Making no-arbitrage discounting-invariant: a new FTAP beyond NFLVR and NUPBR

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Abstract

In the simplest formulation, this paper addresses the following question: Given two positive asset prices on a right-open interval, how can one decide, in an economically natural manner, whether or not this is an arbitrage-free model?

In general multi-asset models of financial markets, the classic notions NFLVR and NUPBR depend crucially on how prices are discounted. To avoid such issues, we introduce a discounting-invariant absence-of-arbitrage concept. Like in earlier work, this rests on zero or some basic strategies being *maximal*; the novelty is that maximality of a strategy is defined in terms of *share* holdings instead of *value*. This allows us to generalise both NFLVR, by dynamic share efficiency, and NUPBR, by dynamic share viability. These concepts are the same for discounted or undiscounted prices, and they can be used in open-ended models under minimal assumptions on asset prices. We establish corresponding versions of the FTAP, i.e., dual characterisations in terms of martingale properties. As one expects, "properly anticipated prices fluctuate randomly", but with an *endogenous* discounting process which must not be chosen a priori. The classic Black–Scholes model on $[0, \infty)$ is arbitrage-free in this sense if and only if its parameters satisfy $m - r \in \{0, \sigma^2\}$ or, equivalently, either bond-discounted or stock-discounted prices are martingales.

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1 Introduction

Consider a financial market with $N \geq 2$ assets on the right-open interval $[0, \infty)$. Prices are modelled by an $\mathbb{R}_+^N$-valued semimartingale $S = (S_t)_{t \geq 0}$, are denominated in some abstract non-tradable unit (one can best think of this as a perishable consumption good), and none of the $S^i \geq 0$ need be strictly positive. There is no extra tradable riskless asset or bank account. We work on $[0, \infty)$ for two reasons. Mathematically, this is the most general setting, including models with finite or random horizons as special cases. Economically, an open-ended model avoids the unnatural setup where everything is described up to and including a final time point. In this setting, we look for an economically natural definition and a subsequent mathematical characterisation of *absence of arbitrage (AOA)*. Such a concept should capture the idea that one cannot get something out of nothing for free, and the denomination of prices should not matter — if a positive process $D = (D_t)_{t \geq 0}$ describes how price units change over time, then S should satisfy AOA if and only if S/D does, for all D from a suitable class $\mathcal{D}$. Finding and describing such an AOA concept is in our view a fundamental question in arbitrage theory.

For models in finite discrete time, and with the extra condition that one asset price remains strictly positive, this has been solved completely; see the textbook by Delbaen/Schachermayer [13, Chapter 2] and in particular Section 2.5 there. In general, surprisingly, the existing literature fails to provide an answer. For markets on $[0, \infty)$, the most general work is due to Delbaen/Schachermayer [9, 12], but they start with already discounted prices of the form $S = (1, X)$. Their AOA concept is no free lunch with vanishing risk (NFLVR), and they also study in [10, 11] for which class $\mathcal{D}$ of discounters/numéraire changes this remains invariant. However, very simple examples of $N = 2$ assets with positive (even continuous) price processes S^1, S^2 show that this approach gives no convincing definition when such an undiscounted model is arbitrage-free. In fact, discounting with $D = S^1$ may lead to a nice arbitrage-free model, while discounting with $D = S^2$ leads to a model with arbitrage (it is enough if the ratio $X = S^2/S^1$ is a martingale converging to 0). An illustrating example already appears in [10], where $S = (1, X)$ for a strict local martingale $X > 0$ with $X_0 = 1$ and $\lim_{t \to \infty} = a < 1$. The interpretation in [10] is from FX trading: 1 is the price of 1 EUR and X is the price of 1 USD, both expressed in EUR, so that $S/X = (1/X, 1)$ models the same FX market with prices in USD. As argued in [10], S satisfies NFLVR but S/X does not. Thus the same FX market viewed in EUR contains no arbitrage, but admits arbitrage when viewed in USD. This looks economically questionable and highlights the need for a discounting-invariant formulation.

Going in that direction, Herdegen [15] studies a general $\mathbb{R}^N$-valued S on $[0, T]$ and develops the AOA concept of numéraire-independent no-arbitrage (NINA) which is invariant for the class $\mathcal{D}$ of all semimartingales $D > 0$. As all processes in [15] are defined on the right-closed interval $[0, T]$, all these D are automatically bounded away from 0 and ∞, P-a.s., and this plays a crucial role in definitions and proofs alike. When S^1 and

S^2 in the above simple example are (possibly correlated) geometric Brownian motions on $[0,\infty)$, they do not satisfy this boundedness condition and hence cannot be used for discounting on $[0,\infty)$ in the manner of [15].

Our approach borrows ideas and techniques from both Delbaen/Schachermayer and Herdegen, and combines them with a key new idea that enables us to solve the above fundamental problem. As in Herdegen [15] and Herdegen/Schweizer [16], we define absence of arbitrage as the property that the zero strategy or a number of basic strategies are *maximal* in the sense that they cannot be "improved" by other strategies. In [15] as well as in earlier work of Delbaen/Schachermayer [9, 10, 11], such improvements are measured in terms of *value* or wealth. This can keep the approach invariant under discounting, but only partially — if a discount factor (or a numéraire) D goes to 0 or explodes to $+\infty$, the invariance breaks down. While this cannot happen on a right-closed time interval, it becomes an issue on a right-open interval, and it is exactly why the above simple example cannot be handled by the approach of [15]. We circumvent this difficulty by measuring "improvements" not in terms of value, but in terms of *shares* compared to a *reference strategy*. This is a self-financing strategy $\eta \geq 0$ whose wealth $V(\eta)$ remains strictly positive at all times. Hence η is *desirable*, because with prices expressed in units of a consumption good, $V(\eta) > 0$ means that an agent using η will never starve completely. We prove as in Delbaen/Schachermayer [9] a key result saying that if one has an AOA property related to η, prices discounted by $V(\eta)$ must converge and hence can be defined on the closed interval $[0,\infty]$. This in turn allows us to exploit the results from Herdegen [15], after we have shown how his and our AOA conditions are related.

Our approach leads to genuinely discounting-invariant concepts in almost fully general frictionless semimartingale models of financial markets. We only assume $S \geq 0$ and the existence of a reference strategy, and the latter already holds as soon as $\sum_{i=1}^N S^i > 0$ and $\sum_{i=1}^N S^i_- > 0$. Two main results are two FTAP versions — one for *dynamic share viability (DSV)*, the discounting-invariant counterpart of no unbounded profit with bounded risk (NUPBR), and one for *dynamic share efficiency (DSE)* which extends NFLVR. In contrast to the classic FTAP formulations of Delbaen/Schachermayer [9, 12] or Karatzas/Kardaras [24], the discounting process in our results must not be chosen a priori, but is an endogenous part of the dual characterisation of absence of arbitrage.

By providing a discounting-invariant AOA framework for general financial markets, we lay the foundations for many possible future developments. One project we are currently pursuing is a general treatment of the growth-optimal portfolio (GOP) and the benchmark approach, also for an infinite horizon; see Filipović/Platen [14]. We have already shown in Bálint/Schweizer [4] how one can use ideas from the present paper in the context of large financial markets. Stochastic portfolio theory (SPT) might benefit from our general perspective, but this looks at present more speculative. Finally, one can try to study utility maximisation, maybe in a discounting-invariant form similarly as in Kardaras [26], or under DSV instead of NUPBR; see Karatzas/Kardaras [24].

The paper is structured as follows. Section 2 introduces the setup and basic concepts and presents our main results. Section 3 is the mathematical core; it first shows how models on right-open intervals can be closed on the right under a weak AOA assumption, then combines this with Herdegen [15] to prove dual characterisations of value maximality for a general time interval, and connects our new concept of *share maximality* to the value maximality studied in [15]. Section 4 proves the main results from Section 2 by using Section 3. Section 5 discusses the robustness of our approach with respect to the choice of the reference strategy η appearing in the concept of share maximality, connects our work to the classic theory, and provides a comparison to the existing literature. In particular, we explain in detail why our approach and the numéraire-independence studied in Delbaen/ Schachermayer [10, 11] are conceptually very different. Section 6 contains examples and counterexamples, including a detailed study of the example with two geometric Brownian motions, and the Appendix collects some technical proofs and auxiliary results.

2 The main results

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, assume that $\mathcal{F}_0$ is trivial and set $\mathcal{F}_\infty := \bigvee_{t\geq 0} \mathcal{F}_t$. There are N basic assets whose prices are modelled by an $\mathbb{R}^N$-valued semimartingale S. If there is a bank account (we do not assume this in general), it must be one component of S. To have trading possible, we thus must have $N \geq 2$. Prices are expressed in some abstract, non-tradable unit, which is best thought of as a perishable consumption good. Its only role is to make trading of asset shares possible.

We use general stochastic integration (in the sense of [17, Chapter III.6] or [33]), call $L(S)$ the space of all $\mathbb{R}^N$-valued predictable S-integrable processes H and denote the (real-valued) stochastic integral of $H \in L(S)$ with respect to S by $H \bullet S := \int H \,\mathrm{d}S$. For any RCLL process Y, we set $Y_{0-} := Y_0$. The scalar product of $x, y \in \mathbb{R}^N$ is $x \cdot y := x^{\mathrm{tr}} y$.

Remark 2.1. We assume that S is a semimartingale so that we can use general integrands with respect to S. Similarly as in [9, 29], one could start with an $\mathbb{R}^N_+$-valued adapted RCLL process $S \geq 0$ and impose an AOA property on S only with respect to *elementary* (i.e. piecewise constant) integrands. For the AOA concept we introduce below, this implies that $S/V(\vartheta)$ is a semimartingale for any self-financing elementary strategy ϑ whose wealth $V(\vartheta)$ and $V_-(\vartheta)$ are strictly positive. In particular, if $S = (1, X)$, then $X \geq 0$ must be a semimartingale. For precise formulations and results, we refer to Bálint/Schweizer [5].

Many of our results involve *discounting*, i.e. dividing prices by positive processes. We define $\mathcal{S}^m := \{\text{all } \mathbb{R}^m\text{-valued semimartingales}\}$ and set $\mathcal{S} := \mathcal{S}^1$, $\mathcal{S}_+ := \{D \in \mathcal{S} : D \geq 0\}$ and $\mathcal{S}_{++} := \{D \in \mathcal{S} : D > 0,\, D_- > 0\}$. Elements $D \in \mathcal{S}_{++}$ are called *discounters*, and we note that $1/D \in \mathcal{S}_{++}$ if $D \in \mathcal{S}_{++}$. Sometimes, we also need discounters from $\mathcal{S}^{\mathrm{unif}}_{++} := \{D \in \mathcal{S}_{++} : \inf_{t\geq 0} D_t > 0,\, \sup_{t\geq 0} D_t < \infty,\, P\text{-a.s.}\}$. For $D \in \mathcal{S}_{++}$, we call S/D

the *D-discounted prices*. The difference between discounters and deflators is discussed below after Definition 2.10.

Self-financing strategies are integrands $\vartheta \in L(S)$ satisfying $V(\vartheta) := \vartheta \cdot S = \vartheta_0 \cdot S_0 + \vartheta \bullet S$. We write $\vartheta \in \Theta^{\mathrm{sf}}$ and call $V(\vartheta)$ the *value process* of ϑ; this is in the same units as S because ϑ is in numbers of shares. For D-discounted prices $\tilde{S} = S/D$, we analogously have $V(\vartheta, \tilde{S}) := \vartheta \cdot \tilde{S} = V(\vartheta)/D$, the value process of ϑ in the units of $\tilde{S}$. Due to [15, Lemma 2.9], $\vartheta \in \Theta^{\mathrm{sf}}$ implies both that $\vartheta \in L(\tilde{S})$ and $V(\vartheta, \tilde{S}) = \vartheta_0 \cdot \tilde{S}_0 + \vartheta \bullet \tilde{S}$ hold. Thus Θ^{sf} does not depend on units even if value processes do. We also need the spaces $\Theta^{\mathrm{sf}}_{+} := \{\vartheta \in \Theta^{\mathrm{sf}} : V(\vartheta) \in \mathcal{S}_{+}\}$ and $\Theta^{\mathrm{sf}}_{++} := \{\vartheta \in \Theta^{\mathrm{sf}} : V(\vartheta) \in \mathcal{S}_{++}\}$; they do not depend on units either. For any $\eta \in \Theta^{\mathrm{sf}}_{++}$, the *$\eta$-discounted prices*

$$S^{\eta} := \frac{S}{V(\eta)} = \frac{S}{\eta \cdot S}$$

play an important role in the sequel. Note that $V(\eta, S^{\eta}) = \eta \cdot S^{\eta} \equiv 1$ and $(S^{\eta})^{\eta'} = S^{\eta'}$. Finally, a process Y is called *S-tradable* if it is the value process of some self-financing strategy, i.e., $Y = V(\vartheta)$ for some $\vartheta \in \Theta^{\mathrm{sf}}$.

Definition 2.2. A *reference strategy* is an $\eta \in \Theta^{\mathrm{sf}}_{++}$ with $\eta \geq 0$ (η is long-only).

In the sequel, we usually assume that there exists a reference strategy η, and some results impose the extra condition that η is bounded (uniformly in (ω, t)). A reference strategy can be viewed as a *desirable investment*; indeed, given that values are in terms of some perishable consumption good, the property $V(\eta) \in \mathcal{S}_{++}$ means that η keeps us forever from complete starvation. As η is expressed in numbers of shares, it is clearly discounting-invariant. See also the comment below after Definition 2.8.

Remark 2.3. 1) The existence of a reference strategy η is a very weak condition on the price process S. Indeed, consider the *market portfolio*, i.e. the strategy $\mathbb{1} := (1, \ldots, 1) \in \mathbb{R}^N$ of holding one share of each asset. If we have nonnegative prices $S \geq 0$, then $\mathbb{1} \in \Theta^{\mathrm{sf}}_{+}$ and all components of the $\mathbb{1}$-discounted price process $S^{\mathbb{1}} = S / \sum_{i=1}^N S^i$ have values between 0 and 1. (Some authors call $S^{\mathbb{1}}$ the process of *market capitalisations*.) If $S \geq 0$ and the sum $\sum_{i=1}^N S^i$ of all prices is strictly positive and has strictly positive left limits, we even have $\mathbb{1} \in \Theta^{\mathrm{sf}}_{++}$ so that the market portfolio is then a reference strategy. Moreover, $\mathbb{1}$ is of course bounded itself. However, it is useful to work with a general reference strategy η because this gives a clearer view on a number of aspects.

2) A reference strategy is by definition long-only, which looks natural from an economic perspective. Mathematically, $\xi \geq 0$ is used in part 1) of the key Theorem 3.10 and therefore appears indirectly in many results throughout the paper.

Definition 2.4. Fix a strategy $\eta \in \Theta^{\mathrm{sf}}$. A strategy $\vartheta \in \Theta^{\mathrm{sf}}$ is called an *η-buy-and-hold strategy* if it is of the form $\vartheta = c\eta$ (componentwise product) for some $c \in \mathbb{R}^N$.

A strategy ϑ is η-buy-and-hold if and only if it is a coordinatewise nonrandom multiple of η. If $\eta \equiv \mathbb{1}$ is the market portfolio, this reduces to the classic concept of buying and holding a fixed number of shares of each asset, and so the above buy-and-hold concept is a natural generalisation. Note that η itself is always an η-buy-and-hold strategy.

For maximal generality with our time horizon, we fix a stopping time ζ and consider the stochastic interval $[\![0,\zeta]\!] = \{(\omega,t) \in \Omega \times [0,\infty) : 0 \le t \le \zeta(\omega)\}$. This includes models indexed by $[0,T]$ with a nonrandom $T < \infty$ as well as by $[0,\infty)$ where $\zeta \equiv \infty$. We extend all stochastic processes to $[\![0,\infty]\!] = [\![0,\infty[\![= \Omega \times [0,\infty)$, almost always by keeping them constant on $[\![\zeta,\infty]\!]$, with one important exception. To concatenate two strategies $\vartheta^1, \vartheta^2 \in \Theta^{\mathrm{sf}}$ at some stopping time τ, we sometimes define, for a mapping F, a new strategy of the form $I_{[\![0,\tau]\!]}\vartheta^1 + I_{]\!]\tau,\infty]\!]}F(\vartheta^1,\vartheta^2)$. On the set $\{\tau = \zeta < \infty\}$, this is then constant for $t > \zeta(\omega)$, but maybe not for $t \ge \zeta(\omega)$.

From now on, we assume that all processes are defined on $[\![0,\infty]\!]$ (but not necessarily on $\Omega \times [0,\infty]$). If a process Y is constant on $[\![\zeta,\infty]\!]$, we then have

$$\inf_{t\ge 0} Y_t(\omega) = I_{\{\zeta(\omega)=\infty\}} \inf_{0\le t<\infty} Y_t(\omega) + I_{\{\zeta(\omega)<\infty\}} \inf_{0\le t\le \zeta(\omega)} Y_t(\omega),$$

$$\liminf_{t\to\infty} Y_t(\omega) = I_{\{\zeta(\omega)=\infty\}} \liminf_{t\to\infty} Y_t(\omega) + I_{\{\zeta(\omega)<\infty\}} Y_{\zeta}(\omega),$$

etc. Of course, if we write $\lim_{t\to\infty} Y_t$, we must make sure that this limit exists on $\{\zeta = \infty\}$. These notations allow us to handle all time horizons in a unified manner.

Remark 2.5. Because ζ is an $\mathbb{F}$-stopping time, we have $\mathcal{F}_\zeta \subseteq \mathcal{F}_\infty \subseteq \mathcal{F}$. We only distinguish measurabilities when it is relevant, and in particular just write L^0 for $L^0(\mathcal{F})$.

The next concept is fundamental for our paper.

Definition 2.6. Fix a strategy $\eta \in \Theta^{\mathrm{sf}}$. A strategy $\vartheta \in \Theta^{\mathrm{sf}}_+$ is called *strongly share maximal (ssm) for* η if there is no $[0,1]$-valued adapted process $\psi = (\psi_t)_{t\ge 0}$ converging P-a.s. as $t \to \infty$ to some $\psi_\infty \in L^\infty_+ \setminus \{0\}$ and such that for every $\varepsilon > 0$, there exists some $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+$ with $V_0(\hat{\vartheta}^\varepsilon) \le V_0(\vartheta) + \varepsilon$ and

$$\liminf_{t\to\infty} (\hat{\vartheta}^\varepsilon_t - \vartheta_t - \psi_t \eta_t) \ge 0 \qquad P\text{-a.s.}$$

We mostly use this concept when η is a reference strategy. Then having η is desirable, and $\psi\eta$ is a dynamic long-only portfolio whose factor ψ stabilises over time and which asymptotically achieves a significant part of η. Strong share maximality says that even with a little extra initial capital $\varepsilon > 0$, one cannot asymptotically improve ϑ via some $\hat{\vartheta}^\varepsilon$ in such a significant manner.

We also need the following concept inspired by Herdegen [15]; the difference to [15] is that we work here on a possibly right-open time interval. Note that we replace "strongly maximal" from [15] by the more explicit terminology "strongly value maximal".

Definition 2.7. Fix an $\mathbb{R}^N$-valued semimartingale $\tilde{S}$. A strategy $\vartheta \in \Theta_+^{\mathrm{sf}}$ is called *strongly value maximal (svm) for* $\tilde{S}$ if there is no $f \in L_+^0 \setminus \{0\}$ such that for every $\varepsilon > 0$, there exists some $\hat{\vartheta}^\varepsilon \in \Theta_+^{\mathrm{sf}}$ with $V_0(\hat{\vartheta}^\varepsilon, \tilde{S}) \leq V_0(\vartheta, \tilde{S}) + \varepsilon$ and

$$\liminf_{t\to\infty} \Big(V_t(\hat{\vartheta}^\varepsilon, \tilde{S}) - V_t(\vartheta, \tilde{S}) - f \Big) \geq 0 \qquad P\text{-a.s.}$$

Maximality of a strategy ϑ always means that ϑ cannot be improved. The key difference between Definitions 2.6 and 2.7 lies in how improvements are measured. For strong value maximality, the comparison is in terms of *value*, which makes the concept depend on the unit (of $\tilde{S}$). In contrast, strong share maximality looks (via the reference strategy η) at *numbers of shares*, and this is independent of any unit for prices.

Given a maximality concept for strategies, we define viability and efficiency as in [15].

Definition 2.8. Fix $\eta \in \Theta^{\mathrm{sf}}$. We say that S satisfies *dynamic share viability (DSV) for* η if the zero strategy $0 \in \Theta_+^{\mathrm{sf}}$ is strongly share maximal for η, and *dynamic share efficiency (DSE) for* η if every η-buy-and-hold strategy $\vartheta \in \Theta_+^{\mathrm{sf}}$ is strongly share maximal for η.

It is a key observation that for fixed η, *strong share maximality for* η*, dynamic share viability for* η *and dynamic share efficiency for* η *are like* Θ^{sf} *all* discounting-invariant *with respect to* $\mathcal{S}_{++}$*,* in the sense that if we have one of these properties for S, we also have it for any D-discounted $\tilde{S} = S/D$ with any discounter $D \in \mathcal{S}_{++}$, and vice versa. In contrast, the strong (value) maximality for S from [15] (and derived concepts like NINA there) is invariant under discounting by discounters $D \in \mathcal{S}_{++}^{\mathrm{unif}} \subsetneq \mathcal{S}_{++}$ (see Lemma 3.1 below), but not under discounting by $D \in \mathcal{S}_{++}$ (see Example 3.2 below). In that sense, the value-related concepts and results from [15] are only numéraire- or discounting-invariant in a restricted manner. But for a general discounting-invariant framework, having invariance with respect to the full class $\mathcal{S}_{++}$ is crucial because the natural class of discounters on a right-open interval like $[0, \infty)$ is $\mathcal{S}_{++}$ and not only $\mathcal{S}_{++}^{\mathrm{unif}}$.

Remark 2.9. 1) Theorems 2.14 and 2.15 below give equivalent characterisations for DSE for η, assuming among other things that η is a reference strategy and bounded (uniformly in (ω, t)). These results show that equivalent definitions of DSE for η are possible: one could as well stipulate that only η itself, or all bounded $\vartheta \in \Theta_+^{\mathrm{sf}}$, should be ssm for η. We have opted for an intermediate definition to preserve the analogy to [15].

2) All our concepts depend on the choice of η. We discuss this in Section 5.1 and show there in particular that the dependence is quite weak.

3) The idea of treating not value processes, but strategies/portfolios in numbers of shares as central objects has already been promoted by Y. Kabanov in his geometric approach to markets with transaction costs; see the textbook by Kabanov/Safarian [19] and in particular Section 3.1 there. But as also stated in [19, Section 3.6.1], models with transaction costs are much less demanding in terms of stochastic calculus because strategies there are processes of finite variation. We cannot impose this in our frictionless market, and so the tools and techniques developed by Kabanov and his co-authors cannot be used in our setup.

The preceding concepts are all about strategies and hence on the primal side. For a dual characterisation in terms of martingale properties, we need the following concept.

Definition 2.10. For $\mathcal{E} \in \{\sigma$-martingale, local martingale, martingale, UI martingale$\}$, an $\mathcal{E}$*-discounter* for an $\mathbb{R}^N$-valued semimartingale $\tilde{S}$ is a $D \in \mathcal{S}_{++}$ such that $\tilde{S}/D$ is an $\mathcal{E}$.

Remark 2.11. In the literature, an $\mathcal{E}$*-deflator* for a class $\mathcal{Y}$ of processes is a strictly positive local martingale Z (often with $Z_0 = 1$) such that ZY is an $\mathcal{E}$ for all $Y \in \mathcal{Y}$. There are two differences to the notion of an $\mathcal{E}$-discounter. Obviously, a deflator acts by multiplication while a discounter acts by division. More importantly, however, we impose no (local) martingale property on an $\mathcal{E}$-discounter D, nor on $1/D$. (Some definitions of an $\mathcal{E}$-deflator Z do not explicitly ask for Z to be a local martingale. But as $\mathcal{Y}$ invariably contains the process $Y \equiv 1$, this property follows from the definition and $Z > 0$.) In our setup, neither S nor the family $\{V(\vartheta) : \vartheta \in \Theta^{\mathrm{sf}}_{++}\}$ of value processes contains a constant process in general; so discounters are more natural and more general than deflators.

With these preliminaries, we can already state our first two main results.

Theorem 2.12. *Suppose $S \geq 0$ and there exists a reference strategy η. Then S satisfies dynamic share viability for η if and only if there exists a σ-martingale discounter D for S with $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s.*

Remark 2.13. As pointed out in the proof in Section 4, the "only if" part in Theorem 2.12 does not need $S \geq 0$. The same applies to Theorem 2.14.

Theorem 2.14. *Suppose $S \geq 0$ and there exists a reference strategy η such that in addition, η and $S^\eta = S/(\eta \cdot S)$ are bounded (uniformly in (ω, t)). Then S satisfies dynamic share efficiency for η if and only if there exists a UI martingale discounter D for S with $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s.*

The proofs of Theorems 2.12 and 2.14 need extra ideas and additional results. These are developed in Section 3 and used in Section 4 to prove the theorems.

Both Theorems 2.12 and 2.14 are modern formulations of the classic idea due to Samuelson [32] that "properly anticipated prices fluctuate randomly" or, in other words, suitably discounted prices form a martingale. The notion of "properly anticipated" or "suitably discounted" is in our paper captured by the *existence* of the process D which turns S via discounting to S/D into a "martingale". The strength of the martingale property of S/D (σ-martingale or UI martingale) depends on the strength of the initial no-arbitrage condition (viability or efficiency). One key contrast to the classic FTAP formulation of Delbaen/Schachermayer [9, 12] is that the discounting process cannot be chosen a priori, but is an endogenous part of the dual characterisation of absence of arbitrage. A similar idea appears in Herdegen [15] (see also [16]) where the dual objects are not only "martingale transformers" like martingale measures or deflators, but *pairs* consisting of an S-tradable numéraire and a "martingale measure". Our $\mathcal{E}$-discounter combines such a pair into a single process; this is more general than a deflator because the latter's local martingale property still reflects an a priori discounting of prices.

To relate our work to the literature, we next recall or rewrite some notions from the classic Delbaen/Schachermayer [9, 12] approach. For any $\mathbb{R}^N$-valued semimartingale $\tilde{S}$, we define

$$L^a_{\mathrm{adm}}(\tilde{S}) := \{H \in L(\tilde{S}) : H \bullet \tilde{S} \geq -a\}$$

and introduce the sets

- $\mathcal{G}^a_{\mathrm{adm}}(\tilde{S}) := \Big\{ \lim_{t\to\infty} V_t(\vartheta, \tilde{S}) - V_0(\vartheta, \tilde{S}) : \vartheta \in \Theta^{\mathrm{sf}}_+,\ V_0(\vartheta, \tilde{S}) = a \text{ and } \lim_{t\to\infty} V_t(\vartheta, \tilde{S}) \text{ exists}\Big\}$,
- $\mathcal{G}_{\mathrm{adm}}(\tilde{S}) := \bigcup_{a\geq 0} \mathcal{G}^a_{\mathrm{adm}}(\tilde{S}) = \Big\{ \lim_{t\to\infty} V_t(\vartheta, \tilde{S}) - V_0(\vartheta, \tilde{S}) : \vartheta \in \Theta^{\mathrm{sf}}_+ \text{ and } \lim_{t\to\infty} V_t(\vartheta, \tilde{S}) \text{ exists}\Big\}$,
- $\mathcal{C}_{\mathrm{adm}}(\tilde{S}) := \mathcal{G}_{\mathrm{adm}}(\tilde{S}) - L^0_+$,
- $\overline{\mathcal{C}}^{\infty}_{\mathrm{adm}}(\tilde{S}) := \overline{\mathcal{C}_{\mathrm{adm}}(\tilde{S}) \cap L^\infty}^{\infty}$;

the bar ${}^{-\infty}$ denotes the norm closure in L^∞. Each $g \in \mathcal{G}^a_{\mathrm{adm}}(\tilde{S})$ is the net outcome (final minus initial value) of a self-financing strategy ϑ whose value is always $\geq -a$, with all quantities in the same units as $\tilde{S}$. Then we say that

- $\mathrm{NA}_\infty(\tilde{S})$ holds if $\mathcal{C}_{\mathrm{adm}}(\tilde{S}) \cap L^\infty_+ = \{0\}$;
- $\mathrm{NUPBR}_\infty(\tilde{S})$ holds if $\mathcal{G}^1_{\mathrm{adm}}(\tilde{S})$ is bounded in L^0;
- $\mathrm{NFLVR}_\infty(\tilde{S})$ holds if $\overline{\mathcal{C}}^{\infty}_{\mathrm{adm}}(\tilde{S}) \cap L^\infty_+ = \{0\}$.

Using [15, Theorem 2.14] (which easily extends to $[\![0,\infty]\!]$) allows us to rewrite things in more familiar form. Fix $\eta \in \Theta^{\mathrm{sf}}_{++}$ and recall the η-discounted prices $S^\eta = S/(\eta \cdot S)$. Then

$$\mathcal{G}^a_{\mathrm{adm}}(S^\eta) = \Big\{ \lim_{t\to\infty} H \bullet S^\eta_t : H \in L^a_{\mathrm{adm}}(S^\eta) \text{ and } \lim_{t\to\infty} H \bullet S^\eta_t \text{ exists}\Big\}. \tag{2.1}$$

If prices $S = (1, X)$ are already discounted, we can take $\eta \equiv \mathrm{e}^1 := (1, 0, \dots, 0) \in \mathbb{R}^N$, getting $S^{\mathrm{e}^1} = (1, X) = S$, and note $\{H \bullet S : H \in L(S)\} = \{H \bullet X : H \in L(X)\}$ to obtain

$$\mathcal{G}^a_{\mathrm{adm}}(1, X) = \Big\{ \lim_{t\to\infty} H \bullet X_t : H \in L(X),\ H \bullet X \geq -a \text{ and } \lim_{t\to\infty} H \bullet X_t \text{ exists}\Big\}.$$

Thus $\mathcal{G}_{\mathrm{adm}}(1, X) = \bigcup_{a\geq 0} \mathcal{G}^a_{\mathrm{adm}}(1, X)$ is precisely the set K_0 (or K) considered in [9] (or [12]), and $\mathrm{NA}_\infty(1, X)$, $\mathrm{NUPBR}_\infty(1, X)$ and $\mathrm{NFLVR}_\infty(1, X)$ recover the standard notions NA, NUPBR and NFLVR in the classic theory following [9, 12]. We remark that the property $\mathrm{NUPBR}_\infty(1, X)$ already appears without a name in [9, Corollary 3.4]; it was later called BK by Kabanov [20] and NUPBR by Karatzas/Kardaras [24].

The next result summarises the main connections between our new results and the classic theory. A full formulation (including more equivalences and a graphical overview) and the proof are in Section 4.

Theorem 2.15. *Suppose $S \geq 0$ and there exists a reference strategy η. Consider the following statements:*

(e1) *S satisfies dynamic share efficiency for η.*

(v1) *S satisfies dynamic share viability for η.*

(e2) *Every bounded $\vartheta \in \Theta_+^{\mathrm{sf}}$ is strongly value maximal for S^η.*

(v2) *0 is strongly value maximal for $S^\eta = S/(\eta \cdot S)$.*

(e3) *There exists a UI martingale discounter D for S with $\eta \cdot (S/D) \in \mathcal{S}_{++}^{\mathrm{unif}}$.*

(v3) *There exists a σ-martingale discounter D for S with $\eta \cdot (S/D) \in \mathcal{S}_{++}^{\mathrm{unif}}$.*

(e4) *$NFLVR_\infty(S^\eta)$ holds, i.e., S^η satisfies $NFLVR_\infty$.*

(v4) *$NUPBR_\infty(S^\eta)$ holds, i.e., S^η satisfies $NUPBR_\infty$.*

Then we have (eK) $\Rightarrow$ (vK) *for* $\mathrm{K} = 1, \ldots, 4$, *and all the statements* (vK), $\mathrm{K} = 1, \ldots, 4$, *are equivalent among themselves. If in addition η and S^η are bounded (uniformly in (ω, t)), then also all the statements* (eK), $\mathrm{K} = 1, \ldots, 4$, *are equivalent among themselves.*

Proving our main results involves several ideas and steps. We give here a short overview and implement this in Section 3. First, because strong share maximality is discounting-invariant with respect to $\mathcal{S}_{++}$, we can work with a discounted price process S/D instead of the original S. We choose $D := V(\xi) = \xi \cdot S$ and show in Theorem 3.10 that strong share maximality for ξ is equivalent to strong value maximality for $S/D = S^\xi$. This gives us almost access to the results from Herdegen [15] who derived dual characterisations for strong (value) maximality, of 0 or of a fixed strategy, in terms of certain martingale properties for suitably discounted prices. (It is at this point that the endogenous discounter appears.) However, [15] crucially exploits that prices there are defined on a right-closed time interval, and the numéraire-invariance in [15] is only with respect to the smaller, restrictive class $\mathcal{S}_{++}^{\mathrm{unif}}$ of discounters. Overcoming this problem needs an extra step. With a similar argument as in Delbaen/Schachermayer [9], we show that for tradably discounted prices S^ξ and under strong value maximality for S^ξ of 0, the value process $V(\vartheta, S^\xi)$ of any self-financing strategy $\vartheta \in \Theta_+^{\mathrm{sf}}$ converges as $t \to \infty$. In effect, all the $V(\vartheta, S^\xi)$ are therefore well defined on a right-closed interval (even if S or S^ξ is not), and this finally allows us to use the results from [15]. Combining everything yields our assertions.

3 The theory

This section is the mathematical core of the paper. It consists of three subsections which mirror the ideas and steps in the discussion at the end of Section 2. To have a clearer structure, we proceed in the reverse order than the above discussion.

3.1 From a stochastic or right-open interval to a closed interval

In this section, we use absence of arbitrage to pass from a model with a general time horizon (stochastic or not, finite or infinite) to a model effectively defined on $\Omega \times [0, \infty]$. This rests on a convergence result in the spirit of Delbaen/Schachermayer [9, Theorem 3.3] combined with ideas from Herdegen [15] to connect strong (value) maximality and NUPBR.

We begin with an auxiliary result.

Lemma 3.1. *Suppose $\vartheta \in \Theta^{\mathrm{sf}}_{+}$ is strongly value maximal for S.*

1) *Strong value maximality is discounting-invariant with respect to $\mathcal{S}^{\mathrm{unif}}_{++}$: If $D \in \mathcal{S}^{\mathrm{unif}}_{++}$, then ϑ is also strongly value maximal for S/D. (The converse is clear because $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ implies $1/D \in \mathcal{S}^{\mathrm{unif}}_{++}$.)*

2) *Strongly value maximal strategies form a cone: For any $\alpha \geq 0$, $\alpha\vartheta$ is also strongly value maximal for S.*

Proof. See Appendix. $\square$

The next example shows that $\mathcal{S}^{\mathrm{unif}}_{++}$ cannot be replaced by $\mathcal{S}_{++}$ in Lemma 3.1.

Example 3.2. *Strong value maximality is not discounting-invariant with respect to $\mathcal{S}_{++}$.* Consider the standard Black–Scholes model (see Example 6.1) with $m = r = \sigma = 1$, so that $S^1_t = e^t$ and $S^2_t = e^{W_t + \frac{1}{2}t}$. Here, 0 is not svm for S because for any $\varepsilon > 0$, the strategy $\hat{\vartheta}^\varepsilon := \varepsilon \mathrm{e}^1 = (\varepsilon, 0)$ of buying and holding ε units of S^1 has $V_0(\hat{\vartheta}^\varepsilon) = \varepsilon$, but $\lim_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon) = +\infty$. But taking $D := S^1 \in \mathcal{S}_{++} \setminus \mathcal{S}^{\mathrm{unif}}_{++}$ yields $\tilde{S} := S/D = (1, e^{W_t - \frac{1}{2}t})$. This is a ($\sigma$-)martingale, and therefore 0 is svm for $\tilde{S}$; see Theorem 3.8 below (applied to $\tilde{S} = S^\xi$ for $\xi \equiv \mathrm{e}^1$).

We first connect strong value maximality and NUPBR; this is similar to Herdegen [15, Proposition 3.24].

Proposition 3.3. *Fix $\xi \in \Theta^{\mathrm{sf}}_{++}$ and recall the ξ-discounted price process $S^\xi = S/(\xi \cdot S)$. Then the following are equivalent:*

(a) *The zero strategy $0 \in \Theta^{\mathrm{sf}}_{+}$ is strongly value maximal for S^ξ.*

(b) *The set $\{\lim_{t\to\infty} H \bullet S^\xi_t : H \in L^1_{\mathrm{adm}}(S^\xi),\ H$ has bounded support on $[0, \infty)\}$ is bounded in L^0.*

(c) *The set $\{\liminf_{t\to\infty} H \bullet S^\xi_t : H \in L^1_{\mathrm{adm}}(S^\xi)\}$ is bounded in L^0.*

(d) *The set* $\{\lim_{t\to\infty} H\bullet S^\xi_t : H \in L^1_{\mathrm{adm}}(S^\xi)$ *and* $\lim_{t\to\infty} H\bullet S^\xi_t$ *exists*$\}$ *is bounded in* L^0.

(e) *NUPBR*$_\infty(S^\xi)$ *holds.*

Proof. (c) $\Rightarrow$ (d) $\Rightarrow$ (b) is clear; (b) $\Rightarrow$ (c) is from the proof of [9, Proposition 3.2]; and (d) $\Leftrightarrow$ (e) follows directly from (2.1) and the definition of $\mathrm{NUPBR}_\infty(S^\xi)$.

We prove (c) $\Rightarrow$ (a) indirectly. If 0 is not svm for S^ξ, we can find $f \in L^0_+ \setminus \{0\}$ and for every $\varepsilon = 1/n$ some $\hat{\vartheta}^n \in \Theta^{\mathrm{sf}}_+$ with $V_0(\hat{\vartheta}^n, S^\xi) \le 1/n$ and $\liminf_{t\to\infty} V_t(\hat{\vartheta}^n, S^\xi) \ge f$ P-a.s. Then $\tilde{\vartheta}^n := n\hat{\vartheta}^n$ is in Θ^{sf}_+ with $V_0(\tilde{\vartheta}^n, S^\xi) \le 1$, and $\tilde{\vartheta}^n$ is also in $L^1_{\mathrm{adm}}(S^\xi)$ because

$$0 \le V(\tilde{\vartheta}^n, S^\xi) = V_0(\tilde{\vartheta}^n, S^\xi) + \tilde{\vartheta}^n \bullet S^\xi \le 1 + \tilde{\vartheta}^n \bullet S^\xi.$$

Therefore, $\liminf_{t\to\infty} \tilde{\vartheta}^n \bullet S^\xi_t = \liminf_{t\to\infty} V_t(\tilde{\vartheta}^n, S^\xi) - V_0(\tilde{\vartheta}^n, S^\xi) \ge nf - 1$ P-a.s. implies that (c) cannot hold as $f \in L^0_+ \setminus \{0\}$.

Finally, for (a) $\Rightarrow$ (b), suppose that (b) is not true. Then also the convex set

$$C := \Big\{ \lim_{t\to\infty} H\bullet S^\xi_t + 1 : H \in L^1_{\mathrm{adm}}(S^\xi),\ H \text{ has bounded support on } [0,\infty)\Big\} \subseteq L^0_+$$

is not bounded in L^0. Lemma A.2 yields a sequence $(H^n)_{n\in\mathbb{N}} \subseteq L^1_{\mathrm{adm}}(S^\xi)$, with each H^n of bounded support on $[0,\infty)$, and some $f \in L^0_+ \setminus \{0\}$ with $\lim_{t\to\infty} H^n \bullet S^\xi_t + 1 \ge nf$ P-a.s. for all $n \in \mathbb{N}$. Note that the limit exists because H^n has bounded support. Consider the integrand $H^n \in L^1_{\mathrm{adm}}(S^\xi)$. By [15, Theorem 2.14] (and an easy extension to $[\![0,\infty]\!]$), there exists a corresponding $\vartheta^n \in \Theta^{\mathrm{sf}}_+$ with $V(\vartheta^n, S^\xi) - V_0(\vartheta^n, S^\xi) = H^n \bullet S^\xi$, where we can choose $V_0(\vartheta^n, S^\xi) = 1$. Defining $\tilde{\vartheta}^n := \vartheta^n/n \in \Theta^{\mathrm{sf}}_+$ yields

$$V(\tilde{\vartheta}^n, S^\xi) = V(\vartheta^n, S^\xi)/n = (H^n \bullet S^\xi + 1)/n,$$

hence $V_0(\tilde{\vartheta}^n, S^\xi) = 1/n$ and $\liminf_{t\to\infty} V_t(\tilde{\vartheta}^n, S^\xi) = \lim_{t\to\infty}(H^n \bullet S^\xi_t + 1)/n \ge f$ P-a.s. Thus 0 is not svm for S^ξ. $\square$

Our next result is of crucial importance. It is a variant of the key result in Delbaen/Schachermayer [9, Theorem 3.3] and shows that loosely speaking, *value processes expressed in good units converge under a weak no-arbitrage assumption.*

Theorem 3.4. *Fix* $\xi \in \Theta^{\mathrm{sf}}_{++}$ *and suppose the zero strategy* $0 \in \Theta^{\mathrm{sf}}_+$ *is strongly value maximal for* S^ξ. *Then for any* $\vartheta \in \Theta^{\mathrm{sf}}_+$, $\lim_{t\to\infty} \vartheta \bullet S^\xi_t$ *exists and is finite, P-a.s. In particular,* $V_\infty(\vartheta, S^\xi) := \lim_{t\to\infty} V_t(\vartheta, S^\xi)$ *exists and is finite, P-a.s.*

Proof. Fix ξ as above and $H \in L^1_{\mathrm{adm}}(S^\xi)$. We first claim that $\lim_{t\to\infty} H\bullet S^\xi_t$ exists and is finite, P-a.s. This follows from upcrossing arguments as in Doob's martingale convergence theorem and is based on the proof of [9, Theorem 3.3]. Indeed, by Proposition 3.3, the strong value maximality for S of 0 implies that the set

$$\Big\{ \lim_{t\to\infty} H\bullet S^\xi_t : H \in L^1_{\mathrm{adm}}(S^\xi),\ H \text{ has bounded support on } [0,\infty)\Big\}$$

is bounded in L^0, so that the conclusion of [9, Proposition 3.1] holds (with S in [9] replaced by S^ξ here). A careful look at [9, Proposition 3.2 and Theorem 3.3] shows that all we need for the proofs of these results is the conclusion of [9, Proposition 3.1]. So we can repeat the proof of [9, Theorem 3.3] step by step[1] to obtain our auxiliary claim about the convergence of $H \bullet S^\xi$.

To prove Theorem 3.4, we now fix $\vartheta \in \Theta^{\mathrm{sf}}_+$, set $v_0 := V_0(\vartheta, S^\xi)$ and define the strategy $\tilde\vartheta := I_{\{v_0 \neq 0\}}\vartheta/v_0 + I_{\{v_0=0\}}(\vartheta + \xi)$. Then $\tilde\vartheta$ is in Θ^{sf}_+, and as $V(\xi, S^\xi) = \xi \cdot S^\xi \equiv 1$,

$$V(\tilde\vartheta, S^\xi) = I_{\{v_0\neq 0\}} V(\vartheta, S^\xi)/v_0 + I_{\{v_0=0\}}\Big(V(\vartheta, S^\xi) + 1\Big).$$

This yields $V_0(\tilde\vartheta, S^\xi) = 1$ and hence $V(\tilde\vartheta, S^\xi) = 1 + \tilde\vartheta \bullet S^\xi$. Because $\tilde\vartheta \in \Theta^{\mathrm{sf}}_+$, this shows that $\tilde\vartheta \in L^1_{\mathrm{adm}}(S^\xi)$ so that by the first part, $\lim_{t\to\infty} V_t(\tilde\vartheta, S^\xi) = \lim_{t\to\infty}(1 + \tilde\vartheta \bullet S^\xi_t)$ exists and is finite, P-a.s. The result for $\vartheta = v_0\tilde\vartheta + I_{\{v_0=0\}}(\tilde\vartheta - \xi)$ then directly follows. ☐

Remark 3.5. Both Proposition 3.3 and Theorem 3.4 are formulated for ξ-discounted prices $S^\xi = S/(\xi \cdot S)$; so the discounter $\xi \cdot S = V(\xi)$ is S-tradable. One can ask if $V(\xi)$ could be replaced by an arbitrary $D \in \mathcal{S}_{++}$, and hence S^ξ by S/D. This is possible in Proposition 3.3, but not in Theorem 3.4; if we take for example $D_t = 2+\sin t$ which is even in $\mathcal{S}^{\mathrm{unif}}_{++}$ but does not converge, then $V(\vartheta, S^\xi/D) = V(\vartheta, S^\xi)/D$ also does not converge.

The significance of Theorem 3.4 is that under its assumptions, the limit $V_\infty(\vartheta, S^\xi)$ exists P-a.s. for all $\vartheta \in \Theta^{\mathrm{sf}}_+$. So $V(\vartheta, S^\xi)$ is well defined on the *closed* interval $[0,\infty]$, and as $V(\xi, S^\xi) \equiv 1$, the model S^ξ is on $[0,\infty]$ a numéraire market in the sense of [15]. Hence *in the setting of Theorem 3.4, the situation is as if we had the market from S^ξ defined up to* ∞, and so we can essentially use all results from [15] also for $[\![0,\infty]\!]$. More precisely, as long as we only use value processes of strategies in Θ^{sf}_+, we do not need S^ξ itself to be defined on $[0,\infty]$.

An important consequence is that the same weak AOA condition as above allows to improve any self-financing strategy asymptotically by a strongly value maximal strategy at no extra cost. This extends a result from Herdegen [15, Theorem 4.1] to $[\![0,\infty]\!]$.

Lemma 3.6. *Fix $\xi \in \Theta^{\mathrm{sf}}_{++}$ and suppose the zero strategy $0 \in \Theta^{\mathrm{sf}}_+$ is strongly value maximal for S^ξ. Then for any $\vartheta \in \Theta^{\mathrm{sf}}_+$, there exists a $\hat\vartheta \in \Theta^{\mathrm{sf}}_+$ which is strongly value maximal for S^ξ and satisfies*

$$V_0(\hat\vartheta, S^\xi) = V_0(\vartheta, S^\xi) \qquad \textit{and} \qquad \liminf_{t\to\infty} V_t(\hat\vartheta - \vartheta, S^\xi) \geq 0 \quad P\text{-a.s.}$$

Proof. Fix ξ as above. For any $\vartheta \in \Theta^{\mathrm{sf}}_+$, the limit $V_\infty(\vartheta, S^\xi)$ exists and is finite, P-a.s., by Theorem 3.4. In Definition 2.7 for S^ξ instead of S, we can thus replace the lim inf by a

[1] There are two minor unclear points or typos in the original proof in [9]. First, a set $A_2 \in \mathcal{F}_{t_2}$ such that $P[A_2 \Delta (B_1 \cap A)] > \alpha - \varepsilon_1 - \varepsilon_2$ is not a good approximation for $B_1 \cap A$; one should rather impose the requirement that $P[A_2\Delta(B_1 \cap A)] < \varepsilon_2/2$. Second, it is not clear why $P[B_1 \cap A] > \alpha - \varepsilon_1$ should be true. However, it is clear that $P[B_1 \cap A] > \alpha - 2\varepsilon_1$, which is still sufficient to obtain the conclusion.

limit, and so our strong value maximality for S^ξ is equivalent to strong maximality of S^ξ on $[0,\infty]$ in the sense of [15]. In particular, having 0 svm for S^ξ is equivalent to having NINA on $[0,\infty]$ for S^ξ in the sense of [15]. Using [15, Theorem 4.1] on $[0,\infty]$ for S^ξ and rewriting $V_\infty(\hat\vartheta, S^\xi) \geq V_\infty(\vartheta, S^\xi)$ as $\liminf_{t\to\infty} V_t(\hat\vartheta - \vartheta, S^\xi) \geq 0$ then gives the result. $\square$

3.2 Dual characterisation of strong value maximality

In this section, we provide dual characterisations of strong value maximality for S^ξ, of the zero strategy 0 or of a given strategy ξ. This uses the results of Herdegen [15] and extends them to a general time horizon by exploiting Section 3.1.

Proposition 3.7. *Fix $\xi \in \Theta^{\mathrm{sf}}_{++}$. Then the following are equivalent:*

(a) *ξ is strongly value maximal for S^ξ.*

(b) *Both NA$_\infty(S^\xi)$ and NUPBR$_\infty(S^\xi)$ hold.*

(c) *NFLVR$_\infty(S^\xi)$ holds.*

Proof. Both $\mathcal{C}_{\mathrm{adm}}(S^\xi)$ and $\mathcal{C}_{\mathrm{adm}}(S^\xi) \cap L^\infty$ are convex, and NUPBR$_\infty(S^\xi)$ means that $\mathcal{G}^1_{\mathrm{adm}}(S^\xi)$ is bounded in L^0. Due to (2.1), (b) $\Leftrightarrow$ (c) can thus be proved like [20, Lemma 2.2].

Both (a) and (c) imply that $0 \in \Theta^{\mathrm{sf}}_{+}$ is svm for S^ξ; indeed, under (a), this follows by Lemma 3.1, 2), and under (c), we combine (c) $\Rightarrow$ (b) with Proposition 3.3. Theorem 3.4 and the subsequent discussion thus allow us to treat S^ξ as if it were defined on $[0,\infty]$, and then the proof of [15, Proposition 3.24, (c)], with T replaced by ∞, gives the conclusion. $\square$

Recall that for $\mathcal{E} \in \{\sigma$-martingale, local martingale, martingale, UI martingale$\}$, an *$\mathcal{E}$-discounter* for an $\mathbb{R}^N$-valued semimartingale $\tilde S$ is a $D \in \mathcal{S}_{++}$ such that $\tilde S / D$ is an $\mathcal{E}$.

Theorem 3.8. *Fix $\xi \in \Theta^{\mathrm{sf}}_{++}$. Then the following are equivalent:*

(a) *The zero strategy $0 \in \Theta^{\mathrm{sf}}_{+}$ is strongly value maximal for S^ξ.*

(b) *There exists a strategy $\hat\vartheta \in \Theta^{\mathrm{sf}}_{++}$ which is strongly value maximal for S^ξ and has $V(\hat\vartheta, S^\xi) \in \mathcal{S}^{\mathrm{unif}}_{++}$.*

(c) *There exists a σ-martingale discounter $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ for S^ξ.*

Proof. (a) $\Rightarrow$ (b) By Lemma 3.6, we can find a $\hat\vartheta \in \Theta^{\mathrm{sf}}_{+}$ which is svm for S^ξ and satisfies $\liminf_{t\to\infty} V_t(\hat\vartheta - \xi, S^\xi) \geq 0$ P-a.s. Superadditivity of the lim inf plus $V(\xi, S^\xi) \equiv 1$ yields

$$\liminf_{t\to\infty} V_t(\hat\vartheta, S^\xi) \geq \liminf_{t\to\infty} V_t(\hat\vartheta - \xi, S^\xi) + \liminf_{t\to\infty} V_t(\xi, S^\xi) \geq 1 > 0 \qquad P\text{-a.s.}$$

But Theorem 3.4 and the subsequent discussion allow us to treat the market given by S^ξ as if it were defined up to ∞, and therefore $\inf_{t\geq 0} V_t(\hat\vartheta, S^\xi) > 0$ P-a.s. follows as in the proof of [15, Proposition 4.4], with T there replaced by ∞. On the other hand, $\limsup_{t\to\infty} V_t(\hat\vartheta, S^\xi) = \lim_{t\to\infty} V_t(\hat\vartheta, S^\xi) < \infty$ P-a.s. by Theorem 3.4, and because $V(\hat\vartheta, S^\xi) = V_0(\hat\vartheta, S^\xi) + \hat\vartheta \mathbin{\bullet} S^\xi$ is RCLL, this implies $\sup_{t\ge 0} V_t(\hat\vartheta, S^\xi) < \infty$ P-a.s. Hence $V(\hat\vartheta, S^\xi)$ is in $\mathcal{S}^{\mathrm{unif}}_{++}$. We note for later use that $V(\hat\vartheta, S^\xi) = \hat\vartheta \cdot S^\xi = V(\hat\vartheta)/V(\xi)$.

(b) $\Rightarrow$ (c) By Proposition 3.7, $\mathrm{NFLVR}_\infty(S^{\hat\vartheta})$ holds. Note that $V(\hat\vartheta, S^{\hat\vartheta}) \equiv 1$. By the discussion after [15, Definition 2.18], we can apply [12, Theorem 1.1] to the price process $(1, X) := (V(\hat\vartheta, S^{\hat\vartheta}), S^{\hat\vartheta})$ of dimension $1 + N$, and so there exists a probability measure $Q \approx P$ (on $\mathcal{F} \supseteq \mathcal{F}_\infty$) such that $S^{\hat\vartheta}$ is a σ-martingale under Q. The density process Z of Q with respect to P is in $\mathcal{S}^{\mathrm{unif}}_{++}$ as it is a strictly positive P-martingale on the *closed* interval $[0, \infty]$. Thus also $D := V(\hat\vartheta, S^\xi)/Z$ is in $\mathcal{S}^{\mathrm{unif}}_{++}$, and $S^\xi/D = ZS^\xi/V(\hat\vartheta, S^\xi) = ZS^{\hat\vartheta}$ is a σ-martingale under P by the Bayes rule for stochastic calculus; see Kallsen [22, Proposition 5.1]. (In classic terminology, Z is a σ-martingale deflator for $S^{\hat\vartheta}$.)

(c) $\Rightarrow$ (a) Because $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ and svm is discounting-invariant with respect to $\mathcal{S}^{\mathrm{unif}}_{++}$ by Lemma 3.1, 1), we can equivalently prove svm of 0 for S^ξ or for S^ξ/D. Hence we can and do assume without loss of generality that S^ξ is a P-σ-martingale. If 0 is not svm for S^ξ, we can find $f \in L^0_+ \setminus \{0\}$ and for every $\varepsilon > 0$ some $\hat\vartheta^\varepsilon \in \Theta^{\mathrm{sf}}_+$ with $V_0(\hat\vartheta^\varepsilon, S^\xi) \le \varepsilon$ and $\liminf_{t\to\infty} V_t(\hat\vartheta^\varepsilon, S^\xi) \ge f$ P-a.s. As $\hat\vartheta^\varepsilon \mathbin{\bullet} S^\xi = V(\hat\vartheta^\varepsilon, S^\xi) - V_0(\hat\vartheta^\varepsilon, S^\xi) \ge -\varepsilon$ on $[0, \infty)$ P-a.s., the Ansel–Stricker lemma [3, Corollary 3.5] implies that $V(\hat\vartheta^\varepsilon, S^\xi)$ is a local P-martingale and a P-supermartingale. Combining this with Fatou's lemma and $f \in L^0_+ \setminus \{0\}$ yields

$$\varepsilon \ge V_0(\hat\vartheta^\varepsilon, S^\xi) \ge \liminf_{t\to\infty} E[V_t(\hat\vartheta^\varepsilon, S^\xi)] \ge E\Big[\liminf_{t\to\infty} V_t(\hat\vartheta^\varepsilon, S^\xi)\Big] \ge E[f] > 0$$

for every $\varepsilon > 0$, which is a contradiction. ☐

Theorem 3.9. *Suppose that $\xi \in \Theta^{\mathrm{sf}}_{++}$ is such that both ξ and S^ξ are bounded (uniformly in (ω, t)). Then the following are equivalent:*

(a) *ξ is strongly value maximal for S^ξ.*

(b) *There exists a UI martingale discounter $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ for S^ξ.*

(c) *Each bounded $\vartheta \in \Theta^{\mathrm{sf}}_+$ is strongly value maximal for S^ξ.*

Proof. (c) $\Rightarrow$ (a) is clear.

(a) $\Rightarrow$ (b) If ξ is svm for S^ξ, the same argument as in the proof of (b) $\Rightarrow$ (c) in Theorem 3.8 yields a $Q \approx P$ such that S^ξ is a σ-martingale under Q. Being uniformly bounded, S^ξ is even a UI martingale under Q, and so the same $D := V(\xi, S^\xi)/Z = 1/Z$ as in the proof of Theorem 3.8 is now a UI martingale discounter for S^ξ and again in $\mathcal{S}^{\mathrm{unif}}_{++}$.

(b) $\Rightarrow$ (c) By Theorem 3.8, 0 is svm for S^ξ. Take any bounded $\vartheta \in \Theta^{\mathrm{sf}}_+$. To show that ϑ is svm for S^ξ, as in the proof of (c) $\Rightarrow$ (a) in Theorem 3.8, we can assume that S^ξ is a UI martingale; so $S^\xi_\infty = \lim_{t\to\infty} S^\xi_t$ exists P-a.s. and in L^1, and then S^ξ is a martingale on $[0,\infty]$. Moreover, $V(\vartheta, S^\xi)$ is P-a.s. convergent as $t \to \infty$ by Theorem 3.4. For any stopping time τ, we have $|V_\tau(\vartheta, S^\xi)| \le \|\vartheta\|_\infty \sum_{i=1}^N |(S^\xi_\tau)^i|$, and the UI property of S^ξ on $[0,\infty]$ implies that $V(\vartheta, S^\xi)$ is of class (D). So $V(\vartheta, S^\xi)$ is even a UI martingale.

If ϑ is not svm for S^ξ, we can find $f \in L^0_+ \setminus \{0\}$ and for every $\varepsilon > 0$ some $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+$ with $V_0(\hat{\vartheta}^\varepsilon, S^\xi) \le V_0(\vartheta, S^\xi) + \varepsilon$ and $\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon - \vartheta, S^\xi) \ge f$ P-a.s. As $\lim_{t\to\infty} V_t(\vartheta, S^\xi)$ exists, we even have $\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon, S^\xi) \ge \lim_{t\to\infty} V_t(\vartheta, S^\xi) + f$ P-a.s., and $V(\hat{\vartheta}^\varepsilon, S^\xi)$ is a supermartingale by the same argument as for ϑ. Combining this with Fatou's lemma, the UI martingale property of $V(\vartheta, S^\xi)$ and $f \in L^0_+ \setminus \{0\}$ then gives a contradiction because

$$\begin{aligned} V_0(\vartheta, S^\xi) + \varepsilon \ge V_0(\hat{\vartheta}^\varepsilon, S^\xi) &\ge \liminf_{t\to\infty} E[V_t(\hat{\vartheta}^\varepsilon, S^\xi)] \ge E\Big[\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon, S^\xi)\Big] \\ &\ge E\Big[\lim_{t\to\infty} V_t(\vartheta, S^\xi)\Big] + E[f] = \lim_{t\to\infty} E[V_t(\vartheta, S^\xi)] + E[f] = V_0(\vartheta, S^\xi) + E[f] \\ &> V_0(\vartheta, S^\xi) \end{aligned}$$

for every $\varepsilon > 0$. $\square$

Propositions 3.3 and 3.7 as well as Theorems 3.8 and 3.9 show a clear pattern: Thanks to the key result in Theorem 3.4, we can fairly easily extend the results from Herdegen [15] to a market with an infinite horizon, as long as we stick to ξ-discounted prices S^ξ. But what can be said if we want to start instead from the original prices S?

According to Lemma 3.1, 1), strong value maximality is discounting-invariant with respect to $\mathcal{S}^{\mathrm{unif}}_{++}$, and $S^\xi = S/V(\xi)$. If we impose the extra condition that $V(\xi)$ is in $\mathcal{S}^{\mathrm{unif}}_{++}$, it is clear that all the results still hold if we replace "strongly value maximal for S^ξ" by "strongly value maximal for S". Moreover, in Lemma 3.6, Theorem 3.8 and in (a)–(c) of Theorem 3.9, we can then also replace S^ξ by S.

In [15], the condition $V(\xi) \in \mathcal{S}^{\mathrm{unif}}_{++}$ is automatically satisfied for any $\xi \in \Theta^{\mathrm{sf}}_{++}$ as the market there is defined on a *closed* time interval. In contrast, on an *right-open* interval like $[0,\infty)$ we consider here, the condition is very restrictive — just think of an undiscounted Black–Scholes model with a positive interest rate $r > 0$ and the market portfolio $\xi \equiv \mathbb{1}$. It is precisely the idea of replacing value maximality by share maximality which allows us to eliminate that restrictive condition and allow general models for S.

3.3 Connecting share maximality and value maximality

In this section, we show that under a very mild condition on the pair (S, ξ) of price process and strategy, strong share maximality for ξ and strong value maximality for S^ξ are equivalent. This is the key for proving our main results.

Theorem 3.10. *Fix $\xi \in \Theta^{\mathrm{sf}}_{++}$.*

1) *If $\xi \ge 0$, then any $\vartheta \in \Theta^{\mathrm{sf}}_+$ which is strongly share maximal for ξ is strongly value maximal for S^ξ.*

2) *If $S \ge 0$, then any $\vartheta \in \Theta^{\mathrm{sf}}_+$ which is strongly value maximal for S^ξ is also strongly share maximal for ξ.*

The proof of Theorem 3.10 needs some preparation.

Lemma 3.11. *Suppose $S \geq 0$ and fix $\xi \in \Theta^{\mathrm{sf}}_{++}$. If there is a strategy $\hat{\vartheta} \in \Theta^{\mathrm{sf}}_{+}$ which is strongly value maximal for S^ξ, then $(\vartheta \cdot S)/(\xi \cdot S)$ is bounded in $t \geq 0$, P-a.s., for every $\vartheta \in \Theta^{\mathrm{sf}}_{+}$. In particular, S^ξ is bounded in $t \geq 0$, P-a.s.*

Proof. If $\hat{\vartheta}$ is svm for S^ξ, then 0 is svm for S^ξ by Lemma 3.1, 2). So Theorem 3.4 implies that for any $\vartheta \in \Theta^{\mathrm{sf}}_{+}$, the process $(\vartheta \cdot S)/(\xi \cdot S) = \vartheta \bullet S^\xi = \vartheta \cdot S^\xi - \vartheta_0 \cdot S^\xi_0$ is P-a.s. convergent as $t \to \infty$ and hence bounded in $t \geq 0$, P-a.s. Choosing $\vartheta := \mathrm{e}^i$ for $i = 1, \dots, N$ gives the second assertion; note that $S \geq 0$ is used here to ensure that $\mathrm{e}^i \in \Theta^{\mathrm{sf}}_{+}$. □

In the proof of Theorem 3.10, we need to concatenate strategies which requires some notation. Fix $\xi \in \Theta^{\mathrm{sf}}_{++}$ and a stopping time τ (as usual with values in $[0, \infty]$). The ξ-*concatenation* at time τ of $\vartheta^1, \vartheta^2 \in \Theta^{\mathrm{sf}}$ is defined by

$$\begin{aligned} \vartheta^1 \oslash^\xi_\tau \vartheta^2 &:= I_{[\![0,\tau]\!]} \vartheta^1 + I_{]\!]\tau,\infty]\!]} \Big(I_\Gamma \vartheta^1 + I_{\Gamma^c} \big(\vartheta^2 + V_\tau(\vartheta^1 - \vartheta^2, S^\xi) \xi \big) \Big) \\ &\text{with } \Gamma := \{ V_\tau(\vartheta^1) < V_\tau(\vartheta^2) \}. \end{aligned} \tag{3.1}$$

The interpretation is as follows. We start with ϑ^1 and follow this strategy until time τ where we compare its value to that of the competitor ϑ^2. If ϑ^1 is strictly cheaper, we stick to it. Otherwise, we liquidate ϑ^1_τ, start with ϑ^2 by buying ϑ^2_τ, and invest the rest of the proceeds (which is nonnegative) into ξ. Note that on $\{\tau = \infty\}$, we have $\vartheta^1 \oslash^\xi_\tau \vartheta^2 = \vartheta^1$ so that the possibly undefined expressions ϑ^1_∞, ϑ^2_∞, S_∞ or S^ξ_∞ never appear.

Lemma 3.12. *Fix $\xi \in \Theta^{\mathrm{sf}}_{++}$ and a stopping time τ. If ϑ^1, ϑ^2 are in Θ^{sf}, then so is $\vartheta^1 \oslash^\xi_\tau \vartheta^2$. If ϑ^1, ϑ^2 are in Θ^{sf}_{+}, then so is $\vartheta^1 \oslash^\xi_\tau \vartheta^2$.*

Proof. See Appendix. □

Proof of Theorem 3.10. 1) Assume ϑ is not svm for S^ξ. So there are $f \in L^0_+ \setminus \{0\}$ and for every $\varepsilon = 1/n$ some $\hat{\vartheta}^n \in \Theta^{\mathrm{sf}}_{+}$ with $\hat{\vartheta}^n_0 \cdot S^\xi_0 = V_0(\hat{\vartheta}^n, S^\xi) \leq \vartheta_0 \cdot S^\xi_0 + 1/n$ and $\liminf_{t\to\infty}((\hat{\vartheta}^n_t - \vartheta_t) \cdot S^\xi_t) \geq f$ P-a.s. Choose $\delta > 0$ and $A \in \mathcal{F}$ with $P[A] > 0$ such that $f \geq 2\delta I_A$ P-a.s., and define

$$\begin{aligned} \sigma'_n &:= \inf\{ t \geq 0 : (\hat{\vartheta}^n_t - \vartheta_t) \cdot S^\xi_t \geq \delta \}, \\ \varphi_n &:= \inf\{ t \geq 0 : P[\sigma'_n \leq t] \geq P[A](1 - 2^{-n+1}) \}, \\ \sigma_n &:= \sigma'_n \wedge \varphi_n \leq \varphi_n. \end{aligned}$$

Then σ'_n is a stopping time, φ_n a bounded nonrandom time and σ_n a bounded stopping time. Moreover, $B_n := \{\sigma'_n \leq \varphi_n\} \in \mathcal{F}_{\varphi_n}$ satisfies $P[B_n] \geq P[A](1 - 2^{-n+1})$ and we have

$$(\hat{\vartheta}^n_{\sigma_n} - \vartheta_{\sigma_n}) \cdot S^\xi_{\sigma_n} = (\hat{\vartheta}^n_{\sigma'_n} - \vartheta_{\sigma'_n}) \cdot S^\xi_{\sigma'_n} \geq \delta \qquad \text{on } B_n,\ P\text{-a.s.} \tag{3.2}$$

by right-continuity. Due to $\liminf_{t\to\infty}((\hat{\vartheta}^n_t - \vartheta_t)\cdot S^\xi_t) \geq f \geq 0$ P-a.s.,

$$\tau_n := \inf\{t \geq \varphi_n : (\hat{\vartheta}^n_t - \vartheta_t)\cdot S^\xi_t \geq -1/n\} \geq \varphi_n$$

is a P-a.s. finite-valued stopping time which satisfies $\tau_n \geq \sigma_n$.

We now consider the strategy

$$\tilde{\vartheta}^n := I_{[\![0,\tau_n]\!]}(\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta) + I_{]\!]\tau_n,\infty]\!]}\Big(\vartheta + V_{\tau_n}(\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta - \vartheta, S^\xi)\xi\Big) + \xi/n, \tag{3.3}$$

with $\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta$ defined in (3.1). In words, we hold a $(1/n)$-multiple of ξ, switch at time σ_n from $\hat{\vartheta}^n$ to ϑ if the value of ϑ is at most the value of $\hat{\vartheta}^n$, and always switch to ϑ at time τ_n; in both cases, any difference in value is invested into ξ. Using $\xi\cdot S^\xi \equiv 1$, this gives

$$V_0(\tilde{\vartheta}^n, S^\xi) = \tilde{\vartheta}^n_0 \cdot S^\xi_0 = \hat{\vartheta}^n_0 \cdot S^\xi_0 + (\xi_0 \cdot S^\xi_0)/n \leq V_0(\vartheta, S^\xi) + 2/n.$$

Next, as $\hat{\vartheta}^n$ and ϑ are in Θ^{sf}_+, Lemma 3.12 yields $\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta \in \Theta^{\mathrm{sf}}_+$, and therefore (3.3) gives $\tilde{\vartheta}^n \cdot S^\xi = V(\tilde{\vartheta}^n, S^\xi) \geq 0$ P-a.s. on $[\![0,\tau_n]\!]$. Using now $V(\xi, S^\xi) \equiv 1$ and the definition (3.1) allows us to compute, as in the proof of Lemma 3.12 in the Appendix, that

$$\begin{aligned} V_{\tau_n}(\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta - \vartheta, S^\xi) &= I_{\{\tau_n=\sigma_n\}} V_{\tau_n}(\hat{\vartheta}^n - \vartheta, S^\xi) \\ &\quad + I_{\{\tau_n>\sigma_n\}}\Big(I_{\Gamma_n} V_{\tau_n}(\hat{\vartheta}^n - \vartheta, S^\xi) + I_{\Gamma_n^c} V_{\sigma_n}(\hat{\vartheta}^n - \vartheta, S^\xi)\Big) \end{aligned} \tag{3.4}$$

with $\Gamma_n := \{V_{\sigma_n}(\hat{\vartheta}^n) < V_{\sigma_n}(\vartheta)\}$. This shows that due to $\tau_n < \infty$ P-a.s., we always have

$$V_{\tau_n}(\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta - \vartheta, S^\xi) \geq \min\Big((\hat{\vartheta}^n_{\tau_n} - \vartheta_{\tau_n})\cdot S^\xi_{\tau_n}, 0\Big) \geq -1/n \qquad P\text{-a.s.} \tag{3.5}$$

Combining (3.3) and (3.5) and using $\xi \geq 0$ implies that on $]\!]\tau_n,\infty]\!]$, we have

$$\tilde{\vartheta}^n - \vartheta = V_{\tau_n}(\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta - \vartheta, S^\xi)\xi + \xi/n \geq 0, \tag{3.6}$$

hence $V(\tilde{\vartheta}^n, S^\xi) \geq V(\vartheta, S^\xi)$, and so $\tilde{\vartheta}^n$ is like ϑ in Θ^{sf}_+.

Now on the set B_n, we have $\sigma_n = \sigma'_n$, hence $V_{\sigma_n}(\hat{\vartheta}^n - \vartheta, S^\xi) = (\hat{\vartheta}^n_{\sigma_n} - \vartheta_{\sigma_n})\cdot S^\xi_{\sigma_n} \geq \delta$ P-a.s. as in (3.2) and therefore by (3.4) also

$$V_{\tau_n}(\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta - \vartheta, S^\xi) = V_{\sigma_n}(\hat{\vartheta}^n - \vartheta, S^\xi) \geq \delta \qquad P\text{-a.s.}$$

Thus (3.6) and $\xi \geq 0$ yield $\tilde{\vartheta}^n - \vartheta \geq \delta\xi$ on B_n on $]\!]\tau_n,\infty]\!]$ and so, as $\tau_n < \infty$ P-a.s.,

$$\liminf_{t\to\infty}(\tilde{\vartheta}^n_t - \vartheta_t - \delta I_{B_n}\xi_t) \geq 0 \qquad P\text{-a.s.} \tag{3.7}$$

If we define the $[0,1]$-valued adapted process $\psi^n = (\psi^n_t)_{t\geq 0}$ by $\psi^n_t := \delta E[I_{B_n}|\mathcal{F}_t]$, then $\varphi_n < \infty$ and $B_n \in \mathcal{F}_{\varphi_n}$ yield $\psi^n_t = \delta I_{B_n}$ for $t \geq \varphi_n$ so that $\psi^n_\infty := \lim_{t\to\infty}\psi^n_t = \delta I_{B_n}$ P-a.s. Moreover, we also obtain via (3.7) that

$$\liminf_{t\to\infty}(\tilde{\vartheta}^n_t - \vartheta_t - \psi^n_t\xi_t) = \liminf_{t\to\infty}(\tilde{\vartheta}^n_t - \vartheta_t - \delta I_{B_n}\xi_t) \geq 0 \qquad P\text{-a.s.}$$

Set $B := \bigcap_{n\in\mathbb{N}} B_n$ and $\psi_t := \delta E[I_B|\mathcal{F}_t]$ for $t \geq 0$. Then $\lim_{t\to\infty} \psi_t = \psi_\infty := \delta I_B$ P-a.s., and $B \subseteq B_n$ for all n implies $\psi \leq \psi^n$ for all n. Moreover, $\psi_\infty \in L^\infty_+ \setminus \{0\}$ because

$$P[B] \geq P[B \cap A] = P[A] - P\left[A \cap \bigcup_{n\in\mathbb{N}} B_n^c\right] \geq P[A] - \sum_{n=1}^{\infty} P[A \cap B_n^c]$$
$$= P[A] - \sum_{n=1}^{\infty}(P[A] - P[A \cap B_n]) \geq P[A]\left(1 - \sum_{n=1}^{\infty} 2^{-n+1}\right) = P[A]/2 > 0.$$

So we have found ψ and for each $n \in \mathbb{N}$ a $\tilde{\vartheta}^n \in \Theta^{\mathrm{sf}}_+$ with $V_0(\tilde{\vartheta}^n, S^\xi) \leq V_0(\vartheta, S^\xi) + 2/n$ and

$$\liminf_{t\to\infty}(\tilde{\vartheta}^n_t - \vartheta_t - \psi_t\xi_t) = \liminf_{t\to\infty}(\tilde{\vartheta}^n_t - \vartheta_t - \psi^n_t\xi_t) \geq 0 \qquad P\text{-a.s.},$$

which contradicts the assumption that ϑ is ssm for ξ.

2) If ϑ is not ssm for ξ, there are a $[0,1]$-valued adapted $\psi = (\psi_t)_{t\geq 0}$ converging P-a.s. to $\psi_\infty := \lim_{t\to\infty} \psi_t \in L^\infty_+ \setminus \{0\}$ and for each $\varepsilon > 0$ a $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+$ with $V_0(\hat{\vartheta}^\varepsilon) \leq V_0(\vartheta) + \varepsilon$, hence $V_0(\hat{\vartheta}^\varepsilon, S^\xi) \leq V_0(\vartheta, S^\xi) + \varepsilon/V_0(\xi)$, and satisfying $\liminf_{t\to\infty}(\hat{\vartheta}^\varepsilon_t - \vartheta_t - \psi_t\xi_t) \geq 0$ P-a.s. By Lemma 3.11, S^ξ is bounded in $t \geq 0$, P-a.s. Superadditivity of the lim inf, Lemma A.1, $V(\xi, S^\xi) = \xi \cdot S^\xi \equiv 1$ and $S^\xi \geq 0$ from $S \geq 0$ thus yield that P-a.s.,

$$\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon - \vartheta, S^\xi) \geq \liminf_{t\to\infty}\left((\hat{\vartheta}^\varepsilon_t - \vartheta_t - \psi_t\xi_t) \cdot S^\xi_t\right) + \liminf_{t\to\infty}\left((\psi_t\xi_t) \cdot S^\xi_t\right)$$
$$\geq \left(\liminf_{t\to\infty}(\hat{\vartheta}^\varepsilon_t - \vartheta_t - \psi_t\xi_t)\right) \cdot \left(\liminf_{t\to\infty} S^\xi_t\right) + \psi_\infty \geq \psi_\infty.$$

So ϑ is not svm for S^ξ, and this completes the proof. □

4 Proofs and a more detailed result

In this section, we prove the main results from Section 2.

Proof of Theorem 2.12. 1) If S satisfies DSV for η, $0 \in \Theta^{\mathrm{sf}}_+$ is ssm for η and hence svm for S^η by Theorem 3.10, 1) for $\xi = \eta$. Theorem 3.8 for $\xi = \eta$ thus yields a $D' \in \mathcal{S}^{\mathrm{unif}}_{++}$ such that S^η/D' is a σ-martingale. Writing $S^\eta/D' = S/((\eta \cdot S)D') =: S/D$ shows that $D = (\eta \cdot S)D' \in \mathcal{S}_{++}$ is a σ-martingale discounter for S. Moreover, $\eta \cdot (S/D) = 1/D'$ is in $\mathcal{S}^{\mathrm{unif}}_{++}$ like D', and in particular, $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s. This argument does not need $S \geq 0$.

2) If $D \in \mathcal{S}_{++}$ is a σ-martingale discounter for S, then $\tilde{S} := S/D$ is a σ-martingale. By [3, Corollary 3.5], $0 \leq V(\eta, \tilde{S}) = V_0(\eta, \tilde{S}) + \eta \bullet \tilde{S}$ is a P-supermartingale so that $\lim_{t\to\infty} V_t(\eta, \tilde{S})$ exists and is finite, P-a.s. (We cannot use Theorem 3.4 here because D need not be S-tradable; see Remark 3.5.) This yields $\sup_{t\geq 0}(\eta_t \cdot \tilde{S}_t) < \infty$ P-a.s., and because also $\inf_{t\geq 0}(\eta_t \cdot \tilde{S}_t) > 0$ P-a.s. by assumption, we obtain $V(\eta, \tilde{S}) = \eta \cdot (S/D) \in \mathcal{S}^{\mathrm{unif}}_{++}$. Now $D' \equiv 1 \in \mathcal{S}^{\mathrm{unif}}_{++}$ is a σ-martingale discounter for $\tilde{S}$, and so Theorem 3.8 applied to $\tilde{S}$ and $\xi = \eta$ implies that 0 is svm for $\tilde{S}^\eta$. By Theorem 3.10, 2) for $\tilde{S} \geq 0$ and $\xi = \eta$, 0 is then ssm for η in the model $\tilde{S}$, and hence also in the model $S = \tilde{S}D$ because strong share maximality is discounting-invariant with respect to $\mathcal{S}_{++}$. So S satisfies DSV for η. □

Proof of Theorem 2.14. This is very similar to the proof of Theorem 2.12, with the main difference that we use Theorem 3.9 instead of Theorem 3.8.

1) If S satisfies DSE for η, then every η-buy-and-hold $\vartheta \in \Theta^{\mathrm{sf}}_+$ and in particular the reference strategy η is ssm for η and hence svm for S^η by Theorem 3.10, 1). Moreover, η and S^η are bounded by assumption, and so Theorem 3.9 for $\xi = \eta$ yields the existence of some $D' \in \mathcal{S}^{\mathrm{unif}}_{++}$ such that S^η/D' is a UI martingale. As before, $D := (\eta \cdot S)D'$ is then a UI martingale discounter for S, and we also again get $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s.

2) If $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ is a UI martingale deflator for S and we set $\tilde{S} := S/D$, then we get $V(\eta, \tilde{S}) \in \mathcal{S}^{\mathrm{unif}}_{++}$ as before. Because η and S^η are bounded by assumption, Theorem 3.9 applied to $\tilde{S}$ and $\xi = \eta$ then yields that each bounded $\vartheta \in \Theta^{\mathrm{sf}}_+$ is svm for $\tilde{S}^\eta$. But every η-buy-and-hold $\vartheta \in \Theta^{\mathrm{sf}}_+$ is bounded like η itself, hence svm for $\tilde{S}^\eta$ and then ssm for η as before. Thus S satisfies DSE for η. $\square$

Before we prove Theorem 2.15, we give a more detailed statement with a number of extra equivalent assertions.

Theorem 4.1. (full version of Theorem 2.15) *Suppose $S \geq 0$ and there exists a reference strategy η. Consider the following statements:*

(e1) *S satisfies dynamic share efficiency for η.*

(v1) *S satisfies dynamic share viability for η.*

(e2) *Every bounded $\vartheta \in \Theta^{\mathrm{sf}}_+$ is strongly value maximal for S^η.*

(e2′) *η is strongly value maximal for $S^\eta = S/(\eta \cdot S)$.*

(v2) *0 is strongly value maximal for $S^\eta = S/(\eta \cdot S)$.*

(e3) *There exists a UI martingale discounter D for S with $\eta \cdot (S/D) \in \mathcal{S}^{\mathrm{unif}}_{++}$.*

(v3) *There exists a σ-martingale discounter D for S with $\eta \cdot (S/D) \in \mathcal{S}^{\mathrm{unif}}_{++}$.*

(e4) *$NFLVR_\infty(S^\eta)$ holds, i.e., S^η satisfies $NFLVR_\infty$.*

(v4) *$NUPBR_\infty(S^\eta)$ holds, i.e., S^η satisfies $NUPBR_\infty$.*

(e5) *Every bounded $\vartheta \in \Theta^{\mathrm{sf}}_+$ is strongly share maximal for η.*

(e5′) *The reference strategy $\eta \in \Theta^{\mathrm{sf}}_+$ is strongly share maximal for η.*

(v5) *There exists some $\vartheta \in \Theta^{\mathrm{sf}}_+$ which is strongly share maximal for η.*

(e6) *For every $D \in \mathcal{S}_{++}$ with $\eta \cdot (S/D) \in \mathcal{S}^{\mathrm{unif}}_{++}$, every bounded $\vartheta \in \Theta^{\mathrm{sf}}_+$ is strongly value maximal for the D-discounted price process S/D.*

(v6) *For every $D \in \mathcal{S}_{++}$ with $\eta \cdot (S/D) \in \mathcal{S}^{\mathrm{unif}}_{++}$, the zero strategy $0 \in \Theta^{\mathrm{sf}}_{+}$ is strongly value maximal for the D-discounted price process S/D.*

Then we have (eK) $\Rightarrow$ (vK) *for* $\mathrm{K} = 1, \dots, 6$*, and the statements* (vK)*,* $\mathrm{K} = 1, \dots, 6$*, are equivalent among themselves. If in addition η and S^η are bounded (uniformly in (ω, t)), then also the statements* (eK)*,* $\mathrm{K} = 1, \dots, 6$*, are equivalent among themselves (including the prime $'$ versions).*

Figure 1 gives a graphical overview of this result.

$$
\begin{array}{ccc}
S \text{ satisfies DSE for } \eta & & S \text{ satisfies DSV for } \eta \\
(\text{every } \eta\text{-buy-and-hold } \vartheta \in \Theta^{\mathrm{sf}}_{+} \text{ is ssm for } \eta) & & (0 \text{ is ssm for } \eta) \\
\Updownarrow & & \Updownarrow \\
\text{every bounded } \vartheta \in \Theta^{\mathrm{sf}}_{+} \text{ is ssm for } \eta & & 0 \in \Theta^{\mathrm{sf}}_{+} \text{ is ssm for } \eta \\
\Updownarrow & & \Updownarrow \\
\eta \text{ is ssm for } \eta & & 0 \text{ is ssm for } \eta \\
\Updownarrow & & \Updownarrow \\
\eta \text{ is svm for } S^\eta & & 0 \text{ is svm for } S^\eta \\
\Updownarrow & & \Updownarrow \\
\text{every bounded } \vartheta \in \Theta^{\mathrm{sf}}_{+} \text{ is svm for } S^\eta & \Longrightarrow & \text{some } \vartheta \in \Theta^{\mathrm{sf}}_{+} \text{ is svm for } S^\eta \\
\Updownarrow & & \Updownarrow \\
\forall D \in \mathcal{S}_{++} \text{ with } \eta \cdot (S/D) \in \mathcal{S}^{\mathrm{unif}}_{++}\text{:} & & \forall D \in \mathcal{S}_{++} \text{ with } \eta \cdot (S/D) \in \mathcal{S}^{\mathrm{unif}}_{++}\text{:} \\
\text{every bounded } \vartheta \in \Theta^{\mathrm{sf}}_{+} \text{ is svm for } S/D & & 0 \in \Theta^{\mathrm{sf}}_{+} \text{ is svm for } S/D \\
\Updownarrow & & \Updownarrow \\
\exists \text{ UI martingale discounter } D \text{ for } S & & \exists\ \sigma\text{-martingale discounter } D \text{ for } S \\
\text{with } \eta \cdot (S/D) \in \mathcal{S}^{\mathrm{unif}}_{++} & & \text{with } \eta \cdot (S/D) \in \mathcal{S}^{\mathrm{unif}}_{++} \\
\Updownarrow & & \Updownarrow \\
S^\eta \text{ satisfies NFLVR}_\infty & & S^\eta \text{ satisfies NUPBR}_\infty
\end{array}
$$

Figure 1: Graphical summary of Theorem 4.1. Assumptions are $S \geq 0$ and that η is a reference strategy (which is assumed to exist). The equivalences on the left side need in addition that η and S^η are bounded (uniformly in (ω, t)).

Proof. While we need $\eta \in \Theta^{\mathrm{sf}}_{++}$ at once to define S^η, the assumption $S \geq 0$ is used only in some implications. We structure the proof to make this apparent and initially only assume that there exists a reference strategy η; so $\eta \geq 0$.

It is clear from the statements or definitions that (eK) $\Rightarrow$ (vK) holds for $\mathrm{K} = 1, \dots, 6$.

Because DSV for η means that 0 is ssm for η, (v1) $\Rightarrow$ (v5) is clear, and (v5) $\Rightarrow$ (v2) follows directly from Lemma 3.1, 2) and Theorem 3.10, 1) for $\xi = \eta$. Next, (v1) $\Rightarrow$ (v3) is the "only if" part of Theorem 2.12, and (v6) $\Rightarrow$ (v2) follows from Lemma 3.1, 1) because $(S/D)/(\eta \cdot (S/D)) = S/(\eta \cdot S) = S^\eta$. Finally, (v2) $\Leftrightarrow$ (v4) is Proposition 3.3 for $\xi = \eta$, and (v1) implies by Theorem 3.10, 1) that 0 is svm for $S^\eta = (S/D)/(\eta \cdot (S/D))$ and hence also svm for S/D by Lemma 3.1, 1) so that we get (v6).

If $S \geq 0$, (v2) $\Rightarrow$ (v1) follows from Theorem 3.10, 2) for $\xi = \eta$, and (v3) $\Rightarrow$ (v1) is the "if" part of Theorem 2.12. This proves all implications for the (vK) statements.

DSE for η means that every η-buy-and-hold $\vartheta \in \Theta^{\mathrm{sf}}_{+}$ is ssm for η. Thus (e1) $\Rightarrow$ (e5$'$) is clear, so is (e5) $\Rightarrow$ (e1) as η is bounded by the assumptions for the statements (eK), and (e5$'$) $\Rightarrow$ (e2$'$) is from Theorem 3.10, 1) for $\xi = \eta$. Moreover, (e2) $\Rightarrow$ (e2$'$) is clear because η is bounded, (e6) $\Rightarrow$ (e2) is clear by taking $D = V(\eta)$ so that $S/D = S^{\eta}$ and $\eta \cdot (S/D) \equiv 1 \in \mathcal{S}^{\mathrm{unif}}_{++}$, and (e2$'$) $\Leftrightarrow$ (e4) is Proposition 3.7 for $\xi = \eta$.

Now suppose $S \geq 0$ and recall the assumption that both η and S^{η} are bounded (uniformly in (ω, t)). Then (e1) $\Leftrightarrow$ (e3) is Theorem 2.14. If (e2$'$) holds, every bounded $\vartheta \in \Theta^{\mathrm{sf}}_{+}$ is svm for S^{η} by Theorem 3.9 for $\eta = \xi$, and hence by Theorem 3.10, 2) ssm for η, so that we get (e2$'$) $\Rightarrow$ (e5). Finally, every such ϑ is also svm for $S/D = S^{\eta}(\eta \cdot (S/D))$ by Lemma 3.1, 1). This gives (e2$'$) $\Rightarrow$ (e6) and completes the proof. $\square$

5 Robustness, classic theory, and literature

This section has three parts. We first discuss to which extent our approach and results are robust with respect to the choice of a reference strategy. We then connect our work to the classic theory, and finally provide a comparison to the existing literature.

5.1 Robustness towards the choice of a reference strategy

As already pointed out in Remark 2.9, 2), our concepts and main results depend on the choice of a reference strategy η. In this section, we show that this dependence is fairly weak, which means that our approach is quite robust towards the choice of η.

Consider two reference strategies η, η'; so both are in $\Theta^{\mathrm{sf}}_{++}$ and ≥ 0. We also consider the *ratio condition*

$$(\eta' \cdot S)/(\eta \cdot S) = V(\eta')/V(\eta) \in \mathcal{S}^{\mathrm{unif}}_{++}, \text{ i.e.,}$$

$$0 < \inf_{t \geq 0} \Big(V_t(\eta')/V_t(\eta)\Big) \leq \sup_{t \geq 0} \Big(V_t(\eta')/V_t(\eta)\Big) < \infty \qquad P\text{-a.s.} \tag{5.1}$$

As $\mathcal{S}^{\mathrm{unif}}_{++}$ is closed under taking reciprocals, (5.1) is symmetric in η and η'.

Lemma 5.1. *Suppose $S \geq 0$ and there exist reference strategies η, η'. Fix $\vartheta \in \Theta^{\mathrm{sf}}_{+}$. If* (5.1) *holds, then ϑ is ssm for η if and only if it is ssm for η'.*

Proof. If ϑ is ssm for η, then it is svm for S^{η} by Theorem 3.10, 1). But $S^{\eta'} = S^{\eta}/D$ with $D := (\eta' \cdot S)/(\eta \cdot S) \in \mathcal{S}^{\mathrm{unif}}_{++}$ due to (5.1). Thus by Lemma 3.1, 1), ϑ is svm for $S^{\eta'}$ as well, and hence ssm for η' by Theorem 3.10, 2). The converse is argued symmetrically. $\square$

Proposition 5.2. *Suppose $S \geq 0$ and there exist reference strategies η, η'.*

1) *If* (5.1) *holds, then DSV for η and DSV for η' are equivalent.*

2) *If η, η' as well as $S^{\eta}, S^{\eta'}$ are bounded (uniformly in (ω, t)), then DSE for η and DSE for η' are equivalent.*

Proof. 1) Apply Lemma 5.1 to $\vartheta \equiv 0$.

2) Because η and $S^{\eta'}$ are bounded, so is $(\eta \cdot S)/(\eta' \cdot S) = \eta \cdot S^{\eta'}$, and analogously, $(\eta' \cdot S)/(\eta \cdot S)$ is bounded. So (5.1) holds. If we have DSE for η, every bounded $\vartheta \in \Theta_+^{\mathrm{sf}}$ and in particular η' is ssm for η by Theorem 2.15, (e1) $\Rightarrow$ (e5). By Lemma 5.1, η' is thus also ssm for η', and so Theorem 2.15, (e5′) $\Rightarrow$ (e1), gives DSE for η'. The converse argument is symmetric. $\square$

The boundedness assumptions in Proposition 5.2, 2) are precisely those we impose in Theorem 2.14 to obtain a dual characterisation for DSE. So DSE is robust with respect to the choice of any reference strategy in that class.

Remark 5.3. Suppose $S \geq 0$ and $\sum_{i=1}^N S^i$ is strictly positive with strictly positive left limits. As seen in Remark 2.3, 1), the market portfolio $\mathbb{1}$ is then a reference strategy with $\mathbb{1}$ and $S^{\mathbb{1}} = S/\sum_{i=1}^N S^i$ bounded (uniformly in (ω, t)). Any $\eta \in \Theta_+^{\mathrm{sf}}$ with $c\mathbb{1} \leq \eta \leq C\mathbb{1}$ for constants $0 < c \leq C < \infty$ is then also a reference strategy with η and S^η bounded (uniformly in (ω, t)); indeed, $\eta \cdot S \geq c\mathbb{1} \cdot S$ and hence $S^\eta \leq \frac{1}{c} S^{\mathbb{1}}$ (coordinatewise). In view of Proposition 5.2, 2), DSE for the market portfolio is thus the same as for any bounded reference strategy which always invests in a uniformly nondegenerate way into all assets. An "extreme" strategy like e^i, buy and hold a single fixed asset i, does not satisfy this.

5.2 Connections to the classic results

Theorem 2.15 indicates that in some way, DSV is related to NUPBR, and DSE to NFLVR. In this section, we study this in more detail in the classic setup $S = (1, X)$. Because our results use the condition $S \geq 0$, we also impose $X \geq 0$.

If $S = (1, X)$ with $X \geq 0$, then $\mathbb{1}$ is a reference strategy (with $V(\mathbb{1}) \geq 1$) and $S^{\mathbb{1}} = S/V(\mathbb{1}) = S/\sum_{i=1}^N S^i$ is bounded (uniformly in (ω, t)). Both these properties still hold if $S \geq 0$ only satisfies $\sum_{i=1}^N S^i > 0$ and $\sum_{i=1}^N S_-^i > 0$. In contrast, e^i is a reference strategy for general S only if $S^i > 0$ and $S_-^i > 0$, and $S^{\mathrm{e}^i} = S/S^i$ has in general no boundedness properties. For $S = (1, X)$, e^1 is always a reference strategy and $S^{\mathrm{e}^1} = S$. But this relies crucially on the particular structure of $S = (1, X)$, and so choosing e^1 as a reference strategy is both more extreme and more delicate than choosing $\mathbb{1}$. The next result reflects this.

Proposition 5.4. *If $S = (1, X)$ for an $\mathbb{R}_+^d$-valued semimartingale $X \geq 0$, classic NUPBR for X is equivalent to S satisfying DSV for e^1 and implies that S satisfies DSV for $\mathbb{1}$.*

Proof. Classic NUPBR for X is the same as $\mathrm{NUPBR}_\infty(1, X)$, and this is by Proposition 3.3 equivalent to 0 being svm for $S^{\mathrm{e}^1} = S = (1, X)$. In turn, this is by Theorem 3.10 equivalent to 0 being ssm for e^1, which is DSV for e^1 by definition. Next, DSV for $\mathbb{1}$ is the same as 0 being ssm for $\mathbb{1}$, which is equivalent to 0 being svm for $S^{\mathbb{1}}$ by Theorem 3.10 again. As $S^{\mathbb{1}} = S/V(\mathbb{1}) = S^{\mathrm{e}^1}/V(\mathbb{1})$, svm for S^{e^1} is by Lemma 3.1, 1) the same as svm for

$S^{\mathbb{1}}$ whenever $V(\mathbb{1}) \in \mathcal{S}^{\text{unif}}_{++}$, and the point is now that this holds if X satisfies NUPBR. Indeed, $V(\mathbb{1}) \geq 1$ due to $X \geq 0$, and as NUPBR for X is equivalent to 0 being svm for $S^{\mathrm{e}^1} = S$, Theorem 3.4 for $\xi \equiv \mathrm{e}^1$ implies that $V(\mathbb{1})$ is convergent and hence bounded in $t \geq 0$, P-a.s. So $V(\mathbb{1}) \in \mathcal{S}^{\text{unif}}_{++}$ and we are done. □

The converse of the implication in Proposition 5.4 is not true in general. A counterexample is given in Example 6.8. Thus our new concept of dynamic share viability, when used for the market portfolio $\mathbb{1}$, is more widely applicable than classic NUPBR.

The situation with DSE versus NFLVR is more subtle. We first give a positive result.

Proposition 5.5. *If $S = (1, X)$ for an $\mathbb{R}^d_+$-valued semimartingale $X \geq 0$, classic NFLVR for X is equivalent to S satisfying DSE for* e^1.

Proof. Classic NFLVR for X is the same as $\mathrm{NFLVR}_\infty(1, X)$, and this is by Proposition 3.3 equivalent to e^1 being svm for $S^{\mathrm{e}^1} = S = (1, X)$. In turn, this is by Theorem 3.10 equivalent to e^1 being ssm for e^1. But any e^1-buy-and-hold strategy ϑ is of the form $\vartheta = \lambda \mathrm{e}^1$ for some $\lambda \in \mathbb{R}$, because $\mathrm{e}^1 = (1, 0, \ldots, 0)$, and so ϑ is in Θ^{sf}_+ if and only if $\lambda \geq 0$. Thanks to Lemma 3.1, 2), e^1 is therefore ssm for e^1 if and only if every e^1-buy-and-hold $\vartheta \in \Theta^{\mathrm{sf}}_+$ is ssm for e^1, which is DSE for e^1 by definition. □

Remark 5.6. It looks tempting to use Theorem 4.1, (e1) ⇔ (e5′), to shorten the above argument. But the proof of that equivalence uses that both η and S^η are bounded (uniformly in (ω, t)), which would place a massive restriction on $S^{\mathrm{e}^1} = S = (1, X)$ for $\eta \equiv \mathrm{e}^1$.

If we want to use the reference strategy $\eta \equiv \mathbb{1}$, the situation for DSE versus NFLVR is different from DSV versus NUPBR. Neither of DSE for $\mathbb{1}$ and NFLVR for X implies the other in general in the classic case $S = (1, X)$. Example 6.10 shows that DSE for $\mathbb{1}$ does not imply NFLVR for X. Conversely, Example 6.7 shows that for $S = (1, X)$, we can have NFLVR for X while DSE for $\mathbb{1}$ fails. (We do note that by Theorem 2.15, S satisfying DSE for $\mathbb{1}$ is equivalent to $S^{\mathbb{1}}$ satisfying NFLVR_∞ with $\eta \equiv \mathbb{1}$, and also to $S^{\mathbb{1}}$ satisfying DSE for $\mathbb{1}$, by discounting-invariance; but this means that we have a result only for $\mathbb{1}$-discounted prices, not for the original prices S.)

The background of this discrepancy is the following: NFLVR for X is equivalent to e^1 being svm for $(1, X)$, whereas DSE for $\mathbb{1}$ is equivalent to $\mathbb{1}$ being svm for $S^{\mathbb{1}}$. Here, "e^1 svm" is weaker than "$\mathbb{1}$ svm" as $\mathrm{e}^1 \leq \mathbb{1}$, but "for $(1, X)$" is stronger than "for $S^{\mathbb{1}}$" as $(1, X) \geq S^{\mathbb{1}}$. Upon reflection, the discrepancy is actually not surprising; in fact, NFLVR is about how e^1 or $V(\mathrm{e}^1)$ fits into the market, whereas DSE for $\mathbb{1}$ looks at all the e^i, $i = 1, \ldots, N$. The next result makes this more precise.

Proposition 5.7. *Suppose that $S \geq 0$ and there exist reference strategies η, η'. Then $\mathrm{NFLVR}_\infty(S^\eta)$ plus $\inf_{t \geq 0}(\eta'_t \cdot S^\eta_t) > 0$ P-a.s. implies that η is strongly share maximal for η'. In particular, if $S = (1, X)$ with $X \geq 0$, then classic NFLVR for X implies that* e^1 *is strongly share maximal for* $\mathbb{1}$.

Proof. The second statement follows from the first for $\eta \equiv \mathrm{e}^1, \eta' \equiv \mathbb{1}$ by observing that $\mathbb{1} \cdot S^{\mathrm{e}^1} = 1 + \sum_{i=1}^d X^i \geq 1$. If we have NFLVR$_\infty(S^\eta)$, then 0 is svm for S^η by Propositions 3.7 and 3.3, and so $V(\eta', S^\eta)$ is convergent and hence bounded in $t \geq 0$, P-a.s., by Theorem 3.4. By assumption, $\inf_{t \geq 0} V_t(\eta', S^\eta) > 0$ P-a.s. so that $V(\eta', S^\eta) \in \mathcal{S}_{++}^{\mathrm{unif}}$. By Proposition 3.7 again, η is svm for S^η and hence by Lemma 3.1, 1) also for $S^\eta / V(\eta', S^\eta) = S^{\eta'}$. By Theorem 3.10, 2), η is then ssm for η'. □

To conclude, we briefly show how our approach yields new results even in the classic case. Note that the next result does not assume that $S \geq 0$.

Proposition 5.8. *Suppose there exists an $\eta \in \Theta_{++}^{\mathrm{sf}}$. Then S^η satisfies NUPBR$_\infty$ if and only if there exists a σ-martingale discounter D for S^η with $D_\infty := \lim_{t \to \infty} D_t < \infty$ P-a.s.*

Proof. By Proposition 3.3 and Theorem 3.8 for $\xi = \eta$, S^η satisfies NUPBR$_\infty$ if and only if it admits a σ-martingale discounter $D \in \mathcal{S}_{++}$ with the extra property $D \in \mathcal{S}_{++}^{\mathrm{unif}}$. Fix any σ-martingale discounter D for S^η. Because $V(\eta, S^\eta) \equiv 1$, writing

$$1/D = V(\eta, S^\eta)/D = V(\eta, S^\eta/D) = V_0(\eta, S^\eta/D) + \eta \bullet (S^\eta/D)$$

shows that $1/D$ is a σ-martingale like S^η/D, and also in $\mathcal{S}_{++}$ like D. So by [3, Corollary 3.5], $1/D$ is a local martingale > 0 and a supermartingale ≥ 0, and hence P-a.s. convergent to some finite limit. D itself is then also P-a.s. convergent and $D_\infty > 0$ P-a.s., which implies $\inf_{t \geq 0} D_t > 0$ P-a.s. The extra property $D \in \mathcal{S}_{++}^{\mathrm{unif}}$ thus holds if and only if $\sup_{t \geq 0} D_t < \infty$ P-a.s. or, equivalently by convergence, $D_\infty < \infty$ P-a.s. □

Corollary 5.9. *Suppose X is an $\mathbb{R}^d$-valued semimartingale. Then X satisfies classic NUPBR if and only if there exists a local martingale $L > 0$ with $L_\infty := \lim_{t \to \infty} L_t > 0$ P-a.s. and such that LX is a σ-martingale.*

Proof. For $S = (1, X)$, $\eta \equiv \mathrm{e}^1$ is in $\Theta_{++}^{\mathrm{sf}}$ with $S^\eta = S^{\mathrm{e}^1} = S$. So we can apply Proposition 5.8 and take $L := 1/D$. The properties of L are all shown in the above proof. □

Corollary 5.9 sharpens the classic characterisation of NUPBR in Karatzas/Kardaras [24, Theorem 4.12] in two ways: X and X_- need not be strictly positive (i.e., we need not assume $X \in \mathcal{S}_{++}^d$), and we get a σ-martingale deflator D for X, not only a supermartingale deflator $\tilde{D}$ for all $H \bullet X$ with $H \in L_{\mathrm{adm}}(X)$; see also Bálint/Schweizer [4, Lemma 2.13] for the connection between these two properties. (Unlike $\tilde{D}$, however, D cannot be chosen S-tradable in general; see Takaoka/Schweizer [35, Remark 2.8] for a counterexample and [4, Propositions 2.19 and 2.20] for related positive results.) Corollary 5.9 also extends [35, Theorem 2.6] from a closed interval $[0, T]$ to a general time horizon.

5.3 Comparison to the literature

This section compares our ideas and results to the existing literature. We first consider absence of arbitrage (AOA) aspects and then discuss numéraire- or discounting-invariance.

5.3.1 Absence of arbitrage

The two most used classic AOA notions in the literature are NFLVR (due to Delbaen/ Schachermayer [9]) and the strictly weaker NUPBR (coined by Karatzas/Kardaras [24]). The latter condition was introduced under different names by different authors — BK in Kabanov [20], no cheap thrills in Loewenstein/Willard [30] or NA1 in Kardaras [25, 27]; see also Kabanov/Kramkov [21] for the notion of NAA1. By [25, Proposition 1] and Kabanov et al. [18, Lemma A.1], all these and NUPBR are equivalent.

Both NFLVR and NUPBR are classically only defined for *discounted* price processes of the form $S = (1, X)$. Dual characterisations, in terms of martingale properties for X, first focused on NFLVR, culminating in the classic FTAP due to Delbaen/Schachermayer [9, 12] that for a general $\mathbb{R}^d$-valued semimartingale X, NFLVR for $S = (1, X)$ is equivalent to the existence of an equivalent σ-martingale measure for the discounted prices X.

Even if NFLVR does not hold, a market can still be sufficiently nice to allow some AOA-type arguments. This has been exploited in several papers. Loewenstein/Willard [30] show in an Itô process setup that already no cheap thrills (NUPBR) is sufficient (and necessary) to solve utility maximization problems; see also Chau et al. [7]. In the benchmark approach presented in Platen/Heath [31], a market may violate NFLVR; but in units of the so-called numéraire portfolio, the theory works as if there was no arbitrage. An excellent discussion with more details can be found in Herdegen [15, Section 5.3]. For stochastic portfolio theory and the study of relative arbitrage (see Karatzas/Fernholz [23] for an overview), a market may have “arbitrage” in the sense of FLVR; but portfolio choice still makes sense, and hedging via superreplication can still work. The comprehensive paper of Karatzas/Kardaras [24] shows that maximising growth rate, asymptotic growth or expected logarithmic utility from terminal wealth all make sense if and only if NUPBR holds. Another overview of the above connections can be found in the recent work of Choulli et al. [8].

In Bálint/Schweizer [4], we have recently studied the dependence of AOA conditions on the time horizon as part of an analysis of large financial markets; see [4, Section 5 and in particular Corollary 5.4]. We also point out in [4, Remark 5.6] that in contrast to common belief, NUPBR on $[0, \infty)$ is not stable under localisation. For related work, we refer to Kardaras [28], Acciaio et al. [1] and Aksamit et al. [2].

Like for NFLVR, the literature contains dual characterisations of NUPBR. Depending on the setting, they vary in the strength of the dual formulation; see Table 1 for an overview. For $S = (1, X)$ on $[0, \infty)$ with $X \in \mathcal{S}^d_{++}$, Karatzas/Kardaras [24] show that NUPBR is equivalent to the existence of an S-tradable supermartingale discounter for all wealth processes of admissible self-financing strategies. On $[0, T]$, this is strengthened by Takaoka/Schweizer [35] to the existence of a σ-martingale discounter for X, where again $S = (1, X)$ but X is an $\mathbb{R}^d$-valued semimartingale. Both Kardaras [27] and Kabanov et al. [18], inspired by the results and a counterexample in [35], work on $[0, T]$ with $S = (1, X)$

for an $\mathbb{R}^d$-valued semimartingale X and characterise NA1 (which is equivalent to NUPBR) by the existence of a local martingale discounter for all wealth processes of admissible self-financing strategies. In [27], this is done for $d = 1$ so that X is real-valued; [18] extend the result to $d \geq 1$ and in addition manage to find an S-*tradable* local martingale discounter under any $R \approx P$ in any neighbourhood of P. An overview of the connections between different types of discounters is given in Bálint/Schweizer [4, Lemma 2.13].

	price process S	time	condition	dual condition
KK [24]	$(1, X) \in \mathcal{S}^{1+d}_{++}$	$[0, \infty)$	NUPBR	$\exists$ S-tradable SMD $D > 0$ for all $H \bullet X$ with $H \in L_{\mathrm{adm}}(X)$, with $D_\infty > 0$
TS [35]	$(1, X) \in \mathcal{S}^{1+d}$	$[0, T]$	NUPBR	$\exists$ σMD $D > 0$ for X
K [27]	$(1, X) \in \mathcal{S}^{1+1}$	$[0, T]$	NA1	$\exists$ LMD $D > 0$ for all $H \bullet X$ with $H \in L_{\mathrm{adm}}(X)$
KKS [18]	$(1, X) \in \mathcal{S}^{1+d}$	$[0, T]$	NA1	$\exists$ S-tradable LMD $D > 0$ for all $H \bullet X$ with $H \in L_{\mathrm{adm}}(X)$, in any neighbourhood of P
H [15]	in $\mathcal{S}^N$	$[0, T]$	NINA	$\exists$ discounter/EσMM pair for S
here	in $\mathcal{S}^N_+$	$[0, \infty)$	DSV for η	$\exists$ LMD $D > 0$ for S with $\inf_{t \geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s.

Table 1: Overview of existing FTAP-type results. Note that NA1 = NUPBR on $[0, T]$.

Table 1 gives an overview of the dual characterisation results discussed above. We recall the space $\mathcal{S}^m$ of $\mathbb{R}^m$-valued semimartingales and use $\mathcal{S}^m_+$, $\mathcal{S}^m_{++}$ as in Section 2. The abbreviations SMD, σMD and LMD denote super-, σ- and local martingale discounters, respectively. The table compares Karatzas/Kardaras [24], Takaoka/Schweizer [35], Kardaras [27], Kabanov et al. [18], Herdegen [15], and the present article. Note that on a right-open interval, the dual characterisation always involves a condition at ∞.

5.3.2 Numéraire- or discounting-invariance

As mentioned above, classic NFLVR and NUPBR are only defined for discounted prices of the form $S = (1, X)$. It is natural to ask in general what happens to an AOA concept if one changes the numéraire, i.e., uses a different process for discounting. This can be done in two different directions, after fixing a price process S:

(A) One can fix a class $\mathcal{D}$ of discounting processes and look for an AOA concept $\mathcal{A}$ which is invariant for the chosen class $\mathcal{D}$, in the sense that $\mathcal{A}$ holds simultaneously for all processes S/D with $D \in \mathcal{D}$.

(B) One can fix an AOA concept $\mathcal{A}'$ and look for a class $\mathcal{D}'$ of discounting processes which leaves the chosen $\mathcal{A}'$ invariant, in the same sense as above.

Both (A) and (B) are concerned with numéraire- or discounting-invariance; but their objectives and results are fundamentally different. In a nutshell, most of the classic results and in particular the work by Delbaen/Schachermayer [10, 11] fall into category (B), whereas both Herdegen's and our approach here address (A). Put differently, we want to be liberal about the class $\mathcal{D}$ of allowed discounters and thus need to look for a suitable new AOA concept $\mathcal{A}$. In contrast, [10, 11] want to keep an established AOA concept $\mathcal{A}'$ and therefore look for restrictions on the class $\mathcal{D}'$ of discounters to achieve this.

Historically, probably the first to study questions of numéraire-invariance for AOA were Delbaen/Schachermayer [10] and Sin [34] (interestingly, these works do not cite each other). [34] studies problem (A) for the special case $\mathcal{D} = \{S^1, \dots, S^N\}$ and replaces for strategies $\vartheta \in \Theta^{\mathrm{sf}}$ the admissibility concept of [10] by the requirement of *feasibility* that the value process should satisfy $V(\vartheta) \geq -V(c\mathbb{1}) = -\sum_{i=1}^N c^i S^i$, where $c^i \geq 0$ is the number of shares of asset i outstanding at time 0 and the product $c\mathbb{1}$ is componentwise. For $S = S^1(1, X^{(1)})$ with $S^1 > 0$ and $X^{(1)} \geq 0$ a semimartingale, the main result is then that $X^{(1)}$ satisfies NFLVR with *feasible* strategies if and only if $X^{(1)}$ admits an equivalent (true) martingale measure, and that this is also equivalent to NFLVR with feasible strategies for any $X^{(k)}$ with $S = S^k(1, X^{(k)})$ whenever $S^k > 0$ and $X^{(k)} \geq 0$ is a semimartingale. Thus one has indeed an answer to (A), and the new concept $\mathcal{A}$ is NFLVR with feasible strategies. Essentially the same approach was redeveloped later in Yan [37] (who was apparently unaware of [34]).

In contrast, Delbaen/Schachermayer [10, 11] study problem (B) and answer the questions appearing there fairly exhaustively. Starting with $S = (1, X)$ and an S-tradable numéraire/discounter $D = V(\vartheta)$, they consider the two markets $S = (1, X)$ and $\tilde{S} = (\frac{1}{D}, \frac{X}{D})$ and show in [10] that if S satisfies classic NFLVR, then $\tilde{S}$ admits an equivalent σ-martingale measure if and only if $D_\infty - D_0$ is maximal in $\mathcal{G}_{\mathrm{adm}}(S)$. In the spirit of (B), this characterises those *S-tradable* discounters which preserve NFLVR. In [11], for such a D and under NFLVR for S, they derive an isometry between two spaces $\mathcal{G}(S)$ and $\mathcal{G}(\tilde{S})$ of (final values of) stochastic integrals. One key assumption for both results is $D_\infty > 0$; so in addition to being S-tradable, D must also be in $\mathcal{S}_{++}^{\mathrm{unif}}$.

After Sin [34], problem (A) was taken up almost 20 years later (without citing [34]) by Herdegen [15] who worked on $[0, T]$ with a general $\mathbb{R}^N$-valued semimartingale S. He used the class $\mathcal{D} = \mathcal{S}_{++}$ of discounters, which on $[0, T]$ coincides with $\mathcal{S}_{++}^{\mathrm{unif}}$ because all processes are defined up to and including T, and introduced the discounting-invariant AOA condition NINA or dynamic (value) viability. It generalises NUPBR and is dually characterised by the existence of a discounter/EσMM pair (D, Q), meaning that $D \in \mathcal{S}_{++}$ and Q is an equivalent σ-martingale measure for S/D. In addition, [15] also presents a discounting-invariant alternative to NFLVR. It is called dynamic (value) efficiency and requires that not one particular asset, but each of the N basic assets (or, equivalently, the market portfolio $\mathbb{1}$) should satisfy (value) maximality. One key insight from Delbaen/Schachermayer [10] also reappears in [15] — NFLVR describes a maximality property of the discounting asset, but does not say much about the market as a whole. (Proposition 5.7 extends that to our framework.)

In the above terminology, the contribution of the present paper can be succinctly described as follows. For an $\mathbb{R}^N_+$-valued semimartingale $S \geq 0$ on the right-open interval $[0, \infty)$, we consider the class $\mathcal{D} = \mathcal{S}_{++}$ of discounters and tackle problem (A). We introduce two new AOA concepts DSV and DSE which are discounting-invariant for $\mathcal{S}_{++}$ and provide dual characterisations.

One interesting related paper in discrete time is Tehranchi [36]. The main result in Theorem 2.10 there is reminiscent of our Theorems 2.12 and 2.14, but has no dual condition at ∞. Moreover, the formulation in [36] hinges crucially on the discrete-time setup.

6 Examples

This section illustrates our results by examples and counterexamples. Most are based on variants of one generic example, and so we start with a general analysis of that setup.

6.1 General results for a two-GBM setup

Example 6.1. For independent Brownian motions B^1, B^2 and $\varrho \in (-1, 1)$, define ϱ-correlated Brownian motions $W^2 := B^2$, $W^1 := \varrho B^2 + \sqrt{1-\varrho^2} B^1$; so (B^1, B^2) and (W^1, W^2) generate the same filtration. For constants $m_1, m_2, \sigma_1, \sigma_2$, define the processes S^1, S^2 by

$$\log S^i_t = \sigma_i W^i_t + \Big(m_i - \frac{1}{2}\sigma_i^2\Big)t, \qquad t \geq 0,\ i = 1, 2. \tag{6.1}$$

We take $m_1, m_2 \in \mathbb{R}$ and $\sigma_1 \geq 0$, but insist on $\sigma_2 > 0$ to avoid degenerate models. The filtration $\mathbb{F}$ is generated by $S = (S^1, S^2)$, made right-continuous and complete. This setup includes two basic cases. If $\sigma_1 = 0$ and we set $m_1 := r \in \mathbb{R}$, $m_2 := m \in \mathbb{R}$, $\sigma_2 := \sigma > 0$, we have the classic *Black–Scholes (BS) model* with a bank account S^1 and one stock S^2. The filtration is then generated by W^2 only. If $\sigma_1 > 0$, we have a symmetric market with two stocks S^1, S^2 (and no bank account), given by correlated *geometric Brownian motions (GBM)*. The filtration is then generated by (W^1, W^2) or equivalently (B^1, B^2).

If we discount all prices by the first asset, this gives the model $S/S^1 = (1, X)$ with

$$\begin{aligned}\log X_t &= \Big(m_2 - m_1 - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)\Big)t + \sigma_2 W^2_t - \sigma_1 W^1_t \\ &= (m_2 - m_1 + \sigma_1^2 - \varrho\sigma_1\sigma_2)t + \bar\sigma \bar W_t - \frac{1}{2}\bar\sigma^2 t,\end{aligned} \tag{6.2}$$

with $\bar\sigma := \sqrt{\sigma_1^2 + \sigma_2^2 - 2\varrho\sigma_1\sigma_2}$ and a new Brownian motion $\bar W := (\sigma_2 W^2 - \sigma_1 W^1)/\bar\sigma$.

For Example 6.1, we can characterise, in terms of the parameters $m_1, m_2, \sigma_1, \sigma_2, \varrho$, when DSV or DSE for $\mathbb{1}$ hold, by using σ-martingale discounters D for S. Then S/D is a σ-martingale > 0 and hence a local martingale > 0. In the filtration generated by (B^1, B^2), all positive local martingales starting at 1 have the form $\mathcal{E}(\xi^1 \bullet B^1 + \xi^2 \bullet B^2)$, and as all coefficients of S are constant, one expects that it is enough to consider only constant processes ξ^1, ξ^2. So we define

$$\mathcal{C} := \{D \in \mathcal{S}_{++} : S^i/D = \mathcal{E}(\alpha_i B^1 + \beta_i B^2) \text{ with constants } \alpha_i, \beta_i,\ i = 1, 2\}.$$

Throughout this section, we consider the setting of Example 6.1. Note that $S_0 = (1, 1)$ implies the normalisation $D_0 = 1$ for any $D \in \mathcal{C}$.

Proposition 6.2. *We always have $\mathcal{C} \neq \emptyset$, and each $D \in \mathcal{C}$ corresponds to a tuple $(\alpha_1, \beta_1, \alpha_2, \beta_2) \in \mathbb{R}^4$ with one free parameter. More precisely, we have the three relations*

$$\alpha_2 = \alpha_1 - \sigma_1\sqrt{1 - \varrho^2}, \tag{6.3}$$

$$\beta_2 = \beta_1 - (\varrho\sigma_1 - \sigma_2), \tag{6.4}$$

$$\alpha_1\sigma_1\sqrt{1 - \varrho^2} + \beta_1(\varrho\sigma_1 - \sigma_2) = m_2 - m_1 + \sigma_1^2 - \varrho\sigma_1\sigma_2. \tag{6.5}$$

In particular, α_2 and β_2 are always determined from α_1 and β_1, respectively. Moreover:

1) If $\sigma_1 = 0$, we must take $\alpha_1 = 0$ and $\beta_1 = -\frac{m_2 - m_1}{\sigma_2}$. This yields

$$D_t^{-1} = \mathcal{E}\Big(-\frac{m_2 - m_1}{\sigma_2} B^2\Big)_t e^{-m_1 t}, \qquad t \geq 0,$$

which is the well-known state price density for the Black–Scholes model.

2) If $\sigma_1 > 0$, we can choose β_1 freely, and α_1 is then determined via (6.5)*.*

Proof. Because $S^i/D = \mathcal{E}(\alpha_i B^1 + \beta_i B^2)$ for $i = 1, 2$ and D is one-dimensional, we have $S^1/\mathcal{E}(\alpha_1 B^1 + \beta_1 B^2) = D = S^2/\mathcal{E}(\alpha_2 B^1 + \beta_2 B^2)$. Plug in (6.1) for S^1, S^2, express W^1, W^2 via B^1, B^2, write out the results and equate the two sides. The coefficients of B^1, B^2, t in the exponents must then coincide, and the resulting three equations yield the claims after straightforward algebra. Note that in case 1), D is not adapted to $\mathbb{F}$ unless $\alpha_1 = 0$. $\square$

Each $D \in \mathcal{C}$ is a (local and) σ-martingale discounter for S in the filtration $\mathbb{F}$. The next result exhibits a particularly useful choice among these.

Proposition 6.3. *There exists a unique S-tradable $\bar{D} \in \mathcal{C}$. In terms of the corresponding parameter tuple from Proposition 6.2, it is given as follows:*

1) If $\sigma_1 = 0$, then $\bar{\alpha}_1 = 0 = \bar{\alpha}_2$ and

$$\bar{\beta}_1 = -\frac{m_2 - m_1}{\sigma_2}, \qquad \bar{\beta}_2 = -\frac{m_2 - m_1 - \sigma_2^2}{\sigma_2}. \tag{6.6}$$

2) *If* $\sigma_1 > 0$, *then*

$$\bar{\alpha}_1\left(\sigma_1\sqrt{1-\varrho^2} + \frac{(\varrho\sigma_1 - \sigma_2)^2}{\sqrt{1-\varrho^2}}\right) = m_2 - m_1 + \sigma_1^2 - \varrho\sigma_1\sigma_2, \tag{6.7}$$

$$\bar{\beta}_1 = \bar{\alpha}_1\frac{\varrho\sigma_1 - \sigma_2}{\sqrt{1-\varrho^2}}, \tag{6.8}$$

$$\bar{\alpha}_2\left(\sigma_1\sqrt{1-\varrho^2} + \frac{(\varrho\sigma_1 - \sigma_2)^2}{\sqrt{1-\varrho^2}}\right) = m_2 - m_1 - \sigma_2^2 + \varrho\sigma_1\sigma_2, \tag{6.9}$$

$$\bar{\beta}_2 = \bar{\alpha}_2\frac{\varrho\sigma_1 - \sigma_2}{\sqrt{1-\varrho^2}}. \tag{6.10}$$

Proof. For $\bar{D}$ to be S-tradable, we must have $\bar{D} = V(\bar{\vartheta})$ for some $\bar{\vartheta} \in \Theta^{\mathrm{sf}}$. Setting $\bar{S} := S/\bar{D}$, this is equivalent to $1 \equiv V(\bar{\vartheta})/\bar{D} = V(\bar{\vartheta}, S)/\bar{D} = V(\bar{\vartheta}, \bar{S})$ or

$$\bar{\vartheta}_t^1\bar{S}_t^1 + \bar{\vartheta}_t^2\bar{S}_t^2 = 1, \qquad t \geq 0. \tag{6.11}$$

Moreover, we also have from the self-financing condition that

$$0 = \mathrm{d}V_t(\bar{\vartheta}, \bar{S}) = \bar{\vartheta}_t^1\,\mathrm{d}\bar{S}_t^1 + \bar{\vartheta}_t^2\,\mathrm{d}\bar{S}_t^2. \tag{6.12}$$

But $\bar{S}^i = \mathcal{E}(\bar{\alpha}_i B^1 + \bar{\beta}_i B^2)$ yields $\mathrm{d}\bar{S}_t^i = \bar{S}_t^i(\bar{\alpha}_i\,\mathrm{d}B_t^1 + \bar{\beta}_i\,\mathrm{d}B_t^2)$. Plugging this into (6.12) and using that B^1, B^2 are independent implies by comparing coefficients that

$$0 = \bar{\vartheta}_t^1\bar{S}_t^1\bar{\alpha}_1 + \bar{\vartheta}_t^2\bar{S}_t^2\bar{\alpha}_2, \tag{6.13}$$

$$0 = \bar{\vartheta}_t^1\bar{S}_t^1\bar{\beta}_1 + \bar{\vartheta}_t^2\bar{S}_t^2\bar{\beta}_2. \tag{6.14}$$

Now we use (6.11) to get $\bar{\vartheta}^1\bar{S}^1 = 1 - \bar{\vartheta}^2\bar{S}^2$, plug this into (6.13) and (6.14), use (6.3) and (6.4) to eliminate $\bar{\alpha}_2$ and $\bar{\beta}_2$ and obtain (after simple calculations)

$$\bar{\alpha}_1 = \bar{\vartheta}^2\bar{S}^2\sigma_1\sqrt{1-\varrho^2}, \tag{6.15}$$

$$\bar{\beta}_1 = \bar{\vartheta}^2\bar{S}^2(\varrho\sigma_1 - \sigma_2). \tag{6.16}$$

Because only one of $\bar{\alpha}_1, \bar{\beta}_1$ can be chosen freely by Proposition 6.2, there is at most one choice of $\bar{D} \in \mathcal{C}$ which is S-tradable. For existence of $\bar{D}$, we consider two cases.

1) If $\sigma_1 = 0$, (6.15) forces $\bar{\alpha}_1 = 0$, hence $\bar{\alpha}_2 = 0$ by (6.3), and Proposition 6.2 and (6.4) yield (6.6). Moreover, (6.16) yields for $\bar{\vartheta}$ the explicit formulas

$$\bar{\vartheta}^2\bar{S}^2 \equiv -\frac{\bar{\beta}_1}{\sigma_2} = \frac{m_2 - m_1}{\sigma_2^2}, \qquad \bar{\vartheta}^1\bar{S}^1 = 1 - \bar{\vartheta}^2\bar{S}^2 \equiv -\frac{m_2 - m_1 - \sigma_2^2}{\sigma_2^2}. \tag{6.17}$$

2) If $\sigma_1 > 0$, solve (6.15) for $\bar{\vartheta}^2\bar{S}^2$ and plug into (6.16) to get (6.8). Insert this into (6.5) to obtain (6.7). Finally, combine (6.7), (6.3) for (6.9), and (6.4), (6.8), (6.3) for (6.10). □

Remark 6.4. 1) In the BS model with parameters m, r, σ, the *proportion of wealth* in the stock S^2 for the strategy $\bar{\vartheta}$ is given by $\bar{\pi}^2 = \bar{\vartheta}^2 S^2/V(\bar{\vartheta}) = \bar{\vartheta}^2\bar{S}^2 = \frac{m-r}{\sigma^2}$. This is exactly the strategy which solves the problem of maximising expected logarithmic utility from final wealth. We therefore call $\bar{\vartheta}$ from (6.17) the *Merton strategy*.

2) The strategy $\bar{\vartheta}^2$ in (6.17) matches intuition quite well. In addition to its buy-low-sell-high property, it goes long S^1 and short S^2 if m_1 is much higher than m_2, short S^1 and long S^2 if m_1 is much lower than m_2, and holds proportional long positions in both assets if the relation between m_1 and m_2 is not extreme.

Our main result about Example 6.1 is now

Theorem 6.5. 1) *If $\sigma_1 = 0$, then S satisfies DSV for $\mathbb{1}$ if and only if*

$$m_2 - m_1 \in \{0, \sigma_2^2\}. \tag{6.18}$$

In particular, the BS model with parameters m, r, σ satisfies DSV for $\mathbb{1}$ if and only if

$$\frac{m-r}{\sigma^2} \in \{0, 1\}.$$

2) *If $\sigma_1 > 0$, the general GBM model satisfies DSV for $\mathbb{1}$ if and only if*

$$m_i - \sigma_i^2 + \varrho\sigma_1\sigma_2 = m_{3-i} \qquad \text{for } i = 1 \text{ or } i = 2. \tag{6.19}$$

3) *S satisfies DSV for $\mathbb{1}$ if and only if one of the two processes $S/S^1 = (1, X)$ or $S/S^2 = (1/X, 1)$ is a martingale.*

4) *S never satisfies DSE for $\mathbb{1}$.*

Proof. Because DSV and DSE are discounting-invariant, we can argue for $\bar{S} = S/\bar{D}$ from Proposition 6.3 instead of S. Write $\bar{S}^i = \mathcal{E}(\bar{\alpha}_i B^1 + \bar{\beta}_i B^2) = \mathcal{E}(\bar{\alpha}_i B^1)\mathcal{E}(\bar{\beta}_i B^2)$.

1) If $\sigma_1 = 0$, then $\bar{\alpha}_1 = 0 = \bar{\alpha}_2$ and $\bar{S}^1 = \mathcal{E}(\bar{\beta}_1 B^1)$ and $\bar{S}^2 = \mathcal{E}((\bar{\beta}_1 + \sigma_2)B^2)$ by (6.4). If either $\bar{\beta}_1 = 0$ or $\bar{\beta}_1 + \sigma_2 = 0$, then $\mathbb{1} \cdot \bar{S} \geq 1$ so that $\bar{S}$ is a (non-UI) martingale with $\inf_{t\geq 0}(\mathbb{1} \cdot \bar{S}_t) > 0$ P-a.s.; so $\bar{S}$ satisfies DSV for $\mathbb{1}$ by Theorem 2.15 with $\eta \equiv \mathbb{1}$. If $\bar{\beta}_1 \neq 0$ and $\bar{\beta}_1 + \sigma_2 \neq 0$, then $\mathbb{1} \cdot \bar{S}_t \to 0$ P-a.s. as $t \to \infty$. Because $\bar{\vartheta} \cdot \bar{S} = V(\bar{\vartheta}, \bar{S}) \equiv 1$, this implies that $(\bar{\vartheta} \cdot \bar{S})/(\mathbb{1} \cdot \bar{S})$ cannot be bounded in $t \geq 0$, P-a.s. So 0 is not svm for $\bar{S}^{\mathbb{1}}$ by Lemma 3.12 for $S = \bar{S}$ and $\xi \equiv \mathbb{1}$, and therefore $\bar{S}$ does not satisfy DSV for $\mathbb{1}$ by Theorem 3.10, 1). In summary, S satisfies DSV for $\mathbb{1}$ if and only if $\bar{\beta}_1 \in \{0, -\sigma_2\}$, which is equivalent to (6.18) in view of (6.6).

2) If $\sigma_1 > 0$, (6.3) shows that $\bar{\alpha}_1$ and $\bar{\alpha}_2$ cannot both be 0, and (6.8), (6.10) imply $\bar{\beta}_i = 0$ if $\bar{\alpha}_i = 0$. So if $\bar{\alpha}_i = 0$, we get $\bar{S}^i \equiv 1$ and hence again $\mathbb{1} \cdot \bar{S} \geq 1$, so that S satisfies DSV for $\mathbb{1}$ by the same argument as in 1). If $\bar{\alpha}_1 \neq 0$ and $\bar{\alpha}_2 \neq 0$, then $\mathbb{1} \cdot \bar{S}_t \to 0$ P-a.s. as $t \to \infty$; so $\bar{S}$ does not satisfy DSV for $\mathbb{1}$, again as in 1). Thus S satisfies DSV for $\mathbb{1}$ if and only if $\bar{\alpha}_i = 0$ for $i = 1$ or $i = 2$, and this translates into (6.19) in view of (6.7), (6.9).

3) The characterisation of DSV for $\mathbb{1}$ in terms of martingale properties follows directly by combining the explicit expression for X in (6.2) with 1) and 2), respectively.

4) Because DSE implies DSV, we can by 3) only have DSE for $\mathbb{1}$ if either $X = S^2/S^1$ or $1/X$ is a martingale. This martingale is by (6.2) always of the form $\exp(\gamma\bar{W}_t - \frac{1}{2}\gamma^2 t)$

for some $\gamma \neq 0$ and some Brownian motion $\bar{W}$, and hence converges to 0 P-a.s. as $t \to \infty$. So if $(1, X) = S/S^1 = S^{e^1}$, say, is a martingale, we have $V_0(e^1, S^{e^1}) = 1 = V_0(e^2, S^{e^1})$, but $\lim_{t\to\infty} V_t(e^1 - e^2, S^{e^1}) = \lim_{t\to\infty}(1 - X_t) = 1 \in L^0_+ \setminus \{0\}$ so that e^2 is not svm for S^{e^1}. But $S^{\mathbb{1}}/S^{e^1} = (S^1 + S^2)/S^1 = 1 + X \geq 1$ is in $\mathcal{S}^{\mathrm{unif}}_{++}$ because $X \geq 0$ is convergent, hence bounded in $t \geq 0$, P-a.s. By Lemma 3.1, 1), e^2 is thus also not svm for $S^{\mathbb{1}}$ and hence not ssm for $\mathbb{1}$ by Theorem 3.10, 1) for $\xi \equiv \mathbb{1}$. As e^2 is a $\mathbb{1}$-buy-and-hold strategy, this implies that S or S^{e^1} does not satisfy DSE for $\mathbb{1}$. If $1/X$ is a martingale, we just interchange e^1 and e^2 in the argument. $\square$

6.2 Explicit examples I

This section gives explicit counterexamples for several wrong statements or implications. All these are based on the general GBM setup from Section 6.1, and for concreteness and simplicity, we work with the BS model. So let $S^1_t = e^{rt}$ and $S^2_t = \exp(\sigma W_t + (m - \frac{1}{2}\sigma^2)t)$ with $m, r \in \mathbb{R}$ and $\sigma > 0$. We also need $X = S^2/S^1$ because $S/S^1 = (1, X)$.

Example 6.6. *DSV for $\mathbb{1}$ does not imply DSE for $\mathbb{1}$.* If we take $m - r \in \{0, \sigma^2\}$, S satisfies DSV for $\mathbb{1}$ by Theorem 6.5, 1). But S never satisfies DSE for $\mathbb{1}$, by Theorem 6.5, 4).

Example 6.7. *NFLVR for $(1, X)$ does not imply DSE for $\mathbb{1}$.* Take $m = r$ so that X is a martingale; then clearly $S/S^1 = (1, X)$ satisfies NFLVR$_\infty$. But again by Theorem 6.5, 4), S never satisfies DSE for $\mathbb{1}$, and neither does S/S^1 because DSE is discounting-invariant.

Example 6.8. *DSV for $\mathbb{1}$ does not imply NUPBR for $(1, X)$.* Now take $m - r = \sigma^2$ so that $X' = 1/X = S^1/S^2$ is a martingale. Then $(1, X) = S/S^1$ satisfies DSV for $\mathbb{1}$ by Theorem 6.5, 1) because S does. However, $X'_t = \exp(-\sigma W_t - \frac{1}{2}\sigma^2 t)$ converges to 0 P-a.s. as $t \to \infty$; so $\lim_{t\to\infty} X_t = +\infty$ P-a.s. and $(1, X)$ does not satisfy NUPBR$_\infty$.

6.3 Explicit examples II

Some of our examples need models S which satisfy DSE, or UI martingales, and both these requirements cannot be satisfied in the setup of Section 6.1. Theorem 6.5 shows that the GBM model never satisfies DSE for $\mathbb{1}$, and the appearing martingales are always stochastic exponentials $\mathcal{E}(\gamma B)$ of some constant multiple of some Brownian motion B. Except for $\gamma = 0$ where $\mathcal{E}(\gamma B) \equiv 1$, such a martingale is never UI because it converges to 0 P-a.s. So we need to construct our examples in a different way.

For ease of exposition, we work in this section in (infinite) discrete time. Via piecewise constant interpolations of processes (LCRL for predictable, RCLL for optional) and piecewise constant filtrations, our models can be embedded in a continuous-time framework. We use (only in this subsection) the notation $\Delta Y_n := Y_n - Y_{n-1}$ for the increment at time n of the discrete-time process $Y = (Y_n)_{n\in\mathbb{N}_0}$. Our examples have two building blocks.

A first basic ingredient is a martingale Y whose increments (or successor values) in each step only take two (different) values. The martingale condition then uniquely determines all one-step transition probabilities as a function of the Y-values, and so we can talk about "the" corresponding martingale. By choosing the increments or values in a suitable way, we can moreover ensure that Y is nonnegative and bounded, hence UI and P-a.s. convergent to some Y_∞ which closes Y on the right as a martingale (i.e., $Y = (Y_n)_{n\in\mathbb{N}_0\cup\{\infty\}}$ is a martingale). Finally, one can also ensure that Y_∞ only takes two values one of which is 0, and thus we obtain a UI martingale which converges to 0 with positive probability.

The second idea is more subtle. We want to work with a two-asset model and trade in such a way that our strategy involves the asymptotic behaviour of both assets in a specific nontrivial way. To this end, we construct $S = (S^1, S^2)$ such that in each step, exactly one of the assets has a price move, and these moves always alternate. This allows to predict which asset coordinate will move in the next step, which can be exploited to construct (switching) strategies with a desired behaviour; and as both coordinates move alternatingly, the resulting wealth process is influenced by each coordinate in turn.

Example 6.9. *DSV for η is not equivalent to the existence of a σ-martingale discounter D for S; the condition $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s. in Theorem 2.12 is indispensable.* To show this, we take $\eta \equiv \mathbb{1}$ and construct a bounded martingale $S \geq 0$ satisfying $P[\lim_{t\to\infty} S_t = 0] > 0$. Then $D \equiv 1$ is a UI martingale discounter for S and we have $P[\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) = 0] \geq P[\lim_{t\to\infty}(\mathbb{1} \cdot S_t) = 0] > 0$. We then show that S does not satisfy DSV for $\mathbb{1}$.

To start the construction, let $Y = (Y_n)_{n\in\mathbb{N}_0}$ be the (unique) martingale with $Y_0 = 1$ which at any time $n \in \mathbb{N}$ only takes the two values $u_n = 2 - 2^{-n}$ or $d_n = 2^{-n}$. Then Y is P-a.s. strictly positive (but not bounded away from 0 uniformly in n) and bounded by 2. So (Y_n) converges to Y_∞ P-a.s., and clearly $P[Y_\infty = 2] = \frac{1}{2} = P[Y_\infty = 0]$.

Now let Y^1, Y^2 be independent copies of Y and define $S = (S^1, S^2)$ by $S^1_0 = 1$ and

$$S^1_{2n-1} = S^1_{2n} = Y^1_n \quad \text{for } n \in \mathbb{N}, \qquad S^2_{2n} = S^2_{2n+1} = Y^2_n \quad \text{for } n \in \mathbb{N}_0.$$

This gives for $n \in \mathbb{N}$ that $\Delta S^1_{2n-1} = \Delta Y^1_n$, $\Delta S^1_{2n} = 0$ and $\Delta S^2_{2n-1} = 0$, $\Delta S^2_{2n} = \Delta Y^2_n$ and in particular yields that the coordinates of S jump alternatingly because

$$\Delta S^2_n I_{\{\Delta S^1_{n-1}=0\}} = 0 = \Delta S^1_n I_{\{\Delta S^2_{n-1}=0\}}. \tag{6.20}$$

Let $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}_0}$ be the filtration generated by S. As S is like Y a bounded martingale, it converges to S_∞ P-a.s., and $B := \{\lim_{n\to\infty}(\mathbb{1} \cdot S_n) = 0\} = \{S_\infty = 0\}$ has $P[B] = \frac{1}{4} > 0$.

Because $(S_n)_{n\in\mathbb{N}_0}$ is strictly positive, $\eta \equiv \mathbb{1}$ is a reference strategy. If S satisfies DSV for $\mathbb{1}$, then 0 is ssm for $\mathbb{1}$, hence svm for $S^{\mathbb{1}}$ by Theorem 3.10, 1) for $\xi \equiv \mathbb{1}$, and so Lemma 3.11 yields $\sup_{n\in\mathbb{N}_0}(\vartheta_n \cdot S_n)/(\mathbb{1} \cdot S_n) < \infty$ P-a.s. for all $\vartheta \in \Theta^{\mathrm{sf}}_+$. Write $(\vartheta \cdot S)/(\mathbb{1} \cdot S) = V(\vartheta)/(\mathbb{1} \cdot S)$. We exhibit below a strategy $\bar{\vartheta} \in \Theta^{\mathrm{sf}}_+$ with $V(\bar{\vartheta}) \equiv \varepsilon > 0$. This yields $\sup_{n\in\mathbb{N}_0}(\bar{\vartheta}_n \cdot S_n)/(\mathbb{1} \cdot S_n) = +\infty$ on B, and so S cannot satisfy DSV for $\mathbb{1}$.

To construct $\bar{\vartheta}$, we fix $\varepsilon > 0$ and consider the strategy which invests the amount ε at time 0 in asset 2 and subsequently reinvests at any time all its wealth into that asset which will not jump in the next period. More formally, we set $\bar{\vartheta}_0 := \bar{\vartheta}_1 := (0, \varepsilon)$ and

$$\bar{\vartheta}_{n+1} := I_{\{\Delta S^1_n = 0\}} \Big(0, \frac{\varepsilon}{S^2_n}\Big) + I_{\{\Delta S^2_n = 0\}} \Big(\frac{\varepsilon}{S^1_n}, 0\Big). \tag{6.21}$$

This is well defined because S^1, S^2 are both strictly positive, and predictable because S is adapted. Moreover, $S^2_0 = S^2_1 = 1$ yields $V_0(\bar{\vartheta}) = V_1(\bar{\vartheta}) = \varepsilon$, and

$$V_{n+1}(\bar{\vartheta}) = I_{\{\Delta S^1_n = 0\}}\, \varepsilon \frac{S^2_{n+1}}{S^2_n} + I_{\{\Delta S^2_n = 0\}}\, \varepsilon \frac{S^1_{n+1}}{S^1_n} = \varepsilon$$

as S^1, S^2 always jump alternatingly. So $V(\bar{\vartheta}) \equiv \varepsilon$, and $\bar{\vartheta}$ is also self-financing because

$$\Delta V_{n+1}(\bar{\vartheta}) - \bar{\vartheta}_{n+1} \cdot \Delta S_{n+1} = 0 - \bar{\vartheta}^1_{n+1} \Delta S^1_{n+1} - \bar{\vartheta}^2_{n+1} \Delta S^2_{n+1} \equiv 0$$

due to (6.21) and (6.20). So $\bar{\vartheta}$ has all the claimed properties, and this ends the example.

Example 6.10. *DSE for η need not imply NFLVR$_\infty$, not even for a classic model of the form $S = (1, X)$.* Similarly as in Example 6.9, let $Y = (Y_n)_{n \in \mathbb{N}_0}$ be the (unique) martingale valued in $(0,1)$ with $Y_0 = \frac{1}{2}$ and $Y_n \in \{\frac{1}{2} 2^{-n}, 1 - \frac{1}{2} 2^{-n}\}$. This converges P-a.s. to Y_∞ which takes the values 0 and 1 each with probability $\frac{1}{2}$. Set $Y' := 1 - Y$ and define

$$S := (1, X) := \Big(1, \frac{Y'}{Y}\Big) = \Big(1, \frac{1 - Y}{Y}\Big).$$

Then $\mathbb{1} \cdot S = \frac{1}{Y}$ and so $S^{\mathbb{1}} = (Y, 1 - Y) = S/(\mathbb{1} \cdot S)$ is a bounded P-martingale with $\mathbb{1} \cdot S^{\mathbb{1}} \equiv 1 \in \mathcal{S}^{\mathrm{unif}}_{++}$. So S satisfies (e3) in Theorem 4.1 with $D = \mathbb{1} \cdot S$ and $\eta \equiv \mathbb{1}$, and this implies that S satisfies DSE for $\mathbb{1}$. However, we clearly have $X \geq 0$ and

$$\lim_{n \to \infty} X_n = \lim_{n \to \infty} \frac{1 - Y_n}{Y_n} = +\infty \qquad \text{on } \{\lim_{n \to \infty} Y_n = 0\} = \{Y_\infty = 0\} =: B.$$

As $P[B] = \frac{1}{2} > 0$, $S = (1, X)$ does not satisfy NUPBR$_\infty$ and thus also not NFLVR$_\infty$.

A Appendix

This section contains some technical proofs and auxiliary results.

For any function $z : [0, \infty) \to \mathbb{R}^N$, set $\underline{z}(\infty) := \liminf_{t \to \infty} z(t)$, with the lim inf taken coordinatewise. If the limit exists, again coordinatewise, we write $z(\infty) := \lim_{t \to \infty} z(t)$. In $\mathbb{R}_+$, the product of ∞ and 0 is 0.

Lemma A.1. *Suppose the functions $x, y : [0, \infty) \to \mathbb{R}^N$ satisfy*

(a) *$y \geq 0$ is bounded (uniformly in $t \geq 0$) by some $C < \infty$.*

(b) *$\underline{x}^i(\infty) \geq 0$ for $i = 1, \ldots, N$.*

Then

$$(\text{A.1}) \qquad \underline{(x\cdot y)}(\infty) \geq \underline{x}(\infty)\cdot \underline{y}(\infty).$$

Proof. Fix $\varepsilon > 0$. Decompose $\{1,\dots,N\}$ into indices ℓ with $\underline{x}^\ell(\infty) = \infty$ and indices m with $\underline{x}^m(\infty) < \infty$. For any ℓ and $t \geq T = T(\ell)$, we have $x^\ell(t) \geq 0$ and $y^\ell(t) \geq \frac{1}{2}\underline{y}^\ell(\infty)$, and for any m, we get $x^m(t) \geq \underline{x}^m(\infty) - \varepsilon$ for $t \geq T = T(m,\varepsilon)$ and $0 \leq y^m(t) \leq C$ for all t. This implies $x^m(t)y^m(t) \geq (\underline{x}^m(\infty)-\varepsilon)y^m(t) \geq \underline{x}^m(\infty)y^m(t) - \varepsilon C$ and therefore

$$(x\cdot y)(t) = \sum_\ell x^\ell(t)y^\ell(t) + \sum_m x^m(t)y^m(t) \geq \frac{1}{2}\sum_\ell x^\ell(t)\underline{y}^\ell(\infty) + \sum_m \Big(\underline{x}^m(\infty)y^m(t) - \varepsilon C\Big).$$

Let $t \to \infty$ and use on the right-hand side the superadditivity of $\liminf$, $y \geq 0$ and the fact that $\underline{x}^m(\infty) \in [0,\infty)$ for all m, to obtain

$$\underline{(x\cdot y)}(\infty) \geq \frac{1}{2}\sum_\ell \underline{x}^\ell(\infty)\underline{y}^\ell(\infty) + \sum_m \underline{x}^m(\infty)\underline{y}^m(\infty) - N\varepsilon C.$$

If there is an ℓ with $\underline{y}^\ell(\infty) > 0$, the right-hand side is $+\infty$ and (A.1) holds trivially. So we can assume for the rest of the proof that $\underline{y}^\ell(\infty) = 0$ for all ℓ; then $\underline{x}^\ell(\infty)\underline{y}^\ell(\infty) = 0$ for all ℓ by our convention, and we end up with

$$\underline{(x\cdot y)}(\infty) \geq \sum_m \underline{x}^m(\infty)\underline{y}^m(\infty) - N\varepsilon C = \sum_{i=1}^N \underline{x}^i(\infty)\underline{y}^i(\infty) - N\varepsilon C.$$

Letting $\varepsilon \searrow 0$ then again gives (A.1) and completes the proof. □

Proof of Lemma 3.1. If ϑ is not svm for S/D, there are $f \in L^0_+ \setminus \{0\}$ and for any $\varepsilon > 0$ some $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+$ with $V_0(\hat{\vartheta}^\varepsilon, S/D) \leq V_0(\vartheta, S/D) + \varepsilon$, hence $V_0(\hat{\vartheta}^\varepsilon) \leq V_0(\vartheta) + \varepsilon D_0$, and

$$(\text{A.2}) \qquad \liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon - \vartheta, S/D) \geq f \geq 0 \qquad P\text{-a.s.}$$

As $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ has $\inf_{t\geq 0} D_t > 0$ P-a.s., $f' := f \liminf_{t\to\infty} D_t$ is in $L^0_+ \setminus \{0\}$. Because $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ also has $\sup_{t\geq 0} D_t < \infty$ P-a.s., (A.2) implies by Lemma A.1 that P-a.s.,

$$\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon - \vartheta) = \liminf_{t\to\infty}\Big(V_t(\hat{\vartheta}^\varepsilon - \vartheta, S/D)D_t\Big) \geq \liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon - \vartheta, S/D)\liminf_{t\to\infty} D_t \geq f'.$$

This shows that ϑ is not svm for S either.

For the second part, if $\alpha\vartheta$ is not svm for S, we can find $f \in L^0_+\setminus\{0\}$ and for every $\varepsilon > 0$ some $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+$ with $V_0(\hat{\vartheta}^\varepsilon) \leq V_0(\alpha\vartheta) + \varepsilon$ and $\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon - \alpha\vartheta) \geq f$ P-a.s. There are two cases. If $\alpha > 0$, then $\tilde{\vartheta} := \hat{\vartheta}^\varepsilon/\alpha \in \Theta^{\mathrm{sf}}_+$ satisfies $V_0(\tilde{\vartheta}) = V_0(\hat{\vartheta}^\varepsilon)/\alpha \leq V_0(\vartheta) + \varepsilon/\alpha$ and $\liminf_{t\to\infty} V_t(\tilde{\vartheta} - \vartheta) = \liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon - \alpha\vartheta)/\alpha \geq f/\alpha$ P-a.s. So ϑ is not svm for S as f/α is in $L^0_+\setminus\{0\}$. If $\alpha = 0$, $\tilde{\vartheta} := \vartheta + \hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+$ has $V_0(\tilde{\vartheta}) \leq V_0(\vartheta) + V_0(\alpha\vartheta) + \varepsilon = V_0(\vartheta) + \varepsilon$ and $\liminf_{t\to\infty} V_t(\tilde{\vartheta} - \vartheta) = \liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon - \alpha\vartheta) \geq f$ P-a.s.; so again ϑ is not svm for S. □

Proof of Lemma 3.12. For brevity, we introduce the set $\Gamma := \{V_\tau(\vartheta^1) < V_\tau(\vartheta^2)\} \in \mathcal{F}_\tau$ and set $\varphi := \vartheta^1 \circledS_\tau^\xi \vartheta^2$. We use $V(\xi, S^\xi) = \xi \cdot S^\xi \equiv 1$, which also gives $\xi \bullet S^\xi \equiv 0$. Then using the definition of φ, the general fact that $XI_{[\![0,\tau]\!]} = X^\tau - X_\tau I_{]\!]\tau,\infty]\!]}$, the fact that ϑ^1, ϑ^2 are self-financing and again the definition of φ yields

$$
\begin{aligned}
&V(\varphi, S^\xi)\\
&= I_{[\![0,\tau]\!]} V(\vartheta^1, S^\xi) + I_{]\!]\tau,\infty]\!]}\Big(I_\Gamma V(\vartheta^1, S^\xi) + I_{\Gamma^c} V(\vartheta^2, S^\xi) + I_{\Gamma^c} V_\tau(\vartheta^1 - \vartheta^2, S^\xi)\Big)\\
&= \Big(V(\vartheta^1, S^\xi)\Big)^\tau + I_{]\!]\tau,\infty]\!]}\bigg(I_\Gamma\Big(V(\vartheta^1, S^\xi) - V_\tau(\vartheta^1, S^\xi)\Big) + I_{\Gamma^c}\Big(V(\vartheta^2, S^\xi) - V_\tau(\vartheta^2, S^\xi)\Big)\bigg)\\
&= V_0(\vartheta^1, S^\xi) + (\vartheta^1 I_{[\![0,\tau]\!]})\bullet S^\xi + \bigg(I_{]\!]\tau,\infty]\!]}\Big(I_\Gamma \vartheta^1 + I_{\Gamma^c}\vartheta^2 + I_{\Gamma^c} V_\tau(\vartheta^1 - \vartheta^2, S^\xi)\xi\Big)\bigg)\bullet S^\xi\\
&= V_0(\varphi, S^\xi) + \varphi \bullet S^\xi.
\end{aligned}
$$

This shows that φ is self-financing. If both ϑ^1, ϑ^2 are in Θ_+^{sf}, the second line above is nonnegative so that also φ is in Θ_+^{sf}. □

The next auxiliary result is extracted from the proof of [25, Proposition 1].

Lemma A.2. *A convex set $C \subseteq L_+^0$ is bounded in L^0 if and only if C contains no sequence $(V^n)_{n\in\mathbb{N}}$ satisfying $V^n \geq n\xi$ P-a.s. for all $n \in \mathbb{N}$ and for some $\xi \in L_+^0 \setminus \{0\}$.*

Proof. The "only if" part is clear. For the "if" part, suppose C is not bounded in L^0 and let $\Omega_u \in \mathcal{F}$ be as in [6, Lemma 2.3]. (In the terminology of [6], C is hereditarily unbounded in probability on Ω_u.) Note that $P[\Omega_u] > 0$ because $P[\Omega_u] = 0$ would imply that C is bounded in L^0. Then [6, Lemma 2.3, part 4)] implies with $\varepsilon := 2^{-n}$ that for each $n \in \mathbb{N}$, there is some $V^n \in C$ such that

$$P[\{V^n \leq n\} \cap \Omega_u] \leq P[\{V^n \leq 2^n\} \cap \Omega_u] \leq 2^{-n}.$$

Take $N \in \mathbb{N}$ with $\sum_{n=N}^\infty 2^{-n} \leq P[\Omega_u]/2$. For $n \geq N$, set $A_n := \{V^n > n\} \cap \Omega_u \in \mathcal{F}$ and define $A := \bigcap_{n\geq N} A_n \in \mathcal{F}$ so that $V^n \geq n I_{A_n} \geq n I_A$ due to $V^n \in C \subseteq L_+^0$. Then

$$P[A] \geq P[\Omega_u] - \sum_{n=N}^\infty P[A_n^c \cap \Omega_u] \geq P[\Omega_u]/2 > 0$$

shows that $\xi := I_A \in L_+^0 \setminus \{0\}$, and we have $V^n \geq n\xi$ P-a.s. for all $n \in \mathbb{N}$. But this contradicts the assumption for the "if" part, and so we are done. □

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