}$$

for some Borel-measurable function $h:[0,\infty)\to\mathbb{R}$. We also write $\vartheta \mathrel{\widehat{=}} (v_0,H) \mathrel{\widehat{=}} (v_0,h)$. By exploiting the form of X in (6.1), we then obtain that with probability 1,

$$V_t(\vartheta,S) = v_0 - \int_0^{t\wedge U} h(s)e^{-s}\,\mathrm{d}s + I_{\{t\geq U\}}h(U)(2-e^{-U}) \qquad \text{for all } t\geq 0. \tag{6.3}$$

If in addition $\vartheta \in \Theta^{\mathrm{sf}}_+(S)$, the expression in (6.3) is also nonnegative.

For any set $A\in\mathcal{F}$, the set $B := U(A)$ is in $\mathcal{B}([0,\infty))$ and satisfies $U^{-1}(B)\supseteq A$ so that $(P\circ U^{-1})[B] \geq P[A]$. Due to $U\sim \mathrm{Exp}(1)$, the law $P\circ U^{-1}$ of U on $[0,\infty)$ is equivalent to Lebesgue measure λ on $[0,\infty)$. If $P[A]=1$, then $\lambda(B^c)=0$ and for every $t\in B$, there is an $\omega\in A$ with $t = U(\omega)$; in other words, λ-almost every t admits an $\omega\in A$ with $U(\omega)=t$. If $P[A]>0$, then also $\lambda(B)>0$ so that there exist (uncountably) infinitely many $t_0\in B$ such that $t_0 = U(\omega_0)$ for some $\omega_0\in A$. Applying this to the set A_0 where (6.3) holds and is nonnegative yields for any $\vartheta\in\Theta^{\mathrm{sf}}_+(S)$ with $\vartheta \mathrel{\widehat{=}} (v_0,h)$ that

$$0 \leq v_0 - \int_0^t h(s)e^{-s}\,\mathrm{d}s + h(t)(2-e^{-t}) \qquad \text{for } \lambda\text{-a.e. } t\geq 0. \tag{6.4}$$

If in addition $V_n(\vartheta,S)\geq v_0$ P-a.s., using $A_0' := A_0\cap\{V_n(\vartheta,S)\geq v_0\}$ and observing that $V_t(\vartheta,S)(\omega) = V_n(\vartheta,S)(\omega)$ if $U(\omega)=t\leq n$ implies that there is a λ-nullset $C\subseteq[0,n]$ with

$$v_0 \leq v_0 - \int_0^t h(s)e^{-s}\,\mathrm{d}s + h(t)(2-e^{-t}) \qquad \text{for all } t\in[0,n]\setminus C. \tag{6.5}$$

Next, for any $n \in \mathbb{N}$, the sets $A_{>n} := A_0 \cap \{U > n\}$ and $A_{\leq n} := A_0 \cap \{U \leq n\}$ satisfy $P[A_{>n}] > 0$ and $P[A_{\leq n}] > 0$. Choosing $t_0 > n$ and $\omega_0 \in A_{>n}$ with $U(\omega_0) = t_0$ then yields

$$0 \leq v_0 - \int_0^t h(s)e^{-s}\,\mathrm{d}s \qquad \text{for all } t \in [0, n]. \tag{6.6}$$

If in addition $V_n(\vartheta, S) \geq v_0$ P-a.s., arguing with $A'_0 \cap \{U > n\}$ instead gives

$$v_0 \leq v_0 - \int_0^t h(s)e^{-s}\,\mathrm{d}s \qquad \text{for all } t \in [0, n]. \tag{6.7}$$

Now consider (6.4) and write the last summand as $h(t) + h(t)(1 - e^{-t})$. If we think of an $h \leq 0$, the preceding sum is $\leq h(t)$ and we obtain $0 \leq v_0 - \int_0^t h(s)e^{-s}\,\mathrm{d}s + h(t)$. Replacing the inequality by an equality leads to the differential (or rather integral) equation

$$0 = v_0 - \int_0^t h(s)e^{-s}\,\mathrm{d}s + h(t), \qquad t \geq 0, \tag{6.8}$$

and using $\frac{\mathrm{d}}{\mathrm{d}t}(e^{-e^{-t}}) = e^{-e^{-t}}e^{-t}$ shows that the function $\hat{h}(t) := -v_0 e^{1-e^{-t}}$ for $t \geq 0$ satisfies (6.8). As $\hat{h}$ is strictly decreasing with $\hat{h}(0) = -v_0$ and $\lim_{t\to\infty} \hat{h}(t) = -v_0 e$, we have

$$-v_0 e \leq \hat{h}(t) \leq -v_0 < 0 \qquad \text{for all } t \geq 0, \tag{6.9}$$

and in particular $\hat{h} < 0$. Note that $\hat{h}$ depends on v_0 even if our notation does not show this.

Remark. If we associate to the function $\hat{h}$ a strategy $\hat{\vartheta}$ via $\hat{\vartheta} \mathrel{\widehat{=}} (v_0, \hat{h})$, then $\hat{\vartheta} \in \Theta^{\mathrm{sf}}(S)$ by construction, but $\hat{\vartheta} \notin \Theta^{\mathrm{sf}}_+(S)$. Indeed, combining (6.3), (6.8) and (6.9) gives

$$V(\hat{\vartheta}, S) = v_0 - \int_0^U \hat{h}(s)e^{-s}\,\mathrm{d}s + \hat{h}(U)(2 - e^{-U}) = \hat{h}(U)(1 - e^{-U}) < 0 \qquad \text{on } [\![U, \infty[\![.$$

Claim 1. For every $\vartheta \in \Theta^{\mathrm{sf}}_+(S)$ with $\vartheta \mathrel{\widehat{=}} (v_0, h)$, we have $h \geq \hat{h}$ λ-a.e.

Proof. If $\lambda(\{h < \hat{h}\}) > 0$, then $\Delta(t) := \operatorname{ess\,inf}_{0\leq s\leq t}(h(s) - \hat{h}(s)) < 0$ on a set C with $\lambda(C) > 0$; the ess inf is of course with respect to λ. Choose C to be of maximal measure and set $t_0 := \inf C = \inf\{t \geq 0 : \Delta(t) < 0\}$. Then $\Delta(t) \geq 0$ for $t < t_0$, hence

$$\int_0^t \big(h(s) - \hat{h}(s)\big)e^{-s}\,\mathrm{d}s \geq \Delta(t) \int_0^t e^{-s}\,\mathrm{d}s \geq 0 \qquad \text{for } t < t_0$$

and therefore by continuity

$$\int_0^{t_0} h(s)e^{-s}\,\mathrm{d}s \geq \int_0^{t_0} \hat{h}(s)e^{-s}\,\mathrm{d}s. \tag{6.10}$$

Moreover, as $t_0 = \inf C$ and $\lambda(C) > 0$, there exist infinitely many $t_2 > t_0$ with

$$\alpha(t_2) := \operatorname*{ess\,inf}_{t_0 < s \leq t_2} \big(h(s) - \hat{h}(s)\big) < 0;$$

note that $\Delta(t_2) = \min(\Delta(t_0), \alpha(t_2))$. So there exists a set $C' \subseteq (t_0, t_2]$ with $\lambda(C') > 0$ such that for all $t_1 \in C'$, we have

$$h(t_1) - \hat{h}(t_1) \le \alpha(t_2) + \delta < 0 \tag{6.11}$$

for some $\delta > 0$ to be chosen. Pick and fix such a t_1. Then (6.10) and $\alpha(t_2) < 0$ give

$$\int_0^{t_1} \big(h(s) - \hat{h}(s)\big) e^{-s}\,\mathrm{d}s \ge \int_{t_0}^{t_1} \big(h(s) - \hat{h}(s)\big) e^{-s}\,\mathrm{d}s \ge \alpha(t_2)(e^{-t_0} - e^{-t_1}) \ge \alpha(t_2) e^{-t_0}. \tag{6.12}$$

Moreover, using (6.11), $\hat{h}(t_1) < 0$ and $t_1 > t_0$ yields

$$(2 - e^{-t_1}) h(t_1) \le (2 - e^{-t_1})\Big(\hat{h}(t_1) + \alpha(t_2) + \delta\Big) \le \hat{h}(t_1) + (2 - e^{-t_0})\Big(\alpha(t_2) + \delta\Big). \tag{6.13}$$

Subtracting (6.12) from (6.13) and using the integral equation (6.8) for $\hat{h}$ leads to

$$\begin{aligned} & v_0 - \int_0^{t_1} h(s) e^{-s}\,\mathrm{d}s + h(t_1)(2 - e^{-t_1}) \\ & \le v_0 - \int_0^{t_1} \hat{h}(s) e^{-s}\,\mathrm{d}s + \hat{h}(t_1) - \alpha(t_2) e^{-t_0} + (2 - e^{-t_0})\Big(\alpha(t_2) + \delta\Big) \\ & = (2 - e^{-t_0})\delta + 2(1 - e^{-t_0})\alpha(t_2) = (1 - e^{-t_0})\alpha(t_2) < 0 \end{aligned} \tag{6.14}$$

for the choice $\delta := -(1 - e^{-t_0})\alpha(t_2)/(2 - e^{-t_0})$, which is independent of t_1 and satisfies $\delta > 0$ due to $\alpha(t_2) < 0$ and $\alpha(t_2) + \delta = \alpha(t_2)/(2 - e^{-t_0}) < 0$. But the strict inequality in (6.14) for all $t_1 \in C'$ contradicts (6.4), and so Claim 1 is proved. □

Claim 2. For any $\vartheta \in \Theta_+^{\mathrm{sf}}(S)$ with $\vartheta \mathrel{\widehat{=}} (v_0, h)$, define $J := J(\vartheta) := h(U)(2 - e^{-U})$. Then $J^- \le v_0(1 + e)$ and $P[J^+ \ge c] \le \frac{2}{c} v_0(1 + e)$ for every $c > 0$.

Proof. Because $U < \infty$ P-a.s., (6.3) implies that $V_\infty(\vartheta, S) = \lim_{t\to\infty} V_t(\vartheta, S)$ exists P-a.s. and satisfies $0 \le V_\infty(\vartheta, S) = v_0 - \int_0^U h(s) e^{-s}\,\mathrm{d}s + J$. Using $h \ge \hat{h} \ge -v_0 e$ λ-a.e. by Claim 1 and (6.9), we get $J \ge -v_0 - v_0 e \int_0^\infty e^{-s}\,\mathrm{d}s = -v_0(1 + e)$ which gives the estimate for J^-. Next, $J^+ = h^+(U)(2 - e^{-U}) \ge c$ implies $h^+(U) \ge \frac{c}{2}$, and then there must be a set $B \in \mathcal{B}([0, \infty))$ with $h^+ I_B \ge \frac{c}{2} I_B$ and U taking values in B. So

$$P[J^+ \ge c] \le \sup\{P[U \in B] : B \text{ satisfies } h^+ I_B \ge \tfrac{c}{2} I_B\},$$

and for any such B, using $U \sim \mathrm{Exp}(1)$ allows us to compute

$$P[U \in B] = \int_0^\infty I_B(s) e^{-s}\,\mathrm{d}s \le \frac{2}{c} \int_0^\infty h^+(s) \mathrm{e}^{-s}\,\mathrm{d}s = \frac{2}{c}\Big(\int_0^\infty h(s) \mathrm{e}^{-s}\,\mathrm{d}s + \int_0^\infty h^-(s) \mathrm{e}^{-s}\,\mathrm{d}s\Big).$$

The first integral is at most v_0 by (6.6) (letting $n \to \infty$), and using once more Claim 1 and (6.9) to obtain $h^- \le \hat{h}^- = -\hat{h} \le v_0 e$, we see that the second integral is at most $v_0 e$. Putting things together yields $P[U \in B] \le \frac{2}{c} v_0(1 + e)$ and gives the estimate for J^+. □

Claim 3. For any sequence of strategies $(\vartheta^k)_{k\in\mathbb{N}} \subseteq \Theta_+^{\mathrm{sf}}(S)$ with $V_0(\vartheta^k, S) = \varepsilon_k \to 0$ as $k \to \infty$, we have $V_\infty(\vartheta^k, S) \to 0$ in L^0 as $k \to \infty$.

Proof. With $\vartheta^k \mathrel{\widehat{=}} (\varepsilon_k, h_k)$, the proof of Claim 2 gives $V_\infty(\vartheta^k, S) = \varepsilon_k - \int_0^U h_k(s)e^{-s}\,\mathrm{d}s + J_k$ with $J_k = J(\vartheta^k)$. Using Claim 1 and (6.9), we then obtain as in the proof of Claim 2 that $-\int_0^U h_k(s)e^{-s}\,\mathrm{d}s \le \int_0^\infty h_k^-(s)e^{-s}\,\mathrm{d}s \le \varepsilon_k e$ so that $V_\infty(\vartheta^k, S) \le \varepsilon_k(1+e) + J_k^+ + J_k^-$. Thus the result follows from Claim 2. ☐

After these preparations, we return to (6.2). Recall that $\vec{\vartheta}$ has $\vartheta^n = \vartheta I_{[\![0,n]\!]} = \mathrm{e}^1 I_{[\![0,n]\!]}$.

Claim 4. For each fixed n, there is no triple $(p, A, \tilde{\vartheta})$ with $p > 0$, $A \in \mathcal{F}$ with $P[A] \ge p$ and $\tilde{\vartheta} \in \Theta_+^{\mathrm{sf}}(S^n)$ with $V_0(\tilde{\vartheta}, S^n) = V_0(\vartheta, S^n) = 1$ and $\liminf_{t\to\infty}(\tilde{\vartheta}_t - \vartheta_t^n) \ge pI_A\mathbb{1}$ P-a.s.

Proof. Note that $\liminf_{t\to\infty}(\tilde{\vartheta}_t - \vartheta_t^n) = \tilde{\vartheta}_n^n - \vartheta_n^n = \tilde{\vartheta}_n^n - \mathrm{e}^1$. Suppose to the contrary that there is such a triple $(p, A, \tilde{\vartheta})$, with $\tilde{\vartheta} \mathrel{\widehat{=}} (1, \tilde{h})$. We first claim that $\tilde{h} \ge 0$ λ-a.e. on $[0, n]$. Indeed, we have (6.5) for $\tilde{h}$ with a set $C \subseteq [0, n]$ with $\lambda(C) = 0$, and so we get

$$\int_0^t \tilde{h}(s)e^{-s}\,\mathrm{d}s \le \tilde{h}(t)(2 - e^{-t}) \qquad \text{for all } t \in [0, n] \setminus C. \tag{6.15}$$

If there is $t_1 \in (0, n]\setminus C$ with $\tilde{h}(t_1) < 0$, setting $m(t) := \inf_{s\in[0,t]\setminus C} \tilde{h}(s)$ gives by (6.15) that $(1-e^{-t_1})m(t_1) \le \tilde{h}(t_1)(2-e^{-t_1})$ and hence $m(t_1) \le (2+e^{-t_1}/(1-e^{-t_1}))\tilde{h}(t_1) < 2\tilde{h}(t_1) < 0$. Choose $\delta > 0$ such that $m(t_1) + \delta < 2\tilde{h}(t_1)$ and $t_2 \in (0, t_1) \setminus C$ with $\tilde{h}(t_2) < m(t_1) + \delta$. Then $\tilde{h}(t_2) < 2\tilde{h}(t_1) < 0$, and iterating produces a sequence $0 < \cdots < t_{k+1} < t_k < \cdots < t_1$ such that $\tilde{h}(t_{k+1}) < 2\tilde{h}(t_k) < \cdots < 2^k\tilde{h}(t_1)$. Plugging this into (6.15) yields

$$-\int_0^{t_{k+1}} \tilde{h}^-(s)e^{-s}\,\mathrm{d}s \le \int_0^{t_{k+1}} \tilde{h}(s)e^{-s}\,\mathrm{d}s \le 2^k\tilde{h}(t_1)$$

and therefore $\int_0^t \tilde{h}^-(s)\mathrm{e}^{-s}\,\mathrm{d}s = +\infty$ for any $t > 0$. But this contradicts the fact that $\tilde{H}$ associated to $\tilde{\vartheta}$ is in $L(X)$, and so we conclude that indeed $\tilde{h} \ge 0$ λ-a.e. on $[0, n]$.

Now $V_n(\tilde{\vartheta}, S) \ge 1 + pI_A(\mathbb{1} \cdot S_n)$ P-a.s. gives via (6.7) that $\int_0^n \tilde{h}(s)e^{-s}\,\mathrm{d}s \le 0$ so that we must have $\tilde{h} = 0$ λ-a.e. on $[0, n]$. But then we get from (6.3) that $V_n(\tilde{\vartheta}, S) = 1$ P-a.s. which contradicts the above inequality as $P[A] \ge p > 0$. This proves Claim 4. ☐

Suppose $\vec{\vartheta}$ is not assm for $\vec{\mathbb{1}}$. Then there exist $p > 0$ and for every $\varepsilon_k = \frac{1}{k}$ some $n = n_k$, $A^n \in \mathcal{F}$ with $P[A^n] \ge p$ and $\hat{\vartheta}^n \in \Theta_+^{\mathrm{sf}}(S^n)$ with $V_0(\hat{\vartheta}^n, S^n) \le V_0(\vartheta^n, S^n) + \frac{1}{k} = \frac{2}{k}$ and

$$\liminf_{t\to\infty}(\hat{\vartheta}_t^n - \vartheta_t^n) \ge pI_{A^n}\mathbb{1} \qquad P\text{-a.s.} \tag{6.16}$$

Note that (6.16) actually says that $\hat{\vartheta}_n^n - \mathrm{e}^1 \ge pI_{A^n}\mathbb{1}$ P-a.s. We also point out that $n = n_k$ depends on k everywhere; but we usually suppress the subscript k for readability.

Claim 5. We have $\hat{\vartheta}^{n_k} - \vartheta^{n_k} \in \Theta_+^{\mathrm{sf}}(S^{n_k})$ for all k.

Proof. This is analogous to the proof of Lemma 2.22; we do not have that ϑ^{n_k} is ssm for $\mathbb{1}$, but can use Claim 4 instead. In more detail, write $n = n_k$ and suppose there is a stopping time $\tau \le n$ such that $C := \{V_\tau(\vartheta^n - \hat{\vartheta}^n, S^n) > 0\}$ has $P[C] > 0$. Then $\tilde{\vartheta}^n := \vartheta^n \oslash^{\mathbb{1}}_\tau \hat{\vartheta}^n$ is in $\Theta^{\mathrm{sf}}_+(S^n)$ by [3, Lemma 3.3], has $V_0(\tilde{\vartheta}^n, S^n) = V_0(\vartheta^n, S^n)$, and computation gives

$$\tilde{\vartheta}^n - \vartheta^n = I_{\rrbracket \tau,\infty\llbracket} I_{\{V_\tau(\vartheta^n - \hat{\vartheta}^n, S^n) \ge 0\}} \Big(\hat{\vartheta}^n - \vartheta^n + V_\tau(\vartheta^n - \hat{\vartheta}^n, \mu^n)\mathbb{1} \Big)$$

so that due to (6.16), we have P-a.s. that

$$\liminf_{t\to\infty} (\tilde{\vartheta}^n_t - \vartheta^n_t) = I_{\{V_\tau(\vartheta^n - \hat{\vartheta}^n, S^n) \ge 0\}} \Big(\liminf_{t\to\infty} (\hat{\vartheta}^n_t - \vartheta^n_t) + V_\tau(\vartheta^n - \hat{\vartheta}^n, \mu^n)\mathbb{1} \Big) \ge I_C p I_{A^n} \mathbb{1}.$$

But this is a contradiction to Claim 4, and so the assertion follows. ☐

Due to Claim 5, the strategy $\bar{\vartheta}^k := (\hat{\vartheta}^{n_k} - \vartheta^{n_k}) \oslash^{\mathbb{1}}_{n_k} 0$ is in $\Theta^{\mathrm{sf}}_+(S^{n_k})$ for all k and has $V_0(\bar{\vartheta}^k, S^{n_k}) = V_0(\hat{\vartheta}^{n_k} - \vartheta^{n_k}, S^{n_k}) \le \frac{2}{k}$. Using $\hat{\vartheta}^{n_k} - \vartheta^{n_k} \in \Theta^{\mathrm{sf}}_+(S^{n_k})$, a computation gives

$$\bar{\vartheta}^k = I_{\llbracket 0,n_k \rrbracket} (\hat{\vartheta}^{n_k} - \vartheta^{n_k}) + I_{\rrbracket n_k,\infty\llbracket} V_{n_k}(\hat{\vartheta}^{n_k} - \vartheta^{n_k}, \mu^{n_k})\mathbb{1}.$$

Because $S^{n_k} \equiv S$ on $\llbracket n_k, \infty\llbracket$, this yields via (6.16), $\mathbb{1}\cdot\mu \equiv 1$ and $U < \infty$ P-a.s. that P-a.s.,

$$V_\infty(\bar{\vartheta}^k, S) = \lim_{t\to\infty} V_t(\bar{\vartheta}^k, S) = V_{n_k}(\hat{\vartheta}^{n_k} - \vartheta^{n_k}, \mu^{n_k}) \lim_{t\to\infty} (\mathbb{1} \cdot S_t) \ge p I_{A^{n_k}} (\mathbb{1} \cdot S_U) \ge 2p I_{A^{n_k}}.$$

But $P[A^{n_k}] \ge p$ and so we have a contradiction to Claim 3. This finally shows that $\vec{\vartheta}$ is assm for $\vec{\mathbb{1}}$ and concludes the example.

A Appendix: Some technical proofs

Proof of Lemma 2.10. The inclusion "$\subseteq$" is clear because $(\vartheta, 0) \in \Theta^{\mathrm{sf}}_+(S, \eta\cdot S)$ for any $\vartheta \in \Theta^{\mathrm{sf}}_+(S)$. For the converse, take $\bar{\vartheta} \in \Theta^{\mathrm{sf}}_+(S, \eta\cdot S)$, write $\bar{\vartheta} = (\vartheta, H)$ where H is the last coordinate of $\bar{\vartheta}$, and note that $V(\bar{\vartheta}, (S, \eta\cdot S)) = \bar{\vartheta}\cdot(S, \eta\cdot S) = (\vartheta + H\eta)\cdot S = V(\vartheta + H\eta, S)$. Now ϑ and H are predictable like $\bar{\vartheta}$, and [34, Lemma 6.1] implies that $\vartheta + H\eta$ is in $L(S)$ with $(\vartheta+H\eta)\bullet S = \bar{\vartheta}\bullet(S, \eta\cdot S)$. Because $\bar{\vartheta}$ is self-financing for $(S, \eta\cdot S)$, we therefore obtain $V(\vartheta+H\eta, S) = V(\bar{\vartheta}, (S, \eta\cdot S)) = V_0(\bar{\vartheta}, (S, \eta\cdot S)) + \bar{\vartheta}\bullet(S, \eta\cdot S) = V_0(\vartheta+H\eta, S) + (\vartheta+H\eta)\bullet S$, and this shows that $\vartheta + H\eta \in \Theta^{\mathrm{sf}}_+(S)$ and "$\supseteq$" follows. ☐

Proof of Lemma 2.13. Local martingales are σ-martingales, and σ-martingales bounded from below are local martingales by Ansel/Stricker [1, Corollary 3.5] and supermartingales by Fatou's lemma. This proves all implications "$\Rightarrow$" and "$\Leftarrow$". If S/D is a local martingale and ϑ is in $\Theta^{\mathrm{sf}}_+(S)$, then ϑ is (S/D)-integrable by [15, Lemma 2.9] and $\vartheta\bullet(S/D)$ is also a local martingale. Moreover, $\mathcal{F}_0$ is trivial and $V(\vartheta, S/D) = V_0(\vartheta, S/D) + \vartheta\bullet(S/D)$ by the self-financing property so that $V(\vartheta, S/D)$ is a local martingale as well. The same reasoning holds for σ-martingales, and this proves every implication "$\Downarrow$". Finally, if $S \ge 0$, then $S^{(i)} = V(\mathrm{e}^i, S) \in \mathcal{X}(S)$ for every i, which proves every implication "$\Uparrow$". ☐

Proof of Lemma 2.22. We have $\hat{\vartheta} - \vartheta \in \Theta^{\mathrm{sf}}(S)$ as $\Theta^{\mathrm{sf}}(S)$ is a linear space; so we only need to show $V(\hat{\vartheta} - \vartheta, S) \geq 0$ on $[\![0, \zeta]\!]$ P-a.s. Suppose to the contrary that there is a stopping time $\tau \leq \zeta$ such that

$$A := \{\tau < \infty\} \cap \{V_\tau(\hat{\vartheta} - \vartheta, S) < 0\} = \{\tau < \infty\} \cap \{V_\tau(\vartheta - \hat{\vartheta}, S) > 0\}$$

has $P[A] > 0$ and define for each $K \in \mathbb{N}$ the set

$$A_K := \{\tau \leq K\} \cap \{V_{\tau \wedge K}(\vartheta - \hat{\vartheta}, S) \geq 1/K\} \in \mathcal{F}_{\tau \wedge K} \subseteq \mathcal{F}_K.$$

Then $A_K \nearrow A$ as $K \to \infty$ and we can choose and fix K large enough to ensure that $P[A_K] > 0$. Define $\tilde{\vartheta} := \vartheta \circledS^{\eta}_{\tau \wedge K} \hat{\vartheta}$ and $\psi_\infty := I_{A_K}/K \in L^\infty_+ \setminus \{0\}$ as well as $\psi_t := E[\psi_\infty | \mathcal{F}_t]$ for $t \geq 0$. Then $\psi_t = \psi_\infty$ P-a.s. for $t \geq K$ so that $\lim_{t\to\infty} \psi_t = \psi_\infty$ P-a.s. Moreover, $\tilde{\vartheta}$ is in $\Theta^{\mathrm{sf}}_+(S)$ by [3, Lemma 3.3] and $V_0(\tilde{\vartheta}, S) = V_0(\vartheta, S)$. Using now $S^\eta = S/(\eta \cdot S)$ and the definition of $\circledS^{\eta}_{\tau \wedge K}$, we obtain first

$$\tilde{\vartheta} - \vartheta = I_{]\!]\tau \wedge K, \infty]\!]} I_{\{V_{\tau \wedge K}(\vartheta - \hat{\vartheta}, S) \geq 0\}} \Big(\hat{\vartheta} - \vartheta + \eta V_{\tau \wedge K}(\vartheta - \hat{\vartheta}, S^\eta)\Big)$$

and then

$$\begin{aligned} \text{(A.1)} \qquad \tilde{\vartheta}_t - \vartheta_t - \psi_t \eta_t = {} & I_{\{t > \tau \wedge K\}} I_{\{V_{\tau \wedge K}(\vartheta - \hat{\vartheta}, S) \geq 0\}} (\hat{\vartheta}_t - \vartheta_t) \\ & + \eta_t \Big(I_{\{t > \tau \wedge K\}} I_{\{V_{\tau \wedge K}(\vartheta - \hat{\vartheta}, S) > 0\}} V_{\tau \wedge K}(\vartheta - \hat{\vartheta}, S^\eta) - \psi_t \Big). \end{aligned}$$

By assumption, $\liminf_{t\to\infty} (\hat{\vartheta}_t - \vartheta_t) \geq 0$ P-a.s. For $t > K$, the second summand in (A.1) equals $\eta_t (I_{\{V_{\tau \wedge K}(\vartheta - \hat{\vartheta}, S) > 0\}} V_{\tau \wedge K}(\vartheta - \hat{\vartheta}, S^\eta) - I_{A_K}/K)$, which is ≥ 0 P-a.s. by the definition of A_K and because $\eta \geq 0$. So we obtain from the superadditivity of the lim inf that

$$\liminf_{t\to\infty} (\tilde{\vartheta}_t - \vartheta_t - \psi_t \eta_t) \geq 0 \qquad P\text{-a.s.},$$

and this contradicts the assumption that ϑ is ssm for η. ☐

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