Making no-arbitrage discounting-invariant: a new FTAP beyond NFLVR and NUPBR

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Abstract

In general multi-asset models of financial markets, the classic no-arbitrage concepts NFLVR and NUPBR have the serious shortcoming that they depend crucially on the way prices are discounted. To avoid this economically unnatural behaviour, we introduce a new way of defining "absence of arbitrage". It rests on the new notion of a strategy being *strongly share maximal* and allows us to generalise both NFLVR (by dynamic share efficiency) and NUPBR (by dynamic share viability). These new absence-of-arbitrage concepts do not change when we look at discounted or undiscounted prices, and they can be used in open-ended models under minimal assumptions on asset prices. We establish corresponding versions of the FTAP, i.e., dual characterisations of our concepts in terms of martingale properties. A key new feature is that as one expects, "properly anticipated prices fluctuate randomly", but with an *endogenous* discounting process which must not be chosen a priori. We show that the classic Black–Scholes model on $[0, \infty)$ is arbitrage-free in our sense if and only if its parameters satisfy $m - r \in \{0, \sigma^2\}$ or, equivalently, either bond-discounted or stock-discounted prices are martingales.

Mathematics subject classification (2010): 91G99, 91B02, 60G48

JEL classification: C00, G10

Keywords: no arbitrage, maximal strategies, semimartingales, discounting, NFLVR, NUPBR, FTAP, σ-martingale discounter, strongly share maximal, dynamic share viability, Black–Scholes model

This version: June 20, 2019

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1 Introduction

The fundamental theorem of asset pricing (FTAP) is one of the cornerstones of mathematical finance and arbitrage theory. Omitting technical details, its usual formulation goes as follows. Consider d risky assets and model their prices by a stochastic process X on a probability space $(\Omega, \mathcal{F}, P)$. It is common to assume that prices are discounted and there is one (extra) riskless asset with constant price 1. Then this model $(1, X)$ is "arbitrage-free" if and only if there exists a probability measure Q equivalent to P under which X is a "martingale". If X is a P-semimartingale, one precise version of "arbitrage-free" is that X satisfies no free lunch with vanishing risk (NFLVR), and "martingale under Q" is then to be read as Q-σ-martingale. For a full exposition, we refer to Delbaen/Schachermayer [5, 8, 9]. A weaker formulation of "arbitrage-free" is that X satisfies no unbounded profit with bounded risk (NUPBR), and the dual "martingale" characterisation is then the existence of an equivalent supermartingale deflator; see Karatzas/Kardaras [17].

One major drawback of the classic results is that *the usual approach of working with discounted prices is not without loss of generality.* We illustrate this in a very simple setting of $N = 2$ assets with prices given by $S = (S^1, S^2)$.

Example 1.1. For independent Brownian motions B^1, B^2 and $\varrho \in (-1, 1)$, define ϱ-correlated Brownian motions $W^2 := B^2$, $W^1 := \varrho B^2 + \sqrt{1-\varrho^2} B^1$; so (B^1, B^2) and (W^1, W^2) generate the same filtration. For constants $m_1, m_2, \sigma_1, \sigma_2$, define the processes S^1, S^2 by

$$\log S^i_t = \sigma_i W^i_t + \Big(m_i - \frac{1}{2}\sigma_i^2\Big)t, \qquad t \geq 0,\ i = 1, 2. \tag{1.1}$$

We take $m_1, m_2 \in \mathbb{R}$ and $\sigma_1 \geq 0$, but insist on $\sigma_2 > 0$ to avoid degenerate models. The filtration $\mathbb{F}$ is generated by $S = (S^1, S^2)$, made right-continuous and complete. This setup includes two basic cases. If $\sigma_1 = 0$ and we set $m_1 := r \in \mathbb{R}$, $m_2 := m \in \mathbb{R}$, $\sigma_2 := \sigma > 0$, we have the classic *Black–Scholes (BS) model* with a bank account S^1 and one stock S^2. The filtration is then generated by W^2 only. If $\sigma_1 > 0$, we have a symmetric market with two stocks S^1, S^2 (and no bank account), given by correlated *geometric Brownian motions (GBM).* The filtration is then generated by (W^1, W^2) or equivalently (B^1, B^2).

If we discount all prices by the first asset, this gives the model $S/S^1 = (1, X)$ with

$$\begin{aligned}\log X_t &= \Big(m_2 - m_1 - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)\Big)t + \sigma_2 W^2_t - \sigma_1 W^1_t \\ &= (m_2 - m_1 + \sigma_1^2 - \varrho\sigma_1\sigma_2)t + \bar{\sigma}\bar{W}_t - \frac{1}{2}\bar{\sigma}^2 t,\end{aligned} \tag{1.2}$$

with $\bar{\sigma} := \sqrt{\sigma_1^2 + \sigma_2^2 - 2\varrho\sigma_1\sigma_2}$ and a new Brownian motion $\bar{W} := (\sigma_2 W^2 - \sigma_1 W^1)/\bar{\sigma}$. Suppose $m_2 - m_1 + \sigma_1^2 - \varrho\sigma_1\sigma_2 = 0$ so that X is a positive martingale converging to 0. (This happens for instance in the BS model when $m = r$.) Then the S^1-discounted model $(1, X)$ is obviously "arbitrage-free"; in fact, $(1, X)$ satisfies NFLVR and hence also NUPBR. But the S^2-discounted model $S/S^2 = (1/X, 1)$ has arbitrage because the strategy $\vartheta \equiv (\varepsilon, -\varepsilon)$ of holding ε units of asset 1 and $-\varepsilon$ units of asset 2 has initial cost 0 and time-t wealth $\varepsilon/X_t - \varepsilon$, which is bounded below by $-\varepsilon$ (so that ϑ is admissible) and tends to $+\infty$ as $t \to \infty$. So $(1/X, 1)$ does not satisfy NUPBR and hence also not NFLVR.

The insight that the classic absence-of-arbitrage (AOA) concepts depend on the choice of discounting is not new. An early but less well known analysis appears in the PhD thesis of Sin [25]. Delbaen/Schachermayer [6] discuss the no-arbitrage property under a change of numéraire, and Herdegen [10] recently developed a theory of no-arbitrage in a numéraire-independent modelling framework where he introduced a number of concepts that are invariant under a change of numéraire, i.e., under discounting. But the results in [10] are not general enough to handle Example 1.1 because [10] assumes that prices S are defined on a right-closed time interval $[0, T]$. A simple extension to a right-open interval or an infinite horizon as in Example 1.1 is not feasible, and a new approach is needed.

The key idea of Herdegen [10] is to define absence of arbitrage as the property that the zero strategy or a number of basic strategies are *maximal* in the sense that they cannot be "improved" by other strategies. In [10] (as well as in earlier work of Delbaen/Schachermayer [5, 6, 7]), improvements are measured in terms of *value* or wealth, in a qualitative (not quantitative) manner. This makes the approach invariant under discounting, but only partially — if a discount factor (or a numéraire) goes to 0 or explodes to $+\infty$, the invariance breaks down. This is not an issue on a right-closed time interval, but may well happen on a right-open time interval, and it is exactly why Example 1.1 cannot be handled by the approach of [10]. We circumvent this difficulty by measuring "improvements" not in terms of value, but in terms of *shares* compared to a *desirable reference strategy*. As we show, this leads to genuinely discounting-invariant concepts in almost fully general frictionless semimartingale models of financial markets. The main results are two FTAP versions — one for *dynamic share viability* which is the discounting-invariant counterpart of NUPBR, and one for *dynamic share efficiency* which extends NFLVR. In contrast to the classic FTAP formulations of Delbaen/Schachermayer [5, 8] or Karatzas/Kardaras [17], the discounting process in our results must not be chosen a priori, but is an endogenous part of the dual chacterisation of absence of arbitrage.

The paper is structured as follows. Section 2 introduces the setup and basic concepts and presents our main results. Section 3 is the mathematical core; it first connects our new concept of *share maximality* of a strategy to the value maximality studied in Herdegen [10], then shows how models on right-open intervals can be closed on the right under a weak AOA assumption, and finally combines this with [10] to prove dual characterisations of value maximality for a general time interval. Section 4 proves the main results from Section 2 by using Section 3. Section 5 discusses the robustness of our approach with respect to the choice of the reference strategy appearing in the concept of share maximality, connects our work to the classic theory and provides a comparison to the existing literature.

Finally, Section 6 contains examples and counterexamples, including a full discussion of Example 1.1, and the Appendix collects some technical proofs and auxiliary results.

2 The main results

We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, assume that $\mathcal{F}_0$ is trivial and set $\mathcal{F}_\infty := \bigvee_{t\geq 0} \mathcal{F}_t$. There are N basic assets whose prices are modelled by an $\mathbb{R}^N$-valued semimartingale S. If there is a bank account (we do not assume this in general), it must be one component of S. To have trading possible, we thus must have $N \geq 2$.

We use general stochastic integration (in the sense of [12, Chapter III.6] or [24]), call $L(S)$ the space of all $\mathbb{R}^N$-valued predictable S-integrable processes H and denote the (real-valued) stochastic integral of $H \in L(S)$ with respect to S by $H \bullet S := \int H \,\mathrm{d}S$. For any RCLL process Y, we set $Y_{0-} := Y_0$. The scalar product of $x, y \in \mathbb{R}^N$ is $x \cdot y := x^{\mathrm{tr}} y$.

Remark 2.1. We assume that S is a semimartingale so that we can use general integrands with respect to S. Similarly as in [5, 20], one could also start with an $\mathbb{R}^N$-valued adapted RCLL process S and impose an AOA type property on S only with respect to *elementary* (i.e. piecewise constant) integrands. For the AOA concept we introduce below, this then implies that $S/f(S)$ is a semimartingale for any linear $f : \mathbb{R}^N \to \mathbb{R}$ with $f(S) > 0$ and $f(S_-) > 0$. In particular, if $S = (1, X)$, then X must be a semimartingale. For more precise formulations and details, we refer to forthcoming work of the first author.

Many of our results involve *discounting*, i.e. dividing prices by positive processes. We define $\mathcal{S} :=$ {all real-valued semimartingales} and set $\mathcal{S}_+ := \{D \in \mathcal{S} : D \geq 0\}$ and $\mathcal{S}_{++} := \{D \in \mathcal{S} : D > 0, D_- > 0\}$. Elements $D \in \mathcal{S}_{++}$ are called *discounters*, and we note that $1/D \in \mathcal{S}_{++}$ if $D \in \mathcal{S}_{++}$. Sometimes, we also need *narrow discounters* $D \in \mathcal{S}^{\mathrm{unif}}_{++} := \{D \in \mathcal{S}_{++} : \inf_{t\geq 0} D_t > 0, \sup_{t\geq 0} D_t < \infty, P\text{-a.s.}\}$. For $D \in \mathcal{S}_{++}$, we call S/D the *D-discounted prices*. The difference between discounters and deflators is discussed below after Definition 2.9.

Self-financing strategies are integrands $\vartheta \in L(S)$ satisfying $V(\vartheta) := \vartheta \cdot S = \vartheta_0 \cdot S_0 + \vartheta \bullet S$. We write $\vartheta \in \Theta^{\mathrm{sf}}$ and call $V(\vartheta)$ the *value process* of ϑ; this is in the same currency units as S because ϑ is in numbers of shares. For D-discounted prices $\tilde{S} = S/D$, we analogously have $V(\vartheta, \tilde{S}) := \vartheta \cdot \tilde{S} = V(\vartheta)/D$, the value process of ϑ in the currency units of $\tilde{S}$. It is a result from [10, Lemma 2.9] that if $\vartheta \in \Theta^{\mathrm{sf}}$, then both $\vartheta \in L(\tilde{S})$ and $V(\vartheta, \tilde{S}) = \vartheta_0 \cdot \tilde{S}_0 + \vartheta \bullet \tilde{S}$ hold. Thus Θ^{sf} does not depend on currency units even if value processes do. We also need the spaces $\Theta^{\mathrm{sf}}_+ := \{\vartheta \in \Theta^{\mathrm{sf}} : V(\vartheta) \in \mathcal{S}_+\}$ and $\Theta^{\mathrm{sf}}_{++} := \{\vartheta \in \Theta^{\mathrm{sf}} : V(\vartheta) \in \mathcal{S}_{++}\}$; they do not depend on currency units either. Finally, a process Y is called *S-tradable* if it is the value process of some self-financing strategy, i.e., $Y = V(\vartheta)$ for some $\vartheta \in \Theta^{\mathrm{sf}}$.

Definition 2.2. A *reference strategy* is an $\eta \in \Theta^{\mathrm{sf}}_{++}$ with $\eta \geq 0$ (η is long-only) and such that the η-discounted price process $S^\eta := S/(\eta \cdot S)$ is bounded uniformly in $t \geq 0$, P-a.s.

In the sequel, we usually work under the assumption that there exists a reference strategy η, and some results impose the extra condition that η is bounded (uniformly in ω, t). Because $V(\eta) \in \mathcal{S}_{++}$ by definition, a reference strategy is a *desirable investment*, and it is expressed in numbers of shares. Note that if we pass from S to discounted prices $\tilde{S} = S/D$ with any $D \in \mathcal{S}_{++}$, we get $\tilde{S}^\eta := \tilde{S}/(\eta \cdot \tilde{S}) = S^\eta$; hence the notion of a reference strategy is discounting-invariant. See also the comment below after Definition 2.7.

Remark 2.3. The existence of a reference strategy η is a very weak condition on the price process S. Indeed, consider the *market portfolio*, i.e. the strategy $\mathbb{1} := (1, \dots, 1) \in \mathbb{R}^N$ of holding one share of each asset. If we have nonnegative prices $S \geq 0$, then $\mathbb{1} \in \Theta_+^{\mathrm{sf}}$ and all components of the $\mathbb{1}$-discounted price process $S^{\mathbb{1}} = S/\sum_{i=1}^N S^i$ have values between 0 and 1. If $S \geq 0$ and the sum $\sum_{i=1}^N S^i$ of all prices is strictly positive and has strictly positive left limits, we even have $\mathbb{1} \in \Theta_{++}^{\mathrm{sf}}$ so that the market portfolio is then a reference strategy. Moreover, $\mathbb{1}$ is of course bounded itself. However, it is useful to work with a general reference strategy η because this gives a clearer view on a number of aspects.

Definition 2.4. Fix a strategy $\eta \in \Theta^{\mathrm{sf}}$. A strategy $\vartheta \in \Theta^{\mathrm{sf}}$ is called an *η-buy-and-hold strategy* if it is of the form $\vartheta^i = c^i \eta^i$ for $i = 1, \dots, N$, where $c \in L^\infty(\mathcal{F}_0; \mathbb{R}^N)$.

Because $\mathcal{F}_0$ is trivial, ϑ is η-buy-and-hold if and only if it is a coordinatewise nonrandom multiple of η. If $\eta \equiv \mathbb{1}$ is the market portfolio, this reduces to the classic concept of buying and holding a fixed number of shares of each asset. More generally, if η is a reference strategy, it is desirable to have η_t^i shares of asset i at time t, and the above buy-and-hold concept is then a natural generalisation from the classic case of the market portfolio. Note that η itself is always an η-buy-and-hold strategy.

For maximal generality with our time horizon, we fix a stopping time ζ and consider the stochastic interval $[\![0, \zeta]\!] = \{(\omega, t) \in \Omega \times [0, \infty) : 0 \leq t \leq \zeta(\omega)\}$. This includes models indexed by $[0, T]$ with a nonrandom $T < \infty$ as well as by $[0, \infty)$ where $\zeta \equiv \infty$. We extend all stochastic processes to $[\![0, \infty]\!] = [\![0, \infty[\![= \Omega \times [0, \infty)$, almost always by keeping them constant on $[\![\zeta, \infty]\!]$, with one important exception. To concatenate two strategies $\vartheta^1, \vartheta^2 \in \Theta^{\mathrm{sf}}$ at some stopping time τ, we sometimes define, for a mapping F, a new strategy of the form $I_{[\![0,\tau]\!]}\vartheta^1 + I_{]\!]\tau,\infty]\!]}F(\vartheta^1, \vartheta^2)$. On the set $\{\tau = \zeta < \infty\}$, this is then constant for $t > \zeta(\omega)$, but maybe not for $t \geq \zeta(\omega)$.

From now on, we assume that all processes are defined on $[\![0, \infty]\!]$ (but not necessarily on $\Omega \times [0, \infty]$). If a process Y is constant on $[\![\zeta, \infty]\!]$, we then have

$$
\begin{aligned}
\inf_{t \geq 0} Y_t(\omega) &= I_{\{\zeta(\omega) = \infty\}} \inf_{0 \leq t < \infty} Y_t(\omega) + I_{\{\zeta(\omega) < \infty\}} \inf_{0 \leq t \leq \zeta(\omega)} Y_t(\omega), \\
\liminf_{t \to \infty} Y_t(\omega) &= I_{\{\zeta(\omega) = \infty\}} \liminf_{t \to \infty} Y_t(\omega) + I_{\{\zeta(\omega) < \infty\}} Y_{\zeta}(\omega),
\end{aligned}
$$

etc. Of course, if we write $\lim_{t \to \infty} Y_t$, we must make sure that this limit exists on $\{\zeta = \infty\}$. These notations allow us to handle all time horizons in a unified manner.

The next concept is fundamental for our paper.

Definition 2.5. Fix a strategy $\eta \in \Theta^{\mathrm{sf}}$. A strategy $\vartheta \in \Theta^{\mathrm{sf}}_{+}$ is called *strongly share maximal (ssm) for* η if there is no $[0,1]$-valued adapted process $\psi = (\psi_t)_{t\ge 0}$ converging P-a.s. as $t \to \infty$ to some $\psi_\infty \in L^\infty_+(\mathcal{F}_\infty) \setminus \{0\}$ and such that for every $\varepsilon > 0$, there exists some $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_{+}$ with $V_0(\hat{\vartheta}^\varepsilon) \le V_0(\vartheta) + \varepsilon$ and

$$\liminf_{t\to\infty} (\hat{\vartheta}^\varepsilon_t - \vartheta_t - \psi_t \eta_t) \ge 0 \qquad P\text{-a.s.}$$

We shall use this concept when η is a reference strategy. Then having η is desirable, and $\psi\eta$ is a dynamic long-only portfolio where the factor ψ stabilises over time and which asymptotically achieves a significant part of η. Strong share maximality then says that even with a little extra initial capital $\varepsilon > 0$, one cannot asymptotically improve ϑ via some $\hat{\vartheta}^\varepsilon$ in such a significant manner.

We also need the following concept inspired by Herdegen [10]; the difference to [10] is that we work here on a possibly open time interval. Note that we replace "strongly maximal" from [10] by the more explicit terminology "strongly value maximal".

Definition 2.6. Fix an $\mathbb{R}^N$-valued semimartingale $\tilde{S}$. A strategy $\vartheta \in \Theta^{\mathrm{sf}}_{+}$ is called *strongly value maximal (svm) for* $\tilde{S}$ if there is no $f \in L^0_+ \setminus \{0\}$ such that for every $\varepsilon > 0$, there exists some $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_{+}$ with $V_0(\hat{\vartheta}^\varepsilon, \tilde{S}) \le V_0(\vartheta, \tilde{S}) + \varepsilon$ and

$$\liminf_{t\to\infty} \Big(V_t(\hat{\vartheta}^\varepsilon, \tilde{S}) - V_t(\vartheta, \tilde{S}) - f \Big) \ge 0 \qquad P\text{-a.s.}$$

Maximality of a strategy ϑ always means that ϑ cannot be improved. The key difference between Definitions 2.5 and 2.6 lies in how improvements are measured. For strong value maximality, the comparison is in terms of *value*, which makes the concept depend on the currency unit (of $\tilde{S}$). In contrast, strong share maximality looks (via the reference strategy η) at *numbers of shares*, and this is independent of any currency unit.

Given a maximality concept for strategies, we define viability and efficiency as in [10].

Definition 2.7. Fix $\eta \in \Theta^{\mathrm{sf}}$. We say that S satisfies *dynamic share viability (DSV) for* η if the zero strategy $0 \in \Theta^{\mathrm{sf}}_{+}$ is strongly share maximal for η, and *dynamic share efficiency (DSE) for* η if every η-buy-and-hold strategy $\vartheta \in \Theta^{\mathrm{sf}}_{+}$ is strongly share maximal for η.

It is a key observation that for fixed η, *strong share maximality for* η*, dynamic share viability for* η *and dynamic share efficiency for* η *are like* Θ^{sf} *all* discounting-invariant *with respect to* $\mathcal{S}_{++}$, in the sense that if we have one of these properties for S, we also have it for any D-discounted $\tilde{S} = S/D$ with any discounter $D \in \mathcal{S}_{++}$, and vice versa. In contrast, the strong (value) maximality for S from [10] (and derived concepts like NINA there) is invariant under discounting by *narrow* discounters $D \in \mathcal{S}^{\mathrm{unif}}_{++} \subsetneq \mathcal{S}_{++}$ (see Lemma 3.1 below), but not under discounting by $D \in \mathcal{S}_{++}$ (see Example 3.2 below). In that sense, the value-related concepts and results from [10] are only numéraire- or discounting-invariant in a restricted manner. But for a general discounting-invariant framework, having invariance with respect to the full class $\mathcal{S}_{++}$ is crucial because the natural class of discounters on an open interval like $[0,\infty)$ is $\mathcal{S}_{++}$ and not $\mathcal{S}^{\mathrm{unif}}_{++}$.

Remark 2.8. **1)** Theorems 2.13 and 2.14 below give equivalent characterisations for DSE for η, assuming among other things that η is a reference strategy and bounded (uniformly in ω, t). These results show that equivalent definitions of DSE for η are possible: one could as well stipulate that only η itself, or all bounded $\vartheta \in \Theta^{\mathrm{sf}}_{+}$, should be ssm for η. We have opted for an intermediate definition to preserve the analogy to [10].

2) All our concepts depend on the choice of η. We discuss this in Section 5.1 and show there in particular that the dependence is quite weak.

The preceding concepts are all about strategies and hence on the primal side. For a dual characterisation in terms of martingale properties, we need the following concept.

Definition 2.9. For $\mathcal{E} \in \{\sigma\text{-martingale, local martingale, martingale, UI martingale}\}$, an *$\mathcal{E}$-discounter* for an $\mathbb{R}^N$-valued semimartingale $\tilde{S}$ is a $D \in \mathcal{S}_{++}$ such that $\tilde{S}/D$ is an $\mathcal{E}$.

Remark 2.10. In the literature, an *$\mathcal{E}$-deflator* for a class $\mathcal{Y}$ of processes is a strictly positive local martingale Z (often with $Z_0 = 1$) such that ZY is an $\mathcal{E}$ for all $Y \in \mathcal{Y}$. There are two differences to the notion of an $\mathcal{E}$-discounter. Obviously, a deflator acts by multiplication while a discounter acts by division. More importantly, however, we impose no (local) martingale property on an $\mathcal{E}$-discounter D, nor on $1/D$. (Some definitions of an $\mathcal{E}$-deflator Z do not explicitly ask for Z to be a local martingale. But as $\mathcal{Y}$ invariably contains the process $Y \equiv 1$, this property follows from the definition and $Z > 0$.) In our setup, neither S nor the family $\{V(\vartheta) : \vartheta \in \Theta^{\mathrm{sf}}_{++}\}$ of value processes contains a constant process in general; so discounters are more natural and more general than deflators.

With these preliminaries, we can already state our first main results.

Theorem 2.11. *Suppose $S \geq 0$ and there exists a reference strategy η. Fix η. Then S satisfies dynamic share viability for η if and only if there exists a σ-martingale discounter D for S with $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s.*

Remark 2.12. As pointed out in the proof in Section 4, the "only if" part in Theorem 2.11 does not need $S \geq 0$. The same applies to Theorem 2.13.

Theorem 2.13. *Suppose $S \geq 0$ and there exists a reference strategy η such that in addition, η and $S^\eta = S/(\eta \cdot S)$ are bounded (uniformly in ω, t). Fix such an η. Then S satisfies dynamic share efficiency for η if and only if there exists a UI martingale discounter D with $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s.*

The proofs of Theorems 2.11 and 2.13 need extra ideas and additional results. These are developed in Section 3 and used in Section 4 to prove Theorems 2.11 and 2.13.

Both Theorems 2.11 and 2.13 are modern formulations of the classic idea due to Samuelson [23] that "properly anticipated prices fluctuate randomly" or, in other words, suitably discounted prices form a martingale. The notion of properly anticipated or suitably discounted is in our paper captured by the *existence* of the process D which turns S via discounting to S/D into a “martingale”. The strength of the martingale property of S/D (σ-martingale or UI martingale) depends on the strength of the initial no-arbitrage condition (viability or efficiency). The main contrast to the classic FTAP formulation of Delbaen/Schachermayer [5, 8] is that the discounting process is not chosen a priori, but an endogenous part of the dual characterisation of absence of arbitrage. A similar idea appears in Herdegen [10] (see also [11]) where the dual objects are not only “martingale transformers” like martingale measures or deflators, but *pairs* consisting of an S-tradable numéraire and a “martingale measure”. Our $\mathcal{E}$-discounter combines such a pair into a single process; this is more general than a deflator because the latter's local martingale property still reflects the effect of an a priori discounting of prices.

We next relate our work to the existing literature. To that end, we recall or rewrite some notions from the classic Delbaen/Schachermayer [5, 8] approach. For any $\mathbb{R}^N$-valued semimartingale $\tilde{S}$, we define $L^a_{\mathrm{adm}}(\tilde{S}) := \{H \in L(\tilde{S}) : H \bullet \tilde{S} \geq -a\}$ and introduce the sets

- $\mathcal{G}^a_{\mathrm{adm}}(\tilde{S}) := \Big\{ \lim_{t\to\infty} V_t(\vartheta, \tilde{S}) - V_0(\vartheta, \tilde{S}) : \vartheta \in \Theta^{\mathrm{sf}}_+,\, V_0(\vartheta, \tilde{S}) = a \text{ and } \lim_{t\to\infty} V_t(\vartheta, \tilde{S}) \text{ exists}\Big\}$,
- $\mathcal{G}_{\mathrm{adm}}(\tilde{S}) := \bigcup_{a\geq 0} \mathcal{G}^a_{\mathrm{adm}}(\tilde{S}) = \Big\{ \lim_{t\to\infty} V_t(\vartheta, \tilde{S}) - V_0(\vartheta, \tilde{S}) : \vartheta \in \Theta^{\mathrm{sf}}_+ \text{ and } \lim_{t\to\infty} V_t(\vartheta, \tilde{S}) \text{ exists}\Big\}$,
- $\mathcal{C}_{\mathrm{adm}}(\tilde{S}) := \mathcal{G}_{\mathrm{adm}}(\tilde{S}) - L^0_+(\mathcal{F}_\zeta)$,
- $\overline{\mathcal{C}}^\infty_{\mathrm{adm}}(\tilde{S}) := \overline{\mathcal{C}_{\mathrm{adm}}(\tilde{S}) \cap L^\infty}^{\infty}$;

the bar $^{-\infty}$ denotes the norm closure in L^∞. Each $g \in \mathcal{G}^a_{\mathrm{adm}}(\tilde{S})$ is the net outcome (final minus initial value) of a self-financing strategy ϑ whose value is always $\geq -a$, with all quantities in the same currency units as $\tilde{S}$. Then we say that

- $\mathrm{NA}_\infty(\tilde{S})$ holds if $\mathcal{C}_{\mathrm{adm}}(\tilde{S}) \cap L^\infty_+ = \{0\}$;
- $\mathrm{NUPBR}_\infty(\tilde{S})$ holds if $\mathcal{G}^1_{\mathrm{adm}}(\tilde{S})$ is bounded in L^0;
- $\mathrm{NFLVR}_\infty(\tilde{S})$ holds if $\overline{\mathcal{C}}^\infty_{\mathrm{adm}}(\tilde{S}) \cap L^\infty_+ = \{0\}$.

Using [10, Theorem 2.14] (which easily extends to $[\![0, \infty]\!]$) allows us to rewrite things in more familiar form. Fix $\eta \in \Theta^{\mathrm{sf}}_{++}$ and recall the η-discounted prices $S^\eta = S/(\eta \cdot S)$. Then

$$\mathcal{G}^a_{\mathrm{adm}}(S^\eta) = \Big\{ \lim_{t\to\infty} H \bullet S^\eta_t : H \in L^a_{\mathrm{adm}}(S^\eta) \text{ and } \lim_{t\to\infty} H \bullet S^\eta_t \text{ exists}\Big\}. \tag{2.1}$$

If prices $S = (1, X)$ are already discounted, we can take $\eta \equiv \mathrm{e}^1 := (1, 0, \dots, 0) \in \mathbb{R}^N$, getting $S^{\mathrm{e}^1} = (1, X) = S$, and note $\{H \bullet S : H \in L(S)\} = \{H \bullet X : H \in L(X)\}$ to obtain

$$\mathcal{G}^a_{\mathrm{adm}}(1, X) = \Big\{ \lim_{t\to\infty} H \bullet X_t : H \in L(X),\, H \bullet X \geq -a \text{ and } \lim_{t\to\infty} H \bullet X_t \text{ exists}\Big\}.$$

Thus $\mathcal{G}_{\mathrm{adm}}(1, X) = \bigcup_{a\geq 0} \mathcal{G}^a_{\mathrm{adm}}(1, X)$ is precisely the set K_0 (or K) considered in [5] (or [8]), and $\mathrm{NA}_\infty(1, X)$, $\mathrm{NUPBR}_\infty(1, X)$ and $\mathrm{NFLVR}_\infty(1, X)$ recover the standard notions in the classic theory following Delbaen/Schachermayer [5, 8]. We remark that the property $\mathrm{NUPBR}_\infty(1, X)$ already appears without a name in [5, Corollary 3.4]; it was later called BK by Kabanov [14] and NUPBR by Karatzas/Kardaras [17].

The next result summarises the connections between our new results and the classic theory. The verbal formulation is better suited for the proof, while the graphical representation gives a better overview.

Theorem 2.14. *Suppose $S \geq 0$ and there exists a reference strategy η. Fix η. Consider the following statements:*

(e1) *S satisfies dynamic share efficiency for η.*

(v1) *S satisfies dynamic share viability for η.*

(e2) *The reference strategy $\eta \in \Theta_+^{\mathrm{sf}}$ is strongly share maximal for η.*

(e2′) *Every bounded $\vartheta \in \Theta_+^{\mathrm{sf}}$ is strongly share maximal for η.*

(v2) *There exists some $\vartheta \in \Theta_+^{\mathrm{sf}}$ which is strongly share maximal for η.*

(e3) *η is strongly value maximal for $S^\eta = S/(\eta \cdot S)$.*

(v3) *0 is strongly value maximal for $S^\eta = S/(\eta \cdot S)$.*

(e4) *For every $D \in \mathcal{S}_{++}$ with $\eta \cdot (S/D) \in \mathcal{S}_{++}^{\mathrm{unif}}$, every bounded $\vartheta \in \Theta_+^{\mathrm{sf}}$ is strongly value maximal for the D-discounted price process S/D.*

(e4′) *Every bounded $\vartheta \in \Theta_+^{\mathrm{sf}}$ is strongly value maximal for S^η.*

(v4) *For every $D \in \mathcal{S}_{++}$ with $\eta \cdot (S/D) \in \mathcal{S}_{++}^{\mathrm{unif}}$, the zero strategy $0 \in \Theta_+^{\mathrm{sf}}$ is strongly value maximal for the D-discounted price process S/D.*

(e5) *There exists a UI martingale discounter D for S with $\eta \cdot (S/D) \in \mathcal{S}_{++}^{\mathrm{unif}}$.*

(v5) *There exists a σ-martingale discounter D for S with $\eta \cdot (S/D) \in \mathcal{S}_{++}^{\mathrm{unif}}$.*

(e6) *$\mathrm{NFLVR}_\infty(S^\eta)$ holds, i.e., S^η satisfies NFLVR_∞.*

(v6) *$\mathrm{NUPBR}_\infty(S^\eta)$ holds, i.e., S^η satisfies NUPBR_∞.*

Then we have (eK) $\Rightarrow$ (vK) *for* $\mathrm{K} = 1, \dots, 6$, *and the statements* (vK), $\mathrm{K} = 1, \dots, 6$, *are equivalent among themselves. If in addition η and S^η are bounded (uniformly in ω, t), then also the statements* (eK), $\mathrm{K} = 1, \dots, 6$, *are equivalent among themselves (including the prime $'$ versions).*

In graphical form, this looks as follows:

$$
\begin{array}{ccc}
S \text{ satisfies DSE for } \eta & & S \text{ satisfies DSV for } \eta \\
(\text{every } \eta\text{-buy-and-hold } \vartheta \in \Theta_+^{\mathrm{sf}} \text{ is ssm for } \eta) & & (0 \text{ is ssm for } \eta) \\
\Updownarrow & & \\
\eta \text{ is ssm for } \eta & & \Updownarrow \\
\Updownarrow & & \\
\text{every bounded } \vartheta \in \Theta_+^{\mathrm{sf}} \text{ is ssm for } \eta & & \text{some } \vartheta \in \Theta_+^{\mathrm{sf}} \text{ is ssm for } \eta \\
\Updownarrow & & \\
\eta \text{ is svm for } S^\eta & & \Updownarrow \\
\Updownarrow & & \\
\text{every bounded } \vartheta \in \Theta_+^{\mathrm{sf}} \text{ is svm for } S^\eta & & \text{some } \vartheta \in \Theta_+^{\mathrm{sf}} \text{ is svm for } S^\eta \\
\Updownarrow & & \Updownarrow \\
\forall D \in \mathcal{S}_{++} \text{ with } \eta \cdot (S/D) \in \mathcal{S}_{++}^{\mathrm{unif}}\text{:} & & \forall D \in \mathcal{S}_{++} \text{ with } \eta \cdot (S/D) \in \mathcal{S}_{++}^{\mathrm{unif}}\text{:} \\
\text{every bounded } \vartheta \in \Theta_+^{\mathrm{sf}} \text{ is svm for } S/D & \Longrightarrow & 0 \in \Theta_+^{\mathrm{sf}} \text{ is svm for } S/D \\
\Updownarrow & & \Updownarrow \\
\exists \text{ UI martingale discounter } D & & \exists\ \sigma\text{-martingale discounter } D \\
\text{with } \eta \cdot (S/D) \in \mathcal{S}_{++}^{\mathrm{unif}} & & \text{with } \eta \cdot (S/D) \in \mathcal{S}_{++}^{\mathrm{unif}} \\
\Updownarrow & & \Updownarrow \\
S^\eta \text{ satisfies NFLVR}_\infty & & S^\eta \text{ satisfies NUPBR}_\infty
\end{array}
$$

Figure 1: Graphical summary of Theorem 2.14. Assumptions are $S \geq 0$ and that η is a reference strategy (which is assumed to exist). The equivalences on the left side need in addition that η and S^η are bounded.

Proving our main results involves several ideas and steps. We give here a short overview and implement this in Section 3. First, because strong share maximality is discounting-invariant with respect to $\mathcal{S}_{++}$, we can work with a discounted price process $\tilde{S}$ instead of the original S. If the pair $(\tilde{S}, \eta)$ has good properties, we show that strong share maximality for η is equivalent to strong value maximality for $\tilde{S}$. Choosing a good discounter (actually, $\eta \cdot S$) thus gives us almost access to the results from Herdegen [10] who derived dual characterisations for strong (value) maximality, of 0 or of a fixed strategy, in terms of certain martingale properties for suitably discounted prices. At this point, the endogenous discounter comes in. However, [10] crucially exploits that prices there are defined on a right-closed time interval, and the numéraire-invariance in [10] is only with respect to the smaller, restrictive class $\mathcal{S}_{++}^{\mathrm{unif}}$ of discounters. Overcoming this problem needs an extra step. With a similar argument as in Delbaen/Schachermayer [5], we show that for suitably (and tradably) discounted prices $\tilde{S}$ and under strong value maximality for $\tilde{S}$ of 0, the value process $V(\vartheta, \tilde{S})$ of any self-financing strategy $\vartheta \in \Theta_+^{\mathrm{sf}}$ converges as $t \to \infty$. In effect, all the $V(\vartheta, \tilde{S})$ are hence defined on a right-closed interval (even if S or $\tilde{S}$ is not), and this finally allows us to use the results from [10]. Combining everything yields our assertions.

3 The theory

This section is the mathematical core of the paper. It consists of three subsections which mirror the ideas and steps in the discussion at the end of Section 2.

3.1 Connecting share maximality and value maximality

We first show that under an extra assumption on the pair (S, ξ) of price process and strategy, strong share maximality for ξ and strong value maximality for S are equivalent. This uses discounting-invariance properties of strong share maximality and explicit arguments.

Lemma 3.1. *Strong value maximality is discounting-invariant with respect to $\mathcal{S}_{++}^{\mathrm{unif}}$: If $\vartheta \in \Theta_+^{\mathrm{sf}}$ is strongly value maximal for S and $D \in \mathcal{S}_{++}^{\mathrm{unif}}$, then ϑ is also strongly value maximal for S/D. (The converse is clear because $D \in \mathcal{S}_{++}^{\mathrm{unif}}$ implies $1/D \in \mathcal{S}_{++}^{\mathrm{unif}}$.) Moreover, $\alpha\vartheta$ is then also strongly value maximal for S for any $\alpha \geq 0$.*

Proof. See Appendix. □

The next example shows that $\mathcal{S}_{++}^{\mathrm{unif}}$ cannot be replaced by $\mathcal{S}_{++}$ in Lemma 3.1.

Example 3.2. *Strong value maximality is not discounting-invariant with respect to $\mathcal{S}_{++}$.* Consider the BS model from Example 1.1 with $m = r = \sigma = 1$, so that $S_t^1 = e^t$ and $S_t^2 = e^{W_t + \frac{1}{2}t}$. Here, 0 is not svm for S because for any $\varepsilon > 0$, the strategy $\hat{\vartheta}^\varepsilon := \varepsilon \mathrm{e}^1 = (\varepsilon, 0)$ of buying and holding ε units of S^1 has $V_0(\hat{\vartheta}^\varepsilon) = \varepsilon$, but $\lim_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon) = +\infty$. But taking $D := S^1 \in \mathcal{S}_{++} \setminus \mathcal{S}_{++}^{\mathrm{unif}}$ yields $\tilde{S} := S/D = (1, e^{W_t - \frac{1}{2}t})$. This is a ($\sigma$-)martingale, and therefore 0 is svm for $\tilde{S}$; see Theorem 3.11 below (applied to $\tilde{S}$ and with $\xi \equiv \mathrm{e}^1$).

In the proofs in this section, we need to concatenate strategies which requires some notation. Fix $\xi \in \Theta_{++}^{\mathrm{sf}}$ and a stopping time τ (as usual with values in $[0, \infty]$). The *ξ-concatenation* at time τ of $\vartheta^1, \vartheta^2 \in \Theta^{\mathrm{sf}}$ is defined by

$$\begin{aligned} \vartheta^1 \circledS_\tau^\xi \vartheta^2 &:= I_{[\![0,\tau]\!]}\vartheta^1 + I_{]\!]\tau,\infty]\!]}\Big(I_\Gamma \vartheta^1 + I_{\Gamma^c}\big(\vartheta^2 + V_\tau(\vartheta^1 - \vartheta^2, S^\xi)\xi\big)\Big) \\ &\quad \text{with } \Gamma := \{V_\tau(\vartheta^1) < V_\tau(\vartheta^2)\}. \end{aligned} \tag{3.1}$$

The interpretation is as follows. We start with ϑ^1 and follow this strategy until time τ where we compare its value to that of the competitor ϑ^2. If ϑ^1 is strictly cheaper, we stick to it. Otherwise, we liquidate ϑ_τ^1, start with ϑ^2 by buying ϑ_τ^2, and invest the rest of the proceeds (which is nonnegative) into ξ. Note that on $\{\tau = \infty\}$, we have $\vartheta^1 \circledS_\tau^\xi \vartheta^2 = \vartheta^1$ so that the possibly undefined expressions ϑ_∞^1, ϑ_∞^2, S_∞ or S_∞^ξ never appear.

Lemma 3.3. *Fix $\xi \in \Theta_{++}^{\mathrm{sf}}$ and a stopping time τ. If ϑ^1, ϑ^2 are in Θ^{sf}, then so is $\vartheta^1 \circledS_\tau^\xi \vartheta^2$. If ϑ^1, ϑ^2 are in Θ_+^{sf}, then so is $\vartheta^1 \circledS_\tau^\xi \vartheta^2$.*

Proof. See Appendix. □

Theorem 3.4. *Fix $\xi \in \Theta^{\mathrm{sf}}_{++}$ and suppose that $V(\xi) \in \mathcal{S}^{\mathrm{unif}}_{++}$.*

1) If $\xi \geq 0$, then any $\vartheta \in \Theta^{\mathrm{sf}}_{+}$ which is strongly share maximal for ξ is also strongly value maximal for S.

2) If $S \geq 0$ and S^ξ is bounded uniformly in $t \geq 0$, P-a.s., then any $\vartheta \in \Theta^{\mathrm{sf}}_{+}$ which is strongly value maximal for S is also strongly share maximal for ξ.

Before proving Theorem 3.4, we summarise its contents in compact form for future use. Note that the assumptions are jointly on the pair (S, ξ).

Corollary 3.5. *Suppose $S \geq 0$ and there exists a reference strategy ξ with $V(\xi) \in \mathcal{S}^{\mathrm{unif}}_{++}$. Fix such a ξ. Then $\vartheta \in \Theta^{\mathrm{sf}}_{+}$ is strongly share maximal for ξ if and only if it is strongly value maximal for S.*

Proof of Theorem 3.4. 1) Assume ϑ is not svm for S. As $V(\xi) \in \mathcal{S}^{\mathrm{unif}}_{++}$, Lemma 3.1 implies that ϑ is also not svm for $S^\xi = S/V(\xi)$. So there are $f \in L^0_+ \setminus \{0\}$ and for every $\varepsilon = 1/n$ some $\hat{\vartheta}^n \in \Theta^{\mathrm{sf}}_{+}$ with $\hat{\vartheta}^n_0 \cdot S^\xi_0 = V_0(\hat{\vartheta}^n, S^\xi) \leq \vartheta_0 \cdot S^\xi_0 + 1/n$ and $\liminf_{t\to\infty}((\hat{\vartheta}^n_t - \vartheta_t) \cdot S^\xi_t) \geq f$ P-a.s. Choose $\delta > 0$ and $A \in \mathcal{F}$ with $P[A] > 0$ such that $f \geq 2\delta I_A$ P-a.s., and define

$$\begin{aligned}
\sigma'_n &:= \inf\{t \geq 0 : (\hat{\vartheta}^n_t - \vartheta_t) \cdot S^\xi_t \geq \delta\},\\
\varphi_n &:= \inf\{t \geq 0 : P[\sigma'_n \leq t] \geq P[A](1 - 2^{-n+1})\},\\
\sigma_n &:= \sigma'_n \wedge \varphi_n \leq \varphi_n.
\end{aligned}$$

Then σ'_n is a stopping time, φ_n a bounded nonrandom time and σ_n a bounded stopping time. Moreover, $B_n := \{\sigma'_n \leq \varphi_n\} \in \mathcal{F}_{\varphi_n}$ satisfies $P[B_n] \geq P[A](1 - 2^{-n+1})$ and we have

$$(\hat{\vartheta}^n_{\sigma_n} - \vartheta_{\sigma_n}) \cdot S^\xi_{\sigma_n} = (\hat{\vartheta}^n_{\sigma'_n} - \vartheta_{\sigma'_n}) \cdot S^\xi_{\sigma'_n} \geq \delta \qquad \text{on } B_n,\ P\text{-a.s.} \tag{3.2}$$

by right-continuity. Due to $\liminf_{t\to\infty}((\hat{\vartheta}^n_t - \vartheta_t) \cdot S^\xi_t) \geq f \geq 0$ P-a.s.,

$$\tau_n := \inf\{t \geq \varphi_n : (\hat{\vartheta}^n_t - \vartheta_t) \cdot S^\xi_t \geq -1/n\} \geq \varphi_n$$

is a P-a.s. finite-valued stopping time which satisfies $\tau_n \geq \sigma_n$.

We now consider the strategy

$$\tilde{\vartheta}^n := I_{[\![0,\tau_n]\!]}(\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta) + I_{]\!]\tau_n,\infty[\![}\Big(\vartheta + V_{\tau_n}(\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta - \vartheta, S^\xi)\xi\Big) + \xi/n, \tag{3.3}$$

with $\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta$ defined in (3.1). In words, we hold a $(1/n)$-multiple of ξ, switch at time σ_n from $\hat{\vartheta}^n$ to ϑ if the value of ϑ is at most the value of $\hat{\vartheta}^n$, and always switch to ϑ at time τ_n; in both cases, any difference in value is invested into ξ. Using $\xi \cdot S^\xi \equiv 1$, this gives

$$V_0(\tilde{\vartheta}^n, S^\xi) = \tilde{\vartheta}^n_0 \cdot S^\xi_0 = \hat{\vartheta}^n_0 \cdot S^\xi_0 + (\xi_0 \cdot S^\xi_0)/n \leq V_0(\vartheta, S^\xi) + 2/n.$$

Next, as $\hat{\vartheta}^n$ and ϑ are in Θ^{sf}_{+}, Lemma 3.3 yields $\hat{\vartheta}^n \oslash^\xi_{\sigma_n} \vartheta \in \Theta^{\mathrm{sf}}_{+}$, and therefore (3.3) gives $\tilde{\vartheta}^n \cdot S^\xi = V(\tilde{\vartheta}^n, S^\xi) \geq 0$ P-a.s. on $[\![0, \tau_n]\!]$. Using now $V(\xi, S^\xi) \equiv 1$ and the definition (3.1) allows us to compute, as in the proof of Lemma 3.3 in the Appendix, that

$$
\begin{aligned}
V_{\tau_n}(\hat{\vartheta}^n \oslash^{\xi}_{\sigma_n} \vartheta - \vartheta, S^{\xi}) &= I_{\{\tau_n=\sigma_n\}} V_{\tau_n}(\hat{\vartheta}^n - \vartheta, S^{\xi}) \\
&\quad + I_{\{\tau_n>\sigma_n\}}\Big(I_{\Gamma_n} V_{\tau_n}(\hat{\vartheta}^n - \vartheta, S^{\xi}) + I_{\Gamma_n^c} V_{\sigma_n}(\hat{\vartheta}^n - \vartheta, S^{\xi})\Big)
\end{aligned}
\tag{3.4}
$$

with $\Gamma_n := \{V_{\sigma_n}(\hat{\vartheta}^n) < V_{\sigma_n}(\vartheta)\}$. This shows that due to $\tau_n < \infty$ P-a.s., we always have

$$
V_{\tau_n}(\hat{\vartheta}^n \oslash^{\xi}_{\sigma_n} \vartheta - \vartheta, S^{\xi}) \geq \min\Big((\hat{\vartheta}^n_{\tau_n} - \vartheta_{\tau_n}) \cdot S^{\xi}_{\tau_n}, 0\Big) \geq -1/n \qquad P\text{-a.s.} \tag{3.5}
$$

Combining (3.3) and (3.5) and using $\xi \geq 0$ implies that on $]\!]\tau_n, \infty]\!]$, we have

$$
\tilde{\vartheta}^n - \vartheta = V_{\tau_n}(\hat{\vartheta}^n \oslash^{\xi}_{\sigma_n} \vartheta - \vartheta, S^{\xi})\xi + \xi/n \geq 0, \tag{3.6}
$$

hence $V(\tilde{\vartheta}^n, S^{\xi}) \geq V(\vartheta, S^{\xi})$, and so $\tilde{\vartheta}^n$ is like ϑ in Θ^{sf}_+.

Now on the set B_n, we have $\sigma_n = \sigma'_n$, hence $V_{\sigma_n}(\hat{\vartheta}^n - \vartheta, S^{\xi}) = (\hat{\vartheta}^n_{\sigma_n} - \vartheta_{\sigma_n}) \cdot S^{\xi}_{\sigma_n} \geq \delta$ P-a.s. as in (3.2) and therefore by (3.4) also

$$
V_{\tau_n}(\hat{\vartheta}^n \oslash^{\xi}_{\sigma_n} \vartheta - \vartheta, S^{\xi}) = V_{\sigma_n}(\hat{\vartheta}^n - \vartheta, S^{\xi}) \geq \delta \qquad P\text{-a.s.}
$$

Thus (3.6) and $\xi \geq 0$ yield $\tilde{\vartheta}^n - \vartheta \geq \delta\xi$ on B_n on $]\!]\tau_n, \infty]\!]$ and so, as $\tau_n < \infty$ P-a.s.,

$$
\liminf_{t\to\infty}(\tilde{\vartheta}^n_t - \vartheta_t - \delta I_{B_n}\xi_t) \geq 0 \qquad P\text{-a.s.} \tag{3.7}
$$

If we define the $[0,1]$-valued adapted process $\psi^n = (\psi^n_t)_{t\geq 0}$ by $\psi^n_t := \delta E[I_{B_n}|\mathcal{F}_t]$, then $\varphi_n < \infty$ and $B_n \in \mathcal{F}_{\varphi_n}$ yield $\psi^n_t = \delta I_{B_n}$ for $t \geq \varphi_n$ so that $\psi^n_\infty := \lim_{t\to\infty}\psi^n_t = \delta I_{B_n}$ P-a.s. Moreover, we also obtain via (3.7) that

$$
\liminf_{t\to\infty}(\tilde{\vartheta}^n_t - \vartheta_t - \psi^n_t\xi_t) = \liminf_{t\to\infty}(\tilde{\vartheta}^n_t - \vartheta_t - \delta I_{B_n}\xi_t) \geq 0 \qquad P\text{-a.s.}
$$

Set $B := \bigcap_{n\in\mathbb{N}} B_n$ and $\psi_t := \delta E[I_B|\mathcal{F}_t]$ for $t \geq 0$. Then $\lim_{t\to\infty}\psi_t = \psi_\infty := \delta I_B$ P-a.s., and $B \subseteq B_n$ for all n implies $\psi \leq \psi^n$ for all n. Moreover, $\psi_\infty \in L^\infty_+(\mathcal{F}_\infty) \setminus \{0\}$ because

$$
\begin{aligned}
P[B] \geq P[B \cap A] &= P[A] - P\Bigg[A \cap \bigcup_{n\in\mathbb{N}} B^c_n\Bigg] \geq P[A] - \sum_{n=1}^{\infty} P[A \cap B^c_n] \\
&= P[A] - \sum_{n=1}^{\infty}(P[A] - P[A \cap B_n]) \geq P[A]\Bigg(1 - \sum_{n=1}^{\infty} 2^{-n+1}\Bigg) = P[A]/2 > 0.
\end{aligned}
$$

So we have found ψ and for each $n \in \mathbb{N}$ a $\tilde{\vartheta}^n \in \Theta^{\mathrm{sf}}_+$ with $V_0(\tilde{\vartheta}^n, S^{\xi}) \leq V_0(\vartheta, S^{\xi}) + 2/n$ and

$$
\liminf_{t\to\infty}(\tilde{\vartheta}^n_t - \vartheta_t - \psi_t\xi_t) = \liminf_{t\to\infty}(\tilde{\vartheta}^n_t - \vartheta_t - \psi^n_t\xi_t) \geq 0 \qquad P\text{-a.s.},
$$

which contradicts the assumption that ϑ is ssm for ξ.

2) If ϑ is not ssm for ξ, there are a $[0,1]$-valued adapted $\psi = (\psi_t)_{t\geq 0}$ converging P-a.s. to $\psi_\infty := \lim_{t\to\infty}\psi_t \in L^\infty_+(\mathcal{F}_\infty)\setminus\{0\}$ and for each $\varepsilon > 0$ a $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+$ with $V_0(\hat{\vartheta}^\varepsilon) \leq V_0(\vartheta) + \varepsilon$,

hence $V_0(\hat{\vartheta}^\varepsilon, S^\xi) \le V_0(\vartheta, S^\xi) + \varepsilon / V_0(\xi)$, and satisfying $\liminf_{t\to\infty}(\hat{\vartheta}^\varepsilon_t - \vartheta_t - \psi_t \xi_t) \ge 0$ P-a.s. By assumption, S^ξ is bounded uniformly in $t \ge 0$, P-a.s. Superadditivity of the lim inf, Lemma A.1, $V(\xi, S^\xi) = \xi \cdot S^\xi \equiv 1$ and $S^\xi \ge 0$ from $S \ge 0$ thus yield that P-a.s.,

$$\begin{aligned}\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon - \vartheta, S^\xi) &\ge \liminf_{t\to\infty} \Big((\hat{\vartheta}^\varepsilon_t - \vartheta_t - \psi_t \xi_t) \cdot S^\xi_t \Big) + \liminf_{t\to\infty} \Big((\psi_t \xi_t) \cdot S^\xi_t \Big) \\ &\ge \Big(\liminf_{t\to\infty} (\hat{\vartheta}^\varepsilon_t - \vartheta_t - \psi_t \xi_t) \Big) \cdot \Big(\liminf_{t\to\infty} S^\xi_t \Big) + \psi_\infty \ge \psi_\infty.\end{aligned}$$

So ϑ is not svm for S^ξ, and by Lemma 3.1 also not svm for $S = S^\xi V(\xi)$ as $V(\xi) \in \mathcal{S}^{\mathrm{unif}}_{++}$. $\square$

3.2 From a stochastic or open interval to a closed interval

In this section, we show how to pass from a model with a general time horizon (stochastic or not, finite or infinite) to a model effectively defined on $\Omega \times [0, \infty]$. This rests on a convergence result in the spirit of Delbaen/Schachermayer [5, Theorem 3.3] combined with ideas from Herdegen [10] to connect strong (value) maximality and NUPBR.

Proposition 3.6. *Suppose there exists a $\xi \in \Theta^{\mathrm{sf}}_{++}$ with $V(\xi) = \xi \cdot S \in \mathcal{S}^{\mathrm{unif}}_{++}$. Recall the ξ-discounted price process $S^\xi = S/(\xi \cdot S)$. Then the following are equivalent:*

(a) *The zero strategy $0 \in \Theta^{\mathrm{sf}}_+$ is strongly value maximal for S.*

(b) *The zero strategy $0 \in \Theta^{\mathrm{sf}}_+$ is strongly value maximal for S^ξ.*

(c) *The set $\{\lim_{t\to\infty} H \bullet S^\xi_t : H \in L^1_{\mathrm{adm}}(S^\xi)$, H has bounded support on $[0,\infty)\}$ is bounded in L^0.*

(d) *The set $\{\liminf_{t\to\infty} H \bullet S^\xi_t : H \in L^1_{\mathrm{adm}}(S^\xi)\}$ is bounded in L^0.*

(e) *The set $\{\lim_{t\to\infty} H \bullet S^\xi_t : H \in L^1_{\mathrm{adm}}(S^\xi)$ and $\lim_{t\to\infty} H \bullet S^\xi_t$ exists$\}$ is bounded in L^0.*

(f) *NUPBR$_\infty(S^\xi)$ holds.*

Proof. (d) $\Rightarrow$ (e) $\Rightarrow$ (c) is clear, (c) $\Rightarrow$ (d) is from the proof of [5, Proposition 3.2], (e) $\Leftrightarrow$ (f) follows directly from (2.1), and (a) $\Leftrightarrow$ (b) is from Lemma 3.1 because $V(\xi) \in \mathcal{S}^{\mathrm{unif}}_{++}$.

We prove (d) $\Rightarrow$ (b) indirectly. If 0 is not svm for S^ξ, we can find $f \in L^0_+ \setminus \{0\}$ and for every $\varepsilon = 1/n$ some $\hat{\vartheta}^n$ with $V_0(\hat{\vartheta}^n, S^\xi) \le 1/n$ and $\liminf_{t\to\infty} V_t(\hat{\vartheta}^n, S^\xi) \ge f$ P-a.s. Then $\tilde{\vartheta}^n := n\hat{\vartheta}^n$ is in Θ^{sf}_+ with $V_0(\tilde{\vartheta}^n, S^\xi) \le 1$, and $\tilde{\vartheta}^n$ is also in $L^1_{\mathrm{adm}}(S^\xi)$ because

$$0 \le V(\tilde{\vartheta}^n, S^\xi) = V_0(\tilde{\vartheta}^n, S^\xi) + \tilde{\vartheta}^n \bullet S^\xi \le 1 + \tilde{\vartheta}^n \bullet S^\xi.$$

Therefore, $\liminf_{t\to\infty} \tilde{\vartheta}^n \bullet S^\xi_t = \liminf_{t\to\infty} V_t(\tilde{\vartheta}^n, S^\xi) - V_0(\tilde{\vartheta}^n, S^\xi) \ge nf - 1$ P-a.s. implies that (d) cannot hold as $f \in L^0_+ \setminus \{0\}$.

Finally, for (b) $\Rightarrow$ (c), suppose that (c) is not true. Then also the convex set

$$C := \Big\{ \lim_{t\to\infty} H \bullet S^\xi_t + 1 : H \in L^1_{\mathrm{adm}}(S^\xi),\ H \text{ has bounded support on } [0,\infty) \Big\} \subseteq L^0_+$$

is not bounded in L^0. Lemma A.2 yields a sequence $(H^n)_{n\in\mathbb{N}} \subseteq L^1_{\mathrm{adm}}(S^\xi)$, with each H^n of bounded support on $[0,\infty)$, and some $f \in L^0_+ \setminus \{0\}$ with $\lim_{t\to\infty} H^n \bullet S^\xi_t + 1 \geq nf$ P-a.s. for all $n \in \mathbb{N}$. Note that the limit exists because H^n has bounded support. Consider the integrand $H^n \in L^1_{\mathrm{adm}}(S^\xi)$. By [10, Theorem 2.14] (and an easy extension to $[\![0,\infty]\!]$), there exists a corresponding $\vartheta^n \in \Theta^{\mathrm{sf}}_+$ with $V(\vartheta^n, S^\xi) - V_0(\vartheta^n, S^\xi) = H^n \bullet S^\xi$, where we can choose $V_0(\vartheta^n, S^\xi) = 1$. Defining $\tilde{\vartheta}^n := \vartheta^n/n \in \Theta^{\mathrm{sf}}_+$ yields

$$V(\tilde{\vartheta}^n, S^\xi) = V(\vartheta^n, S^\xi)/n = (H^n \bullet S^\xi + 1)/n,$$

hence $V_0(\tilde{\vartheta}^n, S^\xi) = 1/n$ and $\liminf_{t\to\infty} V_t(\tilde{\vartheta}^n, S^\xi) = \lim_{t\to\infty}(H^n \bullet S^\xi_t + 1)/n \geq f$ P-a.s. Thus 0 is not svm for S^ξ. □

Our next result is of crucial importance. It is a variant of the key result in Delbaen/Schachermayer [5, Theorem 3.3] and shows that loosely speaking, *value processes expressed in good currency units converge under a weak no-arbitrage assumption.*

Theorem 3.7. *Suppose the zero strategy $0 \in \Theta^{\mathrm{sf}}_+$ is strongly value maximal for S. Then for any $\xi \in \Theta^{\mathrm{sf}}_{++}$ with $V(\xi) \in \mathcal{S}^{\mathrm{unif}}_{++}$ and any $\vartheta \in \Theta^{\mathrm{sf}}_+$, $\lim_{t\to\infty} \vartheta \bullet S^\xi_t$ exists and is finite, P-a.s. In particular, $V_\infty(\vartheta, S^\xi) := \lim_{t\to\infty} V_t(\vartheta, S^\xi)$ exists and is finite, P-a.s.*

Proof. Fix ξ as above and $H \in L^1_{\mathrm{adm}}(S^\xi)$. We first claim that $\lim_{t\to\infty} H \bullet S^\xi_t$ exists and is finite, P-a.s. This follows from upcrossing arguments as in Doob's martingale convergence theorem and is based on the proof of [5, Theorem 3.3]. Indeed, by Proposition 3.6, the strong value maximality for S of 0 implies that the set

$$\Big\{ \lim_{t\to\infty} H \bullet S^\xi_t : H \in L^1_{\mathrm{adm}}(S^\xi),\ H \text{ has bounded support on } [0,\infty) \Big\}$$

is bounded in L^0, so that the conclusion of [5, Proposition 3.1] holds. A careful look at [5, Proposition 3.2 and Theorem 3.3] shows that all we need for the proofs of these results is the conclusion of [5, Proposition 3.1]. So we can repeat the proof of [5, Theorem 3.3] step by step[1] to obtain our auxiliary claim about the convergence of $H \bullet S^\xi$. (This uses that $V(\xi)$ is S-tradable because $\xi \in \Theta^{\mathrm{sf}}_{++}$ has the value process $V(\xi, S^\xi) = \xi \cdot S^\xi \equiv 1$.)

To prove Theorem 3.7, we now fix $\vartheta \in \Theta^{\mathrm{sf}}_+$, set $v_0 := V_0(\vartheta, S^\xi)$ and define the strategy $\tilde{\vartheta} := I_{\{v_0 \neq 0\}} \vartheta/v_0 + I_{\{v_0 = 0\}}(\vartheta + \xi)$. Then $\tilde{\vartheta}$ is in Θ^{sf}_+, and as $V(\xi, S^\xi) \equiv 1$,

$$V(\tilde{\vartheta}, S^\xi) = I_{\{v_0 \neq 0\}} V(\vartheta, S^\xi)/v_0 + I_{\{v_0 = 0\}}\Big(V(\vartheta, S^\xi) + 1\Big).$$

This yields $V_0(\tilde{\vartheta}, S^\xi) = 1$ and hence $V(\tilde{\vartheta}, S^\xi) = 1 + \tilde{\vartheta} \bullet S^\xi$. Because $\tilde{\vartheta} \in \Theta^{\mathrm{sf}}_+$, this shows that $\tilde{\vartheta} \in L^1_{\mathrm{adm}}(S^\xi)$ so that $\lim_{t\to\infty} V_t(\tilde{\vartheta}, S^\xi) = \lim_{t\to\infty}(1 + \tilde{\vartheta} \bullet S^\xi_t)$ exists and is finite, P-a.s. The result for $\vartheta = v_0\tilde{\vartheta} + I_{\{v_0=0\}}(\tilde{\vartheta} - \xi)$ then directly follows. □

[1] There are two minor unclear points or typos in the original proof in [5]. First, a set $A_2 \in \mathcal{F}_{t_2}$ such that $P[A_2 \Delta (B_1 \cap A)] > \alpha - \varepsilon_1 - \varepsilon_2$ is not a good approximation for $B_1 \cap A$; one should rather impose the requirement that $P[A_2 \Delta (B_1 \cap A)] < \varepsilon_2/2$. Second, it is not clear why $P[B_1 \cap A] > \alpha - \varepsilon_1$ should be true. However, it is clear that $P[B_1 \cap A] > \alpha - 2\varepsilon_1$, which is still sufficient to obtain the conclusion.

Remark 3.8. Both Proposition 3.6 and Theorem 3.7 are formulated for ξ-discounted prices; so the discounter $\xi \cdot S = V(\xi)$ is S-tradable. One can ask if $V(\xi)$ could be replaced by an arbitrary $D \in \mathcal{S}^{\mathrm{unif}}_{++}$, and hence S^ξ by S/D. This is possible in the first, but not in the second result; if we take for example $D_t = 2 + \sin t$ which is in $\mathcal{S}^{\mathrm{unif}}_{++}$ but does not converge, then $V(\vartheta, S^\xi/D) = V(\vartheta, S^\xi)/D$ also does not converge.

The significance of Theorem 3.7 is that under its assumptions, the limit $V_\infty(\vartheta, S^\xi)$ exists P-a.s. for all $\vartheta \in \Theta^{\mathrm{sf}}_+$. So $V(\vartheta, S^\xi)$ is defined on the *closed* interval $[0,\infty]$, and as $V(\xi, S^\xi) \equiv 1$, the model S^ξ is on $[0,\infty]$ a numéraire market in the sense of [10]. Hence *in the setting of Theorem 3.7, the situation is as if we had S^ξ defined up to* ∞, and so we can essentially use all results from [10] also for $[\![0,\infty]\!]$. More precisely, as long as we only use value processes of strategies in Θ^{sf}_+, we do not need S^ξ itself to be defined on $[0,\infty]$.

An important consequence is that the same weak AOA condition as above allows to improve any self-financing strategy asymptotically by a strongly value maximal strategy at no extra cost. This extends a result from Herdegen [10, Theorem 4.1] to $[\![0,\infty]\!]$.

Lemma 3.9. *Suppose the zero strategy $0 \in \Theta^{\mathrm{sf}}_+$ is strongly value maximal for S and there exists a $\xi \in \Theta^{\mathrm{sf}}_{++}$ with $V(\xi) \in \mathcal{S}^{\mathrm{unif}}_{++}$. Then for any $\vartheta \in \Theta^{\mathrm{sf}}_+$, there exists a $\hat{\vartheta} \in \Theta^{\mathrm{sf}}_+$ which is strongly value maximal for S and satisfies*

$$V_0(\hat{\vartheta}) = V_0(\vartheta) \qquad \text{and} \qquad \liminf_{t\to\infty} V_t(\hat{\vartheta} - \vartheta) \geq 0 \quad P\text{-a.s.}$$

Proof. Fix ξ as above. By Lemma 3.1, svm for S is the same as svm for S^ξ. For any $\vartheta \in \Theta^{\mathrm{sf}}_+$, the limit $V_\infty(\vartheta, S^\xi)$ exists and is finite, P-a.s., by Theorem 3.7. For S^ξ instead of S, we can thus replace the lim inf in Definition 2.6 by a limit, and so our strong value maximality for S^ξ is equivalent to strong maximality of S^ξ on $[0,\infty]$ in the sense of [10]. In particular, having 0 svm for S^ξ is equivalent to having NINA on $[0,\infty]$ for S^ξ in the sense of [10]. So we can use [10, Theorem 4.1] on $[0,\infty]$ for S^ξ, and the assertion follows. $\square$

3.3 Dual characterisation of strong value maximality

In this section, we provide dual characterisations of strong value maximality for S, of the zero strategy 0 or of the reference strategy η. This uses the results of Herdegen [10] and extends them to a general time horizon by exploiting Section 3.2.

Proposition 3.10. *If there exists a $\xi \in \Theta^{\mathrm{sf}}_{++}$ with $V(\xi) \in \mathcal{S}^{\mathrm{unif}}_{++}$, the following are equivalent:*

(a) *ξ is strongly value maximal for S.*

(b) *ξ is strongly value maximal for S^ξ.*

(c) *Both $NA_\infty(S^\xi)$ and $NUPBR_\infty(S^\xi)$ hold.*

(d) *NFLVR*$_\infty(S^\xi)$ *holds.*

Proof. (a) $\Leftrightarrow$ (b) is from Lemma 3.1. Next, both $\mathcal{C}_{\mathrm{adm}}(S^\xi)$ and $\mathcal{C}_{\mathrm{adm}}(S^\xi) \cap L^\infty$ are convex, and $\mathrm{NUPBR}_\infty(S^\xi)$ means that $\mathcal{G}^1_{\mathrm{adm}}(S^\xi)$ is bounded in L^0. Due to (2.1), (c) $\Leftrightarrow$ (d) can thus be proved like [14, Lemma 2.2].

Both (a) and (d) imply that $0 \in \Theta^{\mathrm{sf}}_+$ is svm for S; indeed, under (a), this follows by Lemma 3.1, and under (d), we combine (d) $\Rightarrow$ (c) with Proposition 3.6. Theorem 3.7 and the subsequent discussion thus allow us to treat S^ξ as if it were defined on $[0,\infty]$, and then the proof of [10, Proposition 3.24, (c)], with T replaced by ∞, gives the conclusion. □

Recall that for $\mathcal{E} \in \{\sigma\text{-martingale, local martingale, martingale, UI martingale}\}$, an $\mathcal{E}$*-discounter* for an $\mathbb{R}^N$-valued semimartingale $\tilde{S}$ is a $D \in \mathcal{S}_{++}$ such that $\tilde{S}/D$ is an $\mathcal{E}$.

Theorem 3.11. *If there exists a* $\xi \in \Theta^{\mathrm{sf}}_{++}$ *with* $V(\xi) \in \mathcal{S}^{\mathrm{unif}}_{++}$, *the following are equivalent:*

(a) *The zero strategy* $0 \in \Theta^{\mathrm{sf}}_+$ *is strongly value maximal for* S.

(b) *There exists a strategy* $\hat{\vartheta} \in \Theta^{\mathrm{sf}}_{++}$ *which is strongly value maximal for* S *and has* $V(\hat{\vartheta}) \in \mathcal{S}^{\mathrm{unif}}_{++}$.

(c) *There exists a narrow* σ*-martingale discounter* $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ *for* S.

More precisely, (c) $\Rightarrow$ (a) *also holds without the existence of such a* ξ.

Proof. (a) $\Rightarrow$ (b) By Lemma 3.9, we can find a $\hat{\vartheta} \in \Theta^{\mathrm{sf}}_+$ which is svm for S and satisfies $\liminf_{t\to\infty} V_t(\hat{\vartheta} - \xi, S^\xi) \geq 0$ P-a.s. Superadditivity of the lim inf plus $V(\xi, S^\xi) \equiv 1$ yields

$$\liminf_{t\to\infty} V_t(\hat{\vartheta}, S^\xi) \geq \liminf_{t\to\infty} V_t(\hat{\vartheta} - \xi, S^\xi) + \liminf_{t\to\infty} V_t(\xi, S^\xi) \geq 1 > 0 \qquad P\text{-a.s.}$$

But Theorem 3.7 and the subsequent discussion allow us to treat S^ξ as if it were defined up to ∞, and so $\inf_{t\geq 0} V_t(\hat{\vartheta}, S^\xi) > 0$ P-a.s. follows as in the proof of [10, Proposition 4.4], with T there replaced by ∞. On the other hand, $\limsup_{t\to\infty} V_t(\hat{\vartheta}, S^\xi) = \lim_{t\to\infty} V_t(\hat{\vartheta}, S^\xi) < \infty$ P-a.s. by Theorem 3.7, and because $V(\hat{\vartheta}, S^\xi) = V_0(\hat{\vartheta}, S^\xi) + \hat{\vartheta} \bullet S^\xi$ is RCLL, this implies $\sup_{t\geq 0} V_t(\hat{\vartheta}, S^\xi) < \infty$ P-a.s. Hence $V(\hat{\vartheta}, S^\xi)$ is in $\mathcal{S}^{\mathrm{unif}}_{++}$ and so is $V(\hat{\vartheta}) = V(\hat{\vartheta}, S^\xi) V(\xi)$.

(b) $\Rightarrow$ (c) By Proposition 3.10, $\mathrm{NFLVR}_\infty(S^\xi)$ holds. Note that $V(\xi, S^\xi) \equiv 1$. By the discussion after [10, Definition 2.18], we can apply [8, Theorem 1.1] to the price process $(1, X) := (V(\xi, S^\xi), S^\xi)$ of dimension $1 + N$, and so there exists a probability measure $Q \approx P$ (on $\mathcal{F} \supseteq \mathcal{F}_\infty$) such that S^ξ is a σ-martingale under Q. The density process Z of Q with respect to P is in $\mathcal{S}^{\mathrm{unif}}_{++}$ as it is a strictly positive P-martingale on the *closed* interval $[0,\infty]$. Thus also $D := V(\xi)/Z$ is in $\mathcal{S}^{\mathrm{unif}}_{++}$, and $S/D = ZS^\xi$ is a σ-martingale under P by the Bayes rule. (In classic terminology, Z is a σ-martingale deflator for S^ξ.)

(c) $\Rightarrow$ (a) Because $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ and strong value maximality is discounting-invariant with respect to $\mathcal{S}^{\mathrm{unif}}_{++}$ by Lemma 3.1, we can equivalently prove svm of 0 for S or for S/D.

Hence we can and do assume without loss of generality that S is a P-σ-martingale. If 0 is not svm for S, we can find $f \in L^0_+ \setminus \{0\}$ and for every $\varepsilon > 0$ some $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+$ with $V_0(\hat{\vartheta}^\varepsilon) \le \varepsilon$ and $\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon) \ge f$ P-a.s. Because $\hat{\vartheta}^\varepsilon \bullet S = V(\hat{\vartheta}^\varepsilon) - V_0(\hat{\vartheta}^\varepsilon) \ge -\varepsilon$ on $[0,\infty)$ P-a.s., the Ansel–Stricker lemma [1, Corollary 3.5] implies that $V(\hat{\vartheta}^\varepsilon)$ is a local P-martingale and a P-supermartingale. Combining this with Fatou's lemma and $f \in L^0_+ \setminus \{0\}$ yields

$$\varepsilon \ge V_0(\hat{\vartheta}^\varepsilon) \ge \liminf_{t\to\infty} E[V_t(\hat{\vartheta}^\varepsilon)] \ge E\Big[\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon)\Big] \ge E[f] > 0$$

for every $\varepsilon > 0$, which is a contradiction. This argument does not need the existence of a $\xi \in \Theta^{\mathrm{sf}}_{++}$ with $V(\xi) \in \mathcal{S}^{\mathrm{unif}}_{++}$. □

Theorem 3.12. *Suppose that $S \ge 0$ and there exists a reference strategy η with both η and $S^\eta = S/V(\eta)$ bounded (uniformly in ω, t) and $V(\eta) \in \mathcal{S}^{\mathrm{unif}}_{++}$. Fix such an η. Then the following are equivalent:*

(a) *η is strongly value maximal for S.*

(b) *There exists a UI martingale discounter $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ for S with $\eta \cdot (S/D) \in \mathcal{S}^{\mathrm{unif}}_{++}$.*

(c) *There exists a UI martingale discounter $D \in \mathcal{S}^{\mathrm{unif}}_{++}$ for S.*

(d) *Each bounded $\vartheta \in \Theta^{\mathrm{sf}}_+$ is strongly value maximal for S.*

Proof. Both (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (a) are clear.

(a) $\Rightarrow$ (b) If η is svm for S, the same argument as in the proof of (b) $\Rightarrow$ (c) in Theorem 3.11 yields a $Q \approx P$ such that S^η is a σ-martingale under Q. Being uniformly bounded, S^η is even a UI martingale under Q, and so the same $D := V(\eta)/Z$ as in the proof of Theorem 3.11 is now a UI martingale discounter for S^η. Moreover, Z is in $\mathcal{S}^{\mathrm{unif}}_{++}$ and $S/D = ZS^\eta$. Because $\eta \cdot S^\eta \equiv 1$, $\eta \cdot (S/D) = Z$ is in $\mathcal{S}^{\mathrm{unif}}_{++}$.

(c) $\Rightarrow$ (d) By Theorem 3.11, 0 is svm for S. Take any bounded $\vartheta \in \Theta^{\mathrm{sf}}_+$. To show that ϑ is svm for S, as in the proof of (c) $\Rightarrow$ (a) in Theorem 3.11, we can assume that S is a UI martingale; so $S_\infty = \lim_{t\to\infty} S_t$ exists P-a.s. and in L^1, and then S is a martingale on $[0,\infty]$. Moreover, $V(\vartheta) = \vartheta \cdot S = V_0(\vartheta) + \vartheta \bullet S$ is a nonnegative σ-martingale, hence a local martingale and a supermartingale by [1, Corollary 3.5], and thus P-a.s. convergent as $t \to \infty$. For any stopping time τ, we have $|V_\tau(\vartheta)| \le \|\vartheta\|_\infty \sum_{i=1}^N |S^i_\tau|$, and the UI property of S on $[0,\infty]$ implies that $V(\vartheta)$ is of class (D). So $V(\vartheta)$ is even a UI martingale.

If ϑ is not svm for S, we can find $f \in L^0_+ \setminus \{0\}$ and for every $\varepsilon > 0$ some $\hat{\vartheta}^\varepsilon \in \Theta^{\mathrm{sf}}_+$ with $V_0(\hat{\vartheta}^\varepsilon) \le V_0(\vartheta) + \varepsilon$ and $\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon - \vartheta) \ge f$ P-a.s. As $\lim_{t\to\infty} V_t(\vartheta)$ exists, we even have $\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon) \ge \lim_{t\to\infty} V_t(\vartheta) + f$ P-a.s., and $V(\hat{\vartheta}^\varepsilon)$ is a supermartingale by the same argument as for ϑ. Combining this with Fatou's lemma, the UI martingale property of $V(\vartheta)$ and $f \in L^0_+ \setminus \{0\}$ then gives a contradiction because for every $\varepsilon > 0$,

$$\begin{aligned} V_0(\vartheta) + \varepsilon \ge V_0(\hat{\vartheta}^\varepsilon) &\ge \liminf_{t\to\infty} E[V_t(\hat{\vartheta}^\varepsilon)] \ge E\Big[\liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon)\Big] \\ &\ge E\Big[\lim_{t\to\infty} V_t(\vartheta)\Big] + E[f] = \lim_{t\to\infty} E[V_t(\vartheta)] + E[f] = V_0(\vartheta) + E[f] > V_0(\vartheta). \end{aligned}$$

□

Remark 3.13. **1)** A closer look at the proof shows that we do not need that $\eta \geq 0$.

2) Both Theorems 3.11 and 3.12 need a ξ (or η) in $\Theta^{\mathrm{sf}}_{++}$ with $V(\xi)$ (resp. $V(\eta)$) in $\mathcal{S}^{\mathrm{unif}}_{++}$. It is precisely the idea of replacing value maximality by share maximality which allows us to eliminate this restrictive condition and hence consider general models for S.

4 Proofs

In this section, we prove the main results from Section 2.

Proof of Theorem 2.11. We often use that $V(\eta) = \eta \cdot S \in \mathcal{S}_{++}$ as $\eta \in \Theta^{\mathrm{sf}}_{++}$ is a reference strategy, and $S^\eta = S/V(\eta)$. Moreover, $V(\eta, S^\eta) = \eta \cdot S^\eta \equiv 1$ is in $\mathcal{S}^{\mathrm{unif}}_{++}$.

1) By discounting-invariance with respect to $\mathcal{S}_{++}$, if S satisfies DSV for η, so does S^η so that $0 \in \Theta^{\mathrm{sf}}_{+}$ is ssm for η in the model S^η. So Theorem 3.4, 1) used for S^η and $\xi = \eta$ implies that 0 is also svm for S^η. Theorem 3.11, (a) $\Rightarrow$ (b), applied to S^η and $\xi = \eta$ then gives the existence of a $D' \in \mathcal{S}^{\mathrm{unif}}_{++}$ such that S^η/D' is a σ-martingale. Writing $S^\eta/D' = S/((\eta \cdot S)D') = S/D$, we see that $D := (\eta \cdot S)D' \in \mathcal{S}_{++}$ is a σ-martingale discounter for S. Moreover, like D', $\eta \cdot (S/D) = (\eta \cdot S^\eta)/D' = 1/D'$ is in $\mathcal{S}^{\mathrm{unif}}_{++}$, and in particular, $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s. This argument does not need $S \geq 0$.

2) If $D \in \mathcal{S}_{++}$ is a σ-martingale discounter, $\tilde{S} := S/D$ is a σ-martingale. By [1, Corollary 3.5], $0 \leq V(\eta, \tilde{S}) = V_0(\eta, \tilde{S}) + \eta \bullet \tilde{S}$ is a P-supermartingale so that $\lim_{t\to\infty} V_t(\eta, \tilde{S})$ exists and is finite, P-a.s. This yields $\sup_{t\geq 0}(\eta_t \cdot \tilde{S}_t) < \infty$ P-a.s., and because we also have $\inf_{t\geq 0}(\eta_t \cdot \tilde{S}_t) > 0$ P-a.s. by assumption, we obtain $V(\eta, \tilde{S}) = \eta \cdot (S/D) \in \mathcal{S}^{\mathrm{unif}}_{++}$. Now $D' \equiv 1 \in \mathcal{S}^{\mathrm{unif}}_{++}$ is a narrow σ-martingale discounter for $\tilde{S}$, and so Theorem 3.11 applied to $\tilde{S}$ and $\xi = \eta$ implies that 0 is svm for $\tilde{S}$. By Theorem 3.4, 2) for $\tilde{S}$ and $\xi = \eta$, 0 is then ssm for η in the model $\tilde{S}$, and hence also in the model $S = \tilde{S}D$ because strong share maximality is discounting-invariant with respect to $\mathcal{S}_{++}$. So S satisfies DSV for η. □

Proof of Theorem 2.13. This is very similar to the proof of Theorem 2.11, with the main difference that we use Theorem 3.12 instead of Theorem 3.11.

1) If S satisfies DSE for η, so does S^η so that every η-buy-and-hold $\vartheta \in \Theta^{\mathrm{sf}}_{+}$ and in particular the reference strategy η is ssm for η and hence svm for S^η by the same argument as for Theorem 2.11, 1). Moreover, η and S^η are bounded by assumption, and so Theorem 3.12, (a) $\Rightarrow$ (b), applied to S^η yields the existence of some $D' \in \mathcal{S}^{\mathrm{unif}}_{++}$ such that S^η/D' is a UI martingale. As before, $D := (\eta \cdot S)D'$ is then a UI martingale discounter for S, and we also again get $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s.

2) If S admits a UI martingale deflator D as in the assertion and we set $\tilde{S} := S/D$, then $V(\eta, \tilde{S}) \in \mathcal{S}^{\mathrm{unif}}_{++}$ as before. As η and S^η are bounded by assumption, Theorem 3.12 applied to $\tilde{S}$ then yields that each bounded $\vartheta \in \Theta^{\mathrm{sf}}_{+}$ is svm for $\tilde{S}$. But every η-buy-and-hold $\vartheta \in \Theta^{\mathrm{sf}}_{+}$ is bounded like η itself, hence svm for $\tilde{S}$ and then ssm for η in the model S, as before. Thus S satisfies DSE for η. □

For ease of reference in the proof, we repeat the statement of Theorem 2.14 here.

Theorem 2.14. *Suppose $S \geq 0$ and there exists a reference strategy η. Fix η. Consider the following statements:*

(e1) *S satisfies dynamic share efficiency for η.*

(v1) *S satisfies dynamic share viability for η.*

(e2) *The reference strategy $\eta \in \Theta_+^{\mathrm{sf}}$ is strongly share maximal for η.*

(e2′) *Every bounded $\vartheta \in \Theta_+^{\mathrm{sf}}$ is strongly share maximal for η.*

(v2) *There exists some $\vartheta \in \Theta_+^{\mathrm{sf}}$ which is strongly share maximal for η.*

(e3) *η is strongly value maximal for $S^\eta = S/(\eta \cdot S)$.*

(v3) *0 is strongly value maximal for $S^\eta = S/(\eta \cdot S)$.*

(e4) *For every $D \in \mathcal{S}_{++}$ with $\eta \cdot (S/D) \in \mathcal{S}_{++}^{\mathrm{unif}}$, every bounded $\vartheta \in \Theta_+^{\mathrm{sf}}$ is strongly value maximal for the D-discounted price process S/D.*

(e4′) *Every bounded $\vartheta \in \Theta_+^{\mathrm{sf}}$ is strongly value maximal for S^η.*

(v4) *For every $D \in \mathcal{S}_{++}$ with $\eta \cdot (S/D) \in \mathcal{S}_{++}^{\mathrm{unif}}$, the zero strategy $0 \in \Theta_+^{\mathrm{sf}}$ is strongly value maximal for the D-discounted price process S/D.*

(e5) *There exists a UI martingale discounter D for S with $\eta \cdot (S/D) \in \mathcal{S}_{++}^{\mathrm{unif}}$.*

(v5) *There exists a σ-martingale discounter D for S with $\eta \cdot (S/D) \in \mathcal{S}_{++}^{\mathrm{unif}}$.*

(e6) *NFLVR$_\infty(S^\eta)$ holds, i.e., S^η satisfies NFLVR$_\infty$.*

(v6) *NUPBR$_\infty(S^\eta)$ holds, i.e., S^η satisfies NUPBR$_\infty$.*

Then we have (eK) $\Rightarrow$ (vK) *for* $\mathrm{K} = 1, \dots, 6$, *and the statements* (vK), $\mathrm{K} = 1, \dots, 6$, *are equivalent among themselves. If in addition η and S^η are bounded (uniformly in ω, t), then also the statements* (eK), $\mathrm{K} = 1, \dots, 6$, *are equivalent among themselves (including the prime $'$ versions).*

Proof. While we need $\eta \in \Theta_{++}^{\mathrm{sf}}$ at once to define S^η, $S \geq 0$ is used only in some implications. We structure the proof to make this apparent and initially only assume that there exists a reference strategy η; so $\eta \geq 0$.

It is clear from the statements or definitions that (eK) $\Rightarrow$ (vK) holds for $\mathrm{K} = 1, \dots, 5$.

Because DSV for η means that 0 is ssm for η, (v1) $\Rightarrow$ (v2) is clear. By Theorem 3.4, 1) applied to S^η and $\xi = \eta$, which has $V(\eta, S^\eta) = \eta \cdot S^\eta \equiv 1 \in \mathcal{S}_{++}^{\mathrm{unif}}$, any $\vartheta \in \Theta_+^{\mathrm{sf}}$ which is ssm for η is also svm for S^η, and so we get (v2) $\Rightarrow$ (v3) from Lemma 3.1. Next, (v1) $\Rightarrow$ (v5) is the "only if" part of Theorem 2.11, and (v4) $\Rightarrow$ (v3) holds because $D := V(\eta) \in \mathcal{S}_{++}$ and $\eta \cdot (S/D) = \eta \cdot S^\eta \equiv 1 \in \mathcal{S}^{\mathrm{unif}}_{++}$. Finally, (v1) implies that 0 is ssm for η in the model S/D, by discounting-invariance, so that (v4) follows by applying Theorem 3.4, 1) to S/D and $\xi = \eta$, and (v3) $\Leftrightarrow$ (v6) is Proposition 3.6 applied to S^η instead of S and with $\xi = \eta$, noting that $(S^\eta)^\eta = S^\eta/(\eta \cdot S^\eta) = S^\eta$.

If $S \geq 0$, (v3) $\Rightarrow$ (v1) follows from Theorem 3.4, 2) used for S^η and $\xi = \eta$, because η is a reference strategy, and (v5) $\Rightarrow$ (v1) is the "if" part of Theorem 2.11.

DSE for η means that every η-buy-and-hold $\vartheta \in \Theta^{\mathrm{sf}}_+$ is ssm for η. Thus (e1) $\Rightarrow$ (e2) is clear, so is (e2$'$) $\Rightarrow$ (e1) as η is bounded, and (e2) $\Rightarrow$ (e3) is from Theorem 3.4, 1) used for S^η and $\xi = \eta$. Moreover, (e4$'$) $\Rightarrow$ (e3) is argued like (v4) $\Rightarrow$ (v3), (e4) $\Rightarrow$ (e4$'$) is clear by taking $D = V(\eta)$ so that $S/D = S^\eta$ and $\eta \cdot (S/D) \equiv 1$, and (e3) $\Leftrightarrow$ (e6) is Proposition 3.10 applied to S^η instead of S, with $\xi = \eta$.

If $S \geq 0$ holds, (e1) $\Leftrightarrow$ (e5) is Theorem 2.13. If (e3) holds, every bounded $\vartheta \in \Theta^{\mathrm{sf}}_+$ is first svm for S^η by Theorem 3.12 applied for S^η, and then by Theorem 3.4, 2) used for S^η and $\xi = \eta$ also ssm for η, so that we get (e3) $\Rightarrow$ (e2$'$). Finally, because $\eta \cdot (S/D) \in \mathcal{S}^{\mathrm{unif}}_{++}$ and $S/D = S^\eta(\eta \cdot S)/D = S^\eta(\eta \cdot (S/D))$, every such ϑ is also svm for S/D by Lemma 3.1. This gives (e3) $\Rightarrow$ (e4) and completes the proof. $\square$

5 Extensions and connections

This section has three parts. We first discuss to which extent our approach and results are robust with respect to the choice of a reference strategy. We then connect our work to the classic theory, and finally provide a comparison to the existing literature.

5.1 Robustness towards the choice of a reference strategy

As already pointed out in Remark 2.8, 2), our concepts and main results depend on the choice of a reference strategy η. In this section, we show that this dependence is fairly weak, which means that our approach is quite robust towards the choice of η.

Consider two reference strategies η, η'; so both are in $\Theta^{\mathrm{sf}}_{++}$ and ≥ 0, with $S^\eta = S/(\eta \cdot S)$ and $S^{\eta'}$ both bounded uniformly in $t \geq 0$, P-a.s. We also consider the *ratio condition*

$$(\eta' \cdot S)/(\eta \cdot S) = V(\eta')/V(\eta) \in \mathcal{S}^{\mathrm{unif}}_{++}, \text{ i.e.,}$$

$$0 < \inf_{t \geq 0} \Big(V_t(\eta')/V_t(\eta) \Big) \leq \sup_{t \geq 0} \Big(V_t(\eta')/V_t(\eta) \Big) < \infty \qquad P\text{-a.s.} \tag{5.1}$$

(As $\mathcal{S}^{\mathrm{unif}}_{++}$ is closed under taking reciprocals, (5.1) is symmetric in η and η'.)

Lemma 5.1. *Suppose $S \geq 0$ and there exist reference strategies η, η'. Fix $\vartheta \in \Theta^{\mathrm{sf}}_+$.*

1) *If (5.1) holds, then ϑ is ssm for η if and only if it is ssm for η'.*

2) *If both η and η' are bounded uniformly in $t \geq 0$, P-a.s., then (5.1) holds.*

Proof. 1) Suppose ϑ is ssm for η. Because $V(\eta, S^\eta) \equiv 1 \in \mathcal{S}_{++}^{\mathrm{unif}}$, Corollary 3.5 applies for S^η and $\xi = \eta$ and implies that ϑ is strongly value maximal for S^η. But $S^{\eta'} = S^\eta / D$ with $D := (\eta' \cdot S)/(\eta \cdot S) \in \mathcal{S}_{++}^{\mathrm{unif}}$ due to (5.1). Thus by Lemma 3.1, ϑ is strongly value maximal for $S^{\eta'}$ as well. Corollary 3.5 applied now for $S^{\eta'}$ and $\xi = \eta'$ yields that ϑ is ssm for η'. The converse is argued symmetrically.

2) The ratio $(\eta \cdot S)/(\eta' \cdot S) = \eta \cdot S^{\eta'}$ is strictly positive like $V(\eta), V(\eta')$ and bounded uniformly in $t \geq 0$, P-a.s., as both η and $S^{\eta'}$ are. Symmetry in η and η' gives (5.1). $\square$

Corollary 5.2. *Suppose $S \geq 0$ and there exist reference strategies η, η'.*

1) *If* (5.1) *holds, then DSV for η and DSV for η' are equivalent.*

2) *If η, η' as well as $S^\eta, S^{\eta'}$ are bounded (uniformly in ω, t), then DSE for η and DSE for η' are equivalent.*

Proof. 1) Apply Lemma 5.1, 1) to $\vartheta \equiv 0$.

2) By discounting-invariance, DSE and ssm are the same in the models S or S^η. If we have DSE for η, every bounded $\vartheta \in \Theta_+^{\mathrm{sf}}$ and in particular η' is ssm for η by Theorem 2.14, (e1) $\Rightarrow$ (e2′). By Lemma 5.1, 2) and then 1), η' is thus also ssm for η', and so Theorem 2.14, (e2) $\Rightarrow$ (e1), gives DSE for η'. The converse argument is symmetric. $\square$

The assumptions in part 2) of Corollary 5.2 are precisely those we impose in Theorem 2.13 to obtain a dual characterisation for DSE. So DSE is robust with respect to the choice of any reference strategy in that class.

Remark 5.3. Suppose $S \geq 0$ and $\sum_{i=1}^N S^i$ is strictly positive with strictly positive left limits. As seen in Remark 2.3, the market portfolio $\mathbb{1}$ is then a reference strategy with $\mathbb{1}$ and $S^{\mathbb{1}} = S / \sum_{i=1}^N S^i$ bounded (uniformly in ω, t). Any $\eta \in \Theta_+^{\mathrm{sf}}$ with $c\mathbb{1} \leq \eta \leq C\mathbb{1}$ for constants $0 < c \leq C < \infty$ is then also a reference strategy with η and S^η bounded (uniformly in ω, t); indeed, $\eta \cdot S \geq c\mathbb{1} \cdot S$ and hence $S^\eta \leq \frac{1}{c} S^{\mathbb{1}}$ (coordinatewise). In view of Corollary 5.2, 2), DSE is therefore the same for the market portfolio as for any bounded reference strategy which always invests in a uniformly nondegenerate way into all assets. An "extreme" strategy like e^i, buy and hold a single fixed asset i, does not satisfy this.

The result for DSV is even better — this criterion is robust towards the choice of any reference strategies η, η' satisfying the ratio condition (5.1). We can say a bit more.

Lemma 5.4. *Suppose $S \geq 0$ and there exists a reference strategy η. Fix η. If DSV for η holds, then $(\vartheta \cdot S)/(\eta \cdot S)$ is bounded in $t \geq 0$, P-a.s., for every $\vartheta \in \Theta_+^{\mathrm{sf}}$.*

Proof. By Theorem 2.14, 0 is svm for S^η. Now note that $(\vartheta \cdot S)/(\eta \cdot S) = V(\vartheta, S^\eta)$ and apply Theorem 3.7 for S^η and $\xi = \eta$, which has $V(\xi, S^\eta) \equiv 1 \in \mathcal{S}_{++}^{\mathrm{unif}}$ and $(S^\eta)^\xi = S^\eta$. $\square$

Corollary 5.5. *Suppose $S \geq 0$ and there exist reference strategies η, η'. If we have both DSV for η and DSV for η', then* (5.1) *holds.*

Proof. Apply Lemma 5.4 with the pairs $(\eta, \vartheta = \eta')$ and $(\eta', \vartheta = \eta)$. $\square$

Combining Corollary 5.2, 1) and Corollary 5.5 shows that we can interpret the ratio condition (5.1) as saying that η and η' are comparable in some sense. If (5.1) holds, either both or none of η, η' yield DSV. If we have DSV for one of the two reference strategies, then DSV for the other holds if and only if (5.1) holds. If (5.1) does not hold, DSV must fail for at least one of η or η', but we cannot tell whether for both or only for one. Example 6.9 shows that both cases can occur.

5.2 Connections to the classic results

Theorem 2.14 indicates that in some way, DSV is connected to NUPBR, and DSE to NFLVR. In this section, we study this in more detail in the classic setup $S = (1, X)$. Because our results use the condition $S \geq 0$, we also impose $X \geq 0$.

Classic NUPBR for X is the same as $\mathrm{NUPBR}_\infty(1, X)$, with $(1, X) = S = S^{\mathrm{e}^1}$. From Theorem 2.14, we expect that this is equivalent to DSV for $\eta \equiv \mathrm{e}^1$, but this is not clear; in general, S^{e^1} is not bounded uniformly in $t \geq 0$, P-a.s., so that e^1 need not be a reference strategy. But in any case, using one of the e^i as a potential "reference strategy" is quite extreme. For instance, in Example 1.1 with two GBMs as asset prices, there is no reason to prefer either $S^{\mathrm{e}^1} = (1, S^2/S^1)$ or $S^{\mathrm{e}^2} = (S^1/S^2, 1)$. It seems there much more natural to use the market portfolio $\eta \equiv \mathbb{1}$, and in a setup with $S = (1, X)$ and $X \geq 0$, this is a proper reference strategy. So we can ask how classic NUPBR and NFLVR for X relate to DSV and DSE for $\mathbb{1}$.

Proposition 5.6. *If $S = (1, X)$ for an $\mathbb{R}^d_+$-valued semimartingale X, classic NUPBR for X implies that S satisfies dynamic share viability for $\mathbb{1}$.*

Proof. Because $V(\mathrm{e}^1) = V(\mathrm{e}^1, (1, X)) \equiv 1 \in \mathcal{S}^{\mathrm{unif}}_{++}$, Proposition 3.6 with $\xi \equiv \mathrm{e}^1$ implies that $\mathrm{NUPBR}_\infty(1, X)$ is equivalent to 0 being svm for S. This yields by Theorem 3.7 with $\xi \equiv \mathrm{e}^1$ and $S^\xi = S$ that $V_\infty(\vartheta) = \lim_{t \to \infty} V_t(\vartheta)$ exists and is finite, P-a.s., for any $\vartheta \in \Theta^{\mathrm{sf}}_+$. So we obtain $0 \leq \inf_{t \geq 0} V_t(\vartheta) \leq \sup_{t \geq 0} V_t(\vartheta) < \infty$ P-a.s. for any $\vartheta \in \Theta^{\mathrm{sf}}_+$, and $X \geq 0$ allows us to choose $\vartheta \equiv (0, \mathrm{e}^i_d) \in \mathbb{R}^{d+1}$ for any $\mathrm{e}^i_d \in \mathbb{R}^d$, $i = 1, \ldots, d$, and get

$$0 \leq \inf_{t \geq 0} X^i_t \leq \sup_{t \geq 0} X^i_t < \infty \qquad P\text{-a.s., } i = 1, \ldots, d.$$

Thus $0 < 1 \leq \inf_{t \geq 0} V_t(\mathbb{1}) \leq \sup_{t \geq 0} V_t(\mathbb{1}) < \infty$ P-a.s. and so $V(\mathbb{1}) = \mathbb{1} \cdot S = 1 + \sum_{i=1}^d X^i$ is in $\mathcal{S}^{\mathrm{unif}}_{++}$. Part 2) of Theorem 3.4 with $\xi \equiv \mathbb{1}$ therefore implies that 0 is ssm for $\mathbb{1}$, and so S satisfies dynamic share viability for $\mathbb{1}$. $\square$

The converse of Proposition 5.6 is not true in general. A counterexample is given in Example 6.8 c). Thus our new concept of dynamic share viability, when used for the market portfolio $\mathbb{1}$, is more widely applicable than the classic NUPBR concept.

In contrast to DSV and NUPBR, neither of "dynamic share efficiency for $\mathbb{1}$" and "NFLVR for X" implies the other in general in the classic case $S = (1, X)$. Example 6.11 shows that DSE for $\mathbb{1}$ does not imply NFLVR, at least (see Theorem 2.14) not for the same model. Conversely, Example 6.8 e) shows that for $S = (1, X)$, we can have NFLVR for X while DSE for $\mathbb{1}$ fails. This is actually not surprising; in fact, NFLVR is about how e^1 or $V(\mathrm{e}^1)$ fits into the market, whereas dynamic share efficiency for $\mathbb{1}$ looks at all the e^i, $i = 1, \dots, N$. The next result makes this more precise.

Proposition 5.7. *Suppose that $S \geq 0$ and there exist reference strategies η, η'. Then NFLVR$_\infty(S^\eta)$ plus $\inf_{t\geq 0}(\eta'_t \cdot S^\eta_t) > 0$ P-a.s. implies that η is strongly share maximal for η'. In particular, if $S = (1, X)$ with $X \geq 0$, then X satisfies (classic) NFLVR only if e^1 is strongly share maximal for $\mathbb{1}$.*

Proof. The second statement follows from the first by taking $\eta \equiv \mathrm{e}^1, \eta' \equiv \mathbb{1}$ and observing that $\mathbb{1} \cdot S^{\mathrm{e}^1} = 1 + \sum_{i=1}^d X^i \geq 1$. If we have NFLVR$_\infty(S^\eta)$, the discussion after [10, Definition 2.18] allows us to apply the FTAP from [8, Theorem 1.1] to the asset price process $(1, \tilde{X}) := (V(\eta, S^\eta), S^\eta)$ of dimension $1 + N$, and this yields a probability measure $Q \approx P$ such that $\tilde{X} = S^\eta$ is a Q-σ-martingale. The density process D of P with respect to Q is in $\mathcal{S}^{\mathrm{unif}}_{++}$ and S^η/D is a P-σ-martingale by the Bayes rule. Because $\eta' \in \Theta^{\mathrm{sf}}_+$, $\eta' \cdot (S^\eta/D) = V(\eta', S^\eta/D)$ is then a P-σ-martingale ≥ 0, hence a P-supermartingale by [1, Corollary 3.5], and thus P-a.s. convergent as $t \to \infty$. Therefore $\sup_{t\geq 0}(\eta'_t \cdot (S^\eta_t/D_t)) < \infty$ P-a.s. and then also $\sup_{t\geq 0}(\eta'_t \cdot S^\eta_t) < \infty$ P-a.s. because $D \in \mathcal{S}^{\mathrm{unif}}_{++}$. Combining this with $\inf_{t\geq 0}(\eta'_t \cdot S^\eta_t) > 0$ P-a.s. gives $V(\eta', S^\eta) \in \mathcal{S}^{\mathrm{unif}}_{++}$. Proposition 3.10 and then Theorem 3.4, 2), both applied for S^η and $\xi = \eta'$, now imply that η' is svm for S^η and then ssm for η. $\square$

5.3 Comparison to the literature

The two most used classic AOA notions in the literature are NFLVR (due to Delbaen/Schachermayer [5]) and the strictly weaker NUPBR (coined by Karatzas/Kardaras [17]). The latter condition was introduced by different authors under different names — BK in Kabanov [14], no cheap thrills in Loewenstein/Willard [21] or NA1 in Kardaras [19]; see also Kabanov/Kramkov [15]. By [19, Proposition 1], all these and NUPBR are equivalent.

Both NFLVR and NUPBR are classically only defined for discounted price processes of the form $S = (1, X)$. Dual characterisations in terms of martingale properties first focused on NFLVR, culminating in the classic FTAP due to Delbaen/Schachermayer [5, 8] that for a general $\mathbb{R}^d$-valued semimartingale X, NFLVR for $S = (1, X)$ is equivalent to the existence of an equivalent σ-martingale measure for the discounted prices X. The fact that NFLVR depends on the numéraire chosen for discounting was probably first noted in [6] or [25]. Herdegen [10] made this observation more precise; he showed that NFLVR describes a maximality property of the discounting asset, but does not say too much about the market as a whole. Proposition 5.7 extends that to our framework.

Even if NFLVR does not hold, a market can still be sufficiently nice to allow AOA-type arguments. This has been exploited in several papers. Loewenstein/Willard [21] show in an Itô process setup that already no cheap thrills (NUPBR) is sufficient (and necessary) to solve utility maximization problems; see also [3]. In the benchmark approach presented in Platen/Heath [22], a market may violate NFLVR; but in units of the so-called numéraire portfolio, the theory works as if there was no arbitrage. For stochastic portfolio theory and the study of relative arbitrage (see Karatzas/Fernholz [16] for an overview), a market may have "arbitrage" in the sense of FLVR; but portfolio choice still makes sense, and hedging via superreplication can still work. The comprehensive paper of Karatzas/Kardaras [17] shows that maximising growth rate, asymptotic growth or expected logarithmic utility from terminal wealth all make sense if and only if NUPBR holds. Another overview of the above connections can be found in the recent work of Choulli et al. [4].

In addition to the above good properties, NUPBR is also more stable than NFLVR under discounting or changes of numéraire. Proposition 3.6 together with Lemma 3.1 shows that $\mathrm{NUPBR}_\infty(S^\eta)$ is equivalent to $\mathrm{NUPBR}_\infty(S^{\eta'})$ whenever η and η' are reference strategies with $V(\eta)$ and $V(\eta')$ both in $\mathcal{S}^{\mathrm{unif}}_{++}$. But this no longer holds if $V(\eta)$ or $V(\eta')$ are only in $\mathcal{S}_{++} \setminus \mathcal{S}^{\mathrm{unif}}_{++}$; see Examples 1.1 and 3.2 as well as the comment after Proposition 5.6.

Like for NFLVR, the literature contains dual characterisations of NUPBR. Depending on the setting, they vary in the strength of the dual formulation; see Figure 1 for an overview. For $S = (1, X)$ with a d-dimensional semimartingale $X > 0$ on $[0, \infty)$, Karatzas/Kardaras [17] show that NUPBR is equivalent to the existence of an S-tradable supermartingale discounter for all wealth processes of admissible self-financing strategies. On $[0, T]$, this is strengthened by Takaoka/Schweizer [26] to the existence of a σ-martingale discounter for X, where again $S = (1, X)$ but X is an $\mathbb{R}^d$-valued semimartingale. Both Kardaras [19] and Kabanov et al. [13], inspired by the results and a counterexample in [26], work on $[0, T]$ with $S = (1, X)$ for an $\mathbb{R}^d$-valued semimartingale X and characterise NA1 (which is equivalent to NUPBR) by the existence of a local martingale discounter for all wealth processes of admissible self-financing strategies. In [19], this is done for $d = 1$ so that X is real-valued; [13] extend the result to $d \geq 1$ and in addition manage to find an *S-tradable* local martingale discounter under any $R \approx P$ in any neighbourhood of P.

All results above are for the special (discounted) case $S = (1, X)$. This was first dropped by Herdegen [10] who worked on $[0, T]$ with a general $\mathbb{R}^N$-valued semimartingale S. His AOA condition NINA or dynamic (value) viability generalises NUPBR and is dually characterised by the existence of a discounter/EσMM pair (D, Q), meaning that $D \in \mathcal{S}_{++}$ and Q is an equivalent σ-martingale measure for D-discounted prices S/D. Our new concept of dynamic share viability for η extends NINA to the infinite-horizon setting $[0, \infty)$ with prices $S \geq 0$, and its dual characterisation in Theorem 2.11 is the existence of a σ-martingale discounter D for S with $\inf_{t \geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s.

Table 1 gives an overview of the dual characterisation results discussed above. We write $\mathcal{S}^m$ for the space of $\mathbb{R}^m$-valued semimartingales and use $\mathcal{S}^m_+$, $\mathcal{S}^m_{++}$ as in Section 2.

	price process S	time	condition	dual condition
KK [17]	$(1, X) \in \mathcal{S}^{1+d}_{++}$	$[0, \infty)$	NUPBR	$\exists$ S-tradable SMD $D > 0$ for all $H \bullet X$ with $H \in L_{\mathrm{adm}}(X)$, with $D_\infty > 0$
TS [26]	$(1, X) \in \mathcal{S}^{1+d}$	$[0, T]$	NUPBR	$\exists$ σMD $D > 0$ for X
K [19]	$(1, X) \in \mathcal{S}^{1+1}$	$[0, T]$	NA1	$\exists$ LMD $D > 0$ for all $H \bullet X$ with $H \in L_{\mathrm{adm}}(X)$
KKS [13]	$(1, X) \in \mathcal{S}^{1+d}$	$[0, T]$	NA1	$\exists$ S-tradable LMD $D > 0$ for all $H \bullet X$ with $H \in L_{\mathrm{adm}}(X)$, in any neighbourhood of P
H [10]	in $\mathcal{S}^N$	$[0, T]$	NINA	$\exists$ discounter/EσMM pair for S
here	in $\mathcal{S}^N_+$	$[0, \infty)$	DSV for η	$\exists$ LMD $D > 0$ for S with $\inf_{t \geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s.

Table 1: Overview of existing FTAP-type results. Note that NA1 = NUPBR on $[0, T]$.

The abbreviations SMD, σMD and LMD denote super-, σ- and local martingale discounters, respectively. The table compares Karatzas/Kardaras [17], Takaoka/Schweizer [26], Kardaras [19], Kabanov et al. [13], Herdegen [10], and the present article.

The key difference between NUPBR and NINA (= dynamic (value) viability) on $[0, T]$ is that the latter is, by design, stable with respect to discounting or numéraire changes on $[0, T]$. In addition, [10] also presents a discounting-stable alternative to NFLVR. It is called dynamic (value) efficiency and requires that not one particular asset, but each of the N basic assets (or equivalently the market portfolio $\mathbb{1}$) should satisfy (value) maximality. A similar approach was presented in Yan [27] who, without justification, chose to discount with the market wealth $V(\mathbb{1}) = \mathbb{1} \cdot S$ and then verified the validity of known duality results for the resulting discounted model. Very similar ideas and results can also be found earlier in the PhD thesis of Sin [25]. [10, 27, 25] all work on $[0, T]$, and our new concept of dynamic share efficiency for η extends dynamic (value) efficiency to $[0, \infty)$ in a discounting-invariant manner. Section 5.1 discusses the choice of η and shows in Corollary 5.2 and Remark 5.3 that if η uses the whole market in a relevant way, the properties of the $V(\eta)$-discounted model are actually global market properties and do not really depend on η. In particular, this motivates (ex post) the definitions and approach in [27].

To illustrate how our approach yields new results even in the classic case, we first prove

Proposition 5.8. *Suppose there exists an $\eta \in \Theta^{\mathrm{sf}}_{++}$. Then S^η satisfies NUPBR$_\infty$ if and only if there exists a σ-martingale discounter D for S^η with $D_\infty := \lim_{t\to\infty} D_t < \infty$ P-a.s.*

Proof. Because $V(\eta, S^\eta) \equiv 1 \in \mathcal{S}^{\mathrm{unif}}_{++}$, Proposition 3.6 and Theorem 3.11 for S^η and $\xi = \eta$ imply that S^η satisfies NUPBR$_\infty$ if and only if it admits a σ-martingale discounter $D \in \mathcal{S}_{++}$ with the extra property $D \in \mathcal{S}^{\mathrm{unif}}_{++}$. Now D being a σ-martingale discounter is equivalent to S^η/D being a σ-martingale, and therefore

$$1/D = V(\eta, S^\eta)/D = V(\eta, S^\eta/D) = V_0(\eta, S^\eta/D) + \eta \bullet (S^\eta/D)$$

is a σ-martingale. Moreover, $D \in \mathcal{S}_{++}$ implies that $1/D$ is also in $\mathcal{S}_{++}$, hence a local martingale > 0 and a supermartingale ≥ 0 by [1, Corollary 3.5], and therefore P-a.s. convergent to some finite limit. So D is also P-a.s. convergent and $D_\infty > 0$ P-a.s., which implies that $\inf_{t\geq 0} D_t > 0$ P-a.s. The extra property $D \in \mathcal{S}_{++}^{\mathrm{unif}}$ thus holds if and only if $\sup_{t\geq 0} D_t < \infty$ P-a.s. or, equivalently by convergence, $D_\infty < \infty$ P-a.s. □

Corollary 5.9. *Suppose X is an $\mathbb{R}^d$-valued semimartingale. Then X satisfies (classic) NUPBR if and only if there exists a local martingale $L > 0$ with $L_\infty := \lim_{t\to\infty} L_t > 0$ P-a.s. and such that LX is a σ-martingale.*

Proof. For $S = (1, X)$, $\eta := \mathrm{e}^1$ is in $\Theta_{++}^{\mathrm{sf}}$ with $S^\eta = S^{\mathrm{e}^1} = S$. So we can apply Proposition 5.8 and take $L := 1/D$. The properties of L are all shown in the proof above. □

Corollary 5.9 sharpens the classic characterisation of NUPBR in [17, Theorem 4.12] in two ways: We do not need X and X_- to be strictly positive, and we get a σ-martingale deflator for X, not only a supermartingale deflator for all $H \bullet X$ with $H \in L_{\mathrm{adm}}(X)$. The result also extends [26, Theorem 2.6] from a closed interval $[0, T]$ to a general time horizon.

6 Examples

This section illustrates our results by examples and counterexamples. Most are based on variants of Example 1.1, and so we start with a general analysis of that setup.

6.1 General results for a two-GBM setup

For Example 1.1, we can characterise completely, in terms of the model parameters $m_1, m_2, \sigma_1, \sigma_2, \varrho$, when DSV or DSE for $\mathbb{1}$ hold. This needs σ-martingale discounters for S. If D is a σ-martingale discounter for S, then S/D is a σ-martingale > 0 and hence a local martingale > 0. In the filtration generated by (B^1, B^2), all positive local martingales starting at 1 have the form $\mathcal{E}(\xi^1 \bullet B^1 + \xi^2 \bullet B^2)$, and as all coefficients of S are constant, it should be enough to consider only constant processes ξ^1, ξ^2. So we define

$$\mathcal{C} := \{D \in \mathcal{S}_{++} : S^i/D = \mathcal{E}(\alpha_i B^1 + \beta_i B^2) \text{ with constants } \alpha_i, \beta_i,\ i = 1, 2\}.$$

Throughout this section, we consider the setting of Example 1.1. Note that $S_0 = (1, 1)$ implies the normalisation $D_0 = 1$ for any $D \in \mathcal{C}$.

Proposition 6.1. *We always have $\mathcal{C} \neq \emptyset$, and each $D \in \mathcal{C}$ corresponds to a tuple $(\alpha_1, \beta_1, \alpha_2, \beta_2) \in \mathbb{R}^4$ with one free parameter. More precisely, we have the three relations*

$$\alpha_2 = \alpha_1 - \sigma_1\sqrt{1-\varrho^2}, \tag{6.1}$$

$$\beta_2 = \beta_1 - (\varrho\sigma_1 - \sigma_2), \tag{6.2}$$

$$\alpha_1\sigma_1\sqrt{1-\varrho^2} + \beta_1(\varrho\sigma_1 - \sigma_2) = m_2 - m_1 + \sigma_1^2 - \varrho\sigma_1\sigma_2. \tag{6.3}$$

In particular, α_2 and β_2 are always determined from α_1 and β_1, respectively. Moreover:

1) *If $\sigma_1 = 0$, we must take $\alpha_1 = 0$ and $\beta_1 = -\frac{m_2-m_1}{\sigma_2}$. This yields*

$$D_t^{-1} = \mathcal{E}\Big(-\frac{m_2-m_1}{\sigma_2}B^2\Big)_t e^{-m_1 t}, \qquad t \geq 0,$$

which is the well-known state price density for the Black–Scholes model.

2) *If $\sigma_1 > 0$, β_1 can be chosen freely and α_1 is then determined via* (6.3).

Proof. Because $S^i/D = \mathcal{E}(\alpha_i B^1 + \beta_i B^2)$ for $i = 1, 2$ and D is one-dimensional, we have $S^1/\mathcal{E}(\alpha_1 B^1 + \beta_1 B^2) = D = S^2/\mathcal{E}(\alpha_2 B^1 + \beta_2 B^2)$. Plug in (1.1) for S^1, S^2, express W^1, W^2 via B^1, B^2, write out the results and equate the two sides. The coefficients of B^1, B^2, t in the exponents must then coincide, and the resulting three equations yield the claims after straightforward algebra. Note that in case 1), D is not adapted to $\mathbb{F}$ unless $\alpha_1 = 0$. □

Each $D \in \mathcal{C}$ is a local martingale discounter for S in the filtration $\mathbb{F}$. The next result exhibits a particularly useful choice among these.

Proposition 6.2. *There exists a unique S-tradable $\bar{D} \in \mathcal{C}$. In terms of the corresponding parameter tuple from Proposition 6.1, it is given as follows:*

1) *If $\sigma_1 = 0$, then $\bar{\alpha}_1 = 0 = \bar{\alpha}_2$ and*

$$\bar{\beta}_1 = -\frac{m_2 - m_1}{\sigma_2}, \qquad \bar{\beta}_2 = -\frac{m_2 - m_1 - \sigma_2^2}{\sigma_2}. \tag{6.4}$$

2) *If $\sigma_1 > 0$, then*

$$\bar{\alpha}_1\Big(\sigma_1\sqrt{1-\varrho^2} + \frac{(\varrho\sigma_1 - \sigma_2)^2}{\sqrt{1-\varrho^2}}\Big) = m_2 - m_1 + \sigma_1^2 - \varrho\sigma_1\sigma_2, \tag{6.5}$$

$$\bar{\beta}_1 = \bar{\alpha}_1\frac{\varrho\sigma_1 - \sigma_2}{\sqrt{1-\varrho^2}}, \tag{6.6}$$

$$\bar{\alpha}_2\Big(\sigma_1\sqrt{1-\varrho^2} + \frac{(\varrho\sigma_1 - \sigma_2)^2}{\sqrt{1-\varrho^2}}\Big) = m_2 - m_1 - \sigma_2^2 + \varrho\sigma_1\sigma_2, \tag{6.7}$$

$$\bar{\beta}_2 = \bar{\alpha}_2\frac{\varrho\sigma_1 - \sigma_2}{\sqrt{1-\varrho^2}}. \tag{6.8}$$

Proof. For $\bar{D}$ to be S-tradable, we must have $\bar{D} = V(\bar{\vartheta})$ for some $\bar{\vartheta} \in \Theta^{\mathrm{sf}}$. Setting $\bar{S} := S/\bar{D}$, this is equivalent to $1 \equiv V(\bar{\vartheta})/\bar{D} = V(\bar{\vartheta}, S)/\bar{D} = V(\bar{\vartheta}, \bar{S})$ or

$$\bar{\vartheta}_t^1\bar{S}_t^1 + \bar{\vartheta}_t^2\bar{S}_t^2 = 1, \qquad t \geq 0. \tag{6.9}$$

Moreover, the self-financing condition yields also $0 = \mathrm{d}V_t(\bar\vartheta, \bar S) = \bar\vartheta^1_t \, \mathrm{d}\bar S^1_t + \bar\vartheta^2_t \, \mathrm{d}\bar S^2_t$. But $\bar S^i = \mathcal{E}(\bar\alpha_i B^1 + \bar\beta_i B^2)$ yields $\mathrm{d}\bar S^i_t = \bar S^i_t(\bar\alpha_i \, \mathrm{d}B^1_t + \bar\beta_i \, \mathrm{d}B^2_t)$. Plugging this into the self-financing condition and using that B^1, B^2 are independent implies by comparing coefficients that

$$0 = \bar\vartheta^1_t \bar S^1_t \bar\alpha_1 + \bar\vartheta^2_t \bar S^2_t \bar\alpha_2, \tag{6.10}$$

$$0 = \bar\vartheta^1_t \bar S^1_t \bar\beta_1 + \bar\vartheta^2_t \bar S^2_t \bar\beta_2. \tag{6.11}$$

Now use (6.9) to get $\bar\vartheta^1 \bar S^1 = 1 - \bar\vartheta^2 \bar S^2$, plug this into (6.10) and (6.11), and use (6.1) and (6.2) to eliminate $\bar\alpha_2$ and $\bar\beta_2$ and obtain (after simple calculations)

$$\bar\alpha_1 = \bar\vartheta^2 \bar S^2 \sigma_1 \sqrt{1 - \varrho^2}, \tag{6.12}$$

$$\bar\beta_1 = \bar\vartheta^2 \bar S^2 (\varrho\sigma_1 - \sigma_2). \tag{6.13}$$

Because only one of $\bar\alpha_1, \bar\beta_1$ can be chosen freely by Proposition 6.1, there is at most one choice of $\bar D \in \mathcal{C}$ which is S-tradable. For existence of $\bar D$, we consider two cases.

1) If $\sigma_1 = 0$, (6.12) forces $\bar\alpha_1 = 0$, hence $\bar\alpha_2 = 0$ by (6.1), and Proposition 6.1 and (6.2) yield (6.4). Moreover, (6.13) yields for $\bar\vartheta$ the explicit formulas

$$\bar\vartheta^2 \bar S^2 \equiv -\frac{\bar\beta_1}{\sigma_2} = \frac{m_2 - m_1}{\sigma_2^2}, \qquad \bar\vartheta^1 \bar S^1 = 1 - \bar\vartheta^2 \bar S^2 \equiv -\frac{m_2 - m_1 - \sigma_2^2}{\sigma_2^2}. \tag{6.14}$$

2) If $\sigma_1 > 0$, solve (6.12) for $\bar\vartheta^2 \bar S^2$ and plug into (6.13) to get (6.6). Insert this into (6.3) to obtain (6.5). Finally, combine (6.5), (6.1) for (6.7), and (6.2), (6.6), (6.1) for (6.8). □

Remark 6.3. In the BS model with parameters m, r, σ, the proportion of wealth in the stock $\bar S^2$ for the strategy $\bar\vartheta$ is given by $\bar\pi^2 = \bar\vartheta^2 \bar S^2 / V(\bar\vartheta, \bar S) = \bar\vartheta^2 \bar S^2 = \frac{m-r}{\sigma^2}$. This is exactly the strategy which solves the problem of maximising expected logarithmic utility from final wealth. We therefore call $\bar\vartheta$ from (6.14) the *Merton strategy*.

Our main result about Example 1.1 is now

Theorem 6.4. 1) *If $\sigma_1 = 0$, S satisfies DSV for $\mathbb{1}$ if and only if*

$$m_2 - m_1 \in \{0, \sigma_2^2\}. \tag{6.15}$$

In particular, the BS model with parameters m, r, σ satisfies DSV for $\mathbb{1}$ if and only if

$$\frac{m - r}{\sigma^2} \in \{0, 1\}.$$

2) *If $\sigma_1 > 0$, the general GBM model satisfies DSV for $\mathbb{1}$ if and only if*

$$m_i - \sigma_i^2 + \varrho\sigma_1\sigma_2 = m_{3-i} \qquad \textit{for } i = 1 \textit{ or } i = 2. \tag{6.16}$$

3) *S satisfies DSV for $\mathbb{1}$ if and only if one of the two processes $S/S^1 = (1, X)$ or $S/S^2 = (1/X, 1)$ is a martingale.*

4) *S never satisfies DSE for $\mathbb{1}$.*

Proof. Because DSV and DSE are discounting-invariant, we can work with $\bar{S} = S/\bar{D}$ from Proposition 6.2 instead of S. Then $\bar{S}^i = \mathcal{E}(\bar{\alpha}_i B^1 + \bar{\beta}_i B^2) = \mathcal{E}(\bar{\alpha}_i B^1)\mathcal{E}(\bar{\beta}_i B^2)$ and we first look at the two cases for σ_1.

1) If $\sigma_1 = 0$, then $\bar{\alpha}_1 = 0 = \bar{\alpha}_2$ and $\bar{S}^1 = \mathcal{E}(\bar{\beta}_1 B^1)$ and $\bar{S}^2 = \mathcal{E}((\bar{\beta}_1 + \sigma_2)B^2)$ by (6.2). If either $\bar{\beta}_1 = 0$ or $\bar{\beta}_1 + \sigma_2 = 0$, then $\mathbb{1} \cdot \bar{S} \geq 1$ so that $\bar{S}$ is a (non-UI) martingale with $\inf_{t\geq 0}(\mathbb{1} \cdot \bar{S}_t) > 0$ P-a.s.; so S satisfies DSV for $\mathbb{1}$ by Theorem 2.14 with $\eta \equiv \mathbb{1}$. If $\bar{\beta}_1 \neq 0$ and $\bar{\beta}_1 + \sigma_2 \neq 0$, then $\mathbb{1} \cdot \bar{S}_t \to 0$ P-a.s. as $t \to \infty$; so $\bar{S} \geq 0$ has $V(\bar{\vartheta}, \bar{S}) \equiv 1 \in \mathcal{S}^{\mathrm{unif}}_{++}$ by Proposition 6.2, but $\inf_{t\geq 0}(\mathbb{1} \cdot \bar{S}_t) = 0$ P-a.s., and hence does not satisfy DSV for $\mathbb{1}$ by Lemma A.3. In summary, S satisfies DSV for $\mathbb{1}$ if and only if $\bar{\beta}_1 \in \{0, -\sigma_2\}$, which is equivalent to (6.15) in view of (6.4).

2) If $\sigma_1 > 0$, (6.1) shows that $\bar{\alpha}_1$ and $\bar{\alpha}_2$ cannot both be 0, and (6.6), (6.8) imply $\bar{\beta}_i = 0$ if $\bar{\alpha}_i = 0$. So if $\bar{\alpha}_i = 0$, we get $\bar{S}^i \equiv 1$ and hence again $\mathbb{1} \cdot \bar{S} \geq 1$, so that S satisfies DSV for $\mathbb{1}$ by the same argument as in 1). If $\bar{\alpha}_1 \neq 0$ and $\bar{\alpha}_2 \neq 0$, then $\mathbb{1} \cdot \bar{S}_t \to 0$ P-a.s. as $t \to \infty$; so $\bar{S}$ does not satisfy DSV for $\mathbb{1}$, again as in 1). Thus S satisfies DSV for $\mathbb{1}$ if and only if $\bar{\alpha}_i = 0$ for $i = 1$ or $i = 2$, and this translates into (6.16) in view of (6.5), (6.7).

3) The characterisation of DSV for $\mathbb{1}$ in terms of martingale properties follows directly by combining the explicit expression for X in (1.2) with 1) and 2), respectively.

4) Because DSE implies DSV, we can by 3) only have DSE for $\mathbb{1}$ if either $X = S^2/S^1$ or $1/X$ is a martingale. This martingale is by (1.2) always of the form $\exp(\gamma \bar{W}_t - \frac{1}{2}\gamma^2 t)$ for some $\gamma \neq 0$ and some Brownian motion $\bar{W}$, and hence converges to 0 P-a.s. as $t \to \infty$. So if $\tilde{S} := S/S^1 = (1, X)$, say, is a martingale, we have $V_0(\mathrm{e}^1, \tilde{S}) = 1 = V_0(\mathrm{e}^2, \tilde{S})$, but $\lim_{t\to\infty} V_t(\mathrm{e}^1 - \mathrm{e}^2, \tilde{S}) = \lim_{t\to\infty}(1 - X_t) = 1 \in L^0_+ \setminus \{0\}$ so that e^2 is not svm for $\tilde{S}$. But $\tilde{S} \geq 0$ satisfies $\mathbb{1} \cdot \tilde{S} \in \mathcal{S}^{\mathrm{unif}}_{++}$ because $\mathbb{1} \cdot \tilde{S} \geq 1$ and X is convergent, hence bounded uniformly in $t \geq 0$, P-a.s. By Theorem 3.4, 1) for $\tilde{S}$ and $\xi \equiv \mathbb{1}$, the $\mathbb{1}$-buy-and-hold strategy e^2 is then also not ssm for $\mathbb{1}$, and so $\tilde{S}$ does not satisfy DSE for $\mathbb{1}$. If $1/X$ is a martingale, we just interchange e^1 and e^2 in the argument. ☐

Remark 6.5. The strategy $\bar{\vartheta}^2$ in (6.14) matches intuition quite well. In addition to its buy-low-sell-high property, it goes long S^1 and short S^2 if m_1 is much higher than m_2, short S^1 and long S^2 if m_1 is much lower than m_2, and holds proportional long positions in both assets if the relation between m_1 and m_2 is not extreme.

6.2 Explicit examples I

This section gives explicit counterexamples for several wrong statements or implications. All these are based on the general GBM setup from Section 6.1, and for concreteness and simplicity, we work with the BS model. So let $S^1_t = e^{rt}$ and $S^2_t = \exp(\sigma W_t + (m - \frac{1}{2}\sigma^2)t)$ with $m, r \in \mathbb{R}$ and $\sigma > 0$. We also need $X = S^2/S^1$ because $S/S^1 = (1, X)$.

Example 6.6. *DSV for $\mathbb{1}$ does not imply DSE for $\mathbb{1}$.* If we take $m - r \in \{0, \sigma^2\}$, S satisfies DSV for $\mathbb{1}$ by Theorem 6.4, 1). But S never satisfies DSE for $\mathbb{1}$, by Theorem 6.4, 4).

Example 6.7. *NFLVR for* $(1, X)$ *does not imply DSE for* $\mathbb{1}$*.* Take $m = r$ so that X is a martingale; then clearly $S/S^1 = (1, X)$ satisfies NFLVR$_\infty$. But again by Theorem 6.4, 4), S never satisfies DSE for $\mathbb{1}$, and hence neither does S/S^1 because $D := S^1 \in \mathcal{S}_{++}$ and DSE is discounting-invariant.

Example 6.8. *DSV for* $\mathbb{1}$ *does not imply NUPBR for* $(1, X)$*.* Now take $m - r = \sigma^2$ so that $X' = 1/X = S^1/S^2$ is a martingale. Then $(1, X) = S/S^1$ satisfies DSV for $\mathbb{1}$ by Theorem 6.4, 1) because S does. However, $X'_t = \exp(-\sigma W_t - \frac{1}{2}\sigma^2 t)$ converges to 0 P-a.s. as $t \to \infty$; so $\lim_{t\to\infty} X_t = +\infty$ P-a.s. and $(1, X)$ does not satisfy NUPBR$_\infty$.

Example 6.9. *If two reference strategies do not satisfy the ratio condition* (5.1)*, then DSV can fail for only one or for both of them.* Take $r = 0$, $m = \frac{1}{2}$, $\sigma = 1$. Then $\eta \equiv \mathbb{1}$ is a reference strategy and S does not satisfy DSV for η by Theorem 6.4, 1). Now take the Merton strategy $\bar{\vartheta}$ from (6.14) and set $\eta' = \bar{\vartheta}$. Then $S^{\eta'} = S/V(\bar{\vartheta}) = S/\bar{D}$ is a local martingale because $\bar{D} \in \mathcal{C}$, and hence it satisfies NUPBR due to [17, Theorem 4.12]. It follows from Theorem 2.14 that S satisfies DSV for η'. In particular, by Corollary 5.2, 1), the ratio condition (5.1) does not hold between η and η'.

Now define $\eta'' = I_{[\![0,1]\!]}\eta + I_{]\!]1,\infty]\!]}(I_A\eta + I_{A^c}(V_1(\eta)/V_1(\eta'))\eta')$, where $A := \{W_1 > 0\} \in \mathcal{F}_1$ has $P[A] = \frac{1}{2}$. Then η'' is a reference strategy which satisfies the ratio condition (5.1) neither with η nor with η', because η and η' do not satisfy (5.1). From Corollary 5.5 for η' and η'', we obtain that S does not satisfy DSV for η'', and so DSV fails for both η and η''.

6.3 Explicit examples II

Some of our examples need models S which satisfy DSE, or UI martingales, and both these requirements cannot be satisfied in the setup of Section 6.1. Theorem 6.4 shows that the GBM model never satisfies DSE for $\mathbb{1}$, and the appearing martingales are always stochastic exponentials $\mathcal{E}(\gamma B)$ of some constant multiple of some Brownian motion B. Except for $\gamma = 0$ where $\mathcal{E}(\gamma B) \equiv 1$, such a martingale is never UI because it converges to 0 P-a.s. So we need to construct our examples in a different way.

For ease of exposition, we work in this section in (infinite) discrete time. Via piecewise constant interpolations of processes (LCRL for predictable, RCLL for optional) and piecewise constant filtrations, our models can be embedded in a continuous-time framework. We use (only in this subsection) the notation $\Delta Y_n := Y_n - Y_{n-1}$ for the jump at time n of the discrete-time process $Y = (Y_n)_{n\in\mathbb{N}_0}$. Our examples have two building blocks.

A first basic ingredient is a martingale Y whose increments (or successor values) in each step only take two (different) values. The martingale condition then uniquely determines all one-step transition probabilities as a function of the Y-values, and so we can talk about "the" corresponding martingale. By choosing the increments or values in a suitable way, we can moreover ensure that Y is nonnegative and bounded, hence UI and P-a.s. convergent to some Y_∞ which closes Y on the right as a martingale (i.e., $Y = (Y_n)_{n\in\mathbb{N}_0\cup\{\infty\}}$

is a martingale). Finally, one can also ensure that Y_∞ only takes two values one of which is 0, and thus we obtain a UI martingale which converges to 0 with positive probability.

The second idea is more subtle. We want to work with a two-asset model and trade in such a way that our strategy involves the asymptotic behaviour of both assets in a specific nontrivial way. To this end, we construct $S = (S^1, S^2)$ such that in each step, exactly one of the assets has a price move, and these moves always alternate. This allows to predict which asset coordinate will move in the next step, which can be exploited to construct (switching) strategies with a desired behaviour; and as both coordinates move alternatingly, the resulting wealth process is influenced by each coordinate in turn.

Example 6.10. *DSV for η is not equivalent to the existence of a σ-martingale discounter D for S; the condition $\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) > 0$ P-a.s. in Theorem 2.11 is indispensable.* To show this, we construct a bounded martingale $S \geq 0$ satisfying $P[\lim_{t\to\infty} S_t = 0] > 0$. Then $P[\inf_{t\geq 0}(\eta_t \cdot (S_t/D_t)) = 0] \geq P[\lim_{t\to\infty}(\mathbb{1} \cdot S_t) = 0] > 0$ and $D \equiv 1$ is a UI martingale discounter for S. We then show that S does not satisfy DSV for $\eta \equiv \mathbb{1}$.

To start with the construction, let $Y = (Y_n)_{n\in\mathbb{N}_0}$ be the (unique) martingale which starts at $Y_0 = 1$ and at any time $n \in \mathbb{N}$ only takes the two values $u_n = 2-2^{-n}$ or $d_n = 2^{-n}$. Then Y is P-a.s. strictly positive (but not bounded away from 0 uniformly in n) and bounded by 2. So (Y_n) converges to Y_∞ P-a.s., and clearly $P[Y_\infty = 2] = \frac{1}{2} = P[Y_\infty = 0]$.

Now let Y^1, Y^2 be independent copies of Y and define $S = (S^1, S^2)$ by $S^1_0 = 1$ and

$$S^1_{2n-1} = S^1_{2n} = Y^1_n \quad \text{for } n \in \mathbb{N}, \qquad S^2_{2n} = S^2_{2n+1} = Y^2_n \quad \text{for } n \in \mathbb{N}_0.$$

This gives for $n \in \mathbb{N}$ that $\Delta S^1_{2n-1} = \Delta Y^1_n$, $\Delta S^1_{2n} = 0$ and $\Delta S^2_{2n-1} = 0$, $\Delta S^2_{2n} = \Delta Y^2_n$ and in particular yields that the coordinates of S jump alternatingly because

$$\Delta S^2_n I_{\{\Delta S^1_{n-1}=0\}} = 0 = \Delta S^1_n I_{\{\Delta S^2_{n-1}=0\}}. \tag{6.17}$$

Let $\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}_0}$ be the filtration generated by S. Then S is like Y a bounded martingale, hence convergent to S_∞ P-a.s., and $P[\lim_{n\to\infty}(\mathbb{1} \cdot S_n) = 0] = P[S_\infty = 0] = \frac{1}{4} > 0$.

To show S does not satisfy DSV for $\mathbb{1}$, we argue indirectly via Lemma A.3. Because $(S_n)_{n\in\mathbb{N}_0}$ is strictly positive, $\eta \equiv \mathbb{1}$ is a reference strategy, and $B := \{\lim_{n\to\infty}(\mathbb{1} \cdot S_n) = 0\}$ has $P[B] > 0$. If S satisfies DSV for $\mathbb{1}$, Lemma A.3 yields $\sup_{n\in\mathbb{N}_0}(\vartheta_n \cdot S_n)/(\mathbb{1} \cdot S_n) < \infty$ P-a.s. for all $\vartheta \in \Theta^{\mathrm{sf}}_+$, and $(\vartheta_n \cdot S_n)/(\mathbb{1} \cdot S_n) = V_n(\vartheta)/(\mathbb{1} \cdot S_n)$. We exhibit below a strategy $\bar{\vartheta} \in \Theta^{\mathrm{sf}}_+$ with $V(\bar{\vartheta}) \equiv \varepsilon > 0$. Because then $\sup_{n\in\mathbb{N}_0}(\bar{\vartheta}_n \cdot S_n)/(\mathbb{1} \cdot S_n) = +\infty$ on B, we conclude that S cannot satisfy DSV for $\mathbb{1}$.

To construct $\bar{\vartheta}$, we fix $\varepsilon > 0$ and consider the strategy which invests the amount ε at time 0 in asset 2 and subsequently reinvests at any time all its wealth into that asset which will not jump in the next period. More formally, we set $\bar{\vartheta}_0 := \bar{\vartheta}_1 := (0, \varepsilon)$ and

$$\bar{\vartheta}_{n+1} := I_{\{\Delta S^1_n=0\}}\left(0, \frac{\varepsilon}{S^2_n}\right) + I_{\{\Delta S^2_n=0\}}\left(\frac{\varepsilon}{S^1_n}, 0\right). \tag{6.18}$$

This is well defined because S^1, S^2 are both strictly positive, and predictable because S is adapted. Moreover, $S_0^2 = S_1^2 = 1$ yields $V_0(\bar{\vartheta}) = V_1(\bar{\vartheta}) = \varepsilon$, and

$$V_{n+1}(\bar{\vartheta}) = I_{\{\Delta S_n^1 = 0\}}\, \varepsilon \frac{S_{n+1}^2}{S_n^2} + I_{\{\Delta S_n^2 = 0\}}\, \varepsilon \frac{S_{n+1}^1}{S_n^1} = \varepsilon$$

as S^1, S^2 always jump alternatingly. So $V(\bar{\vartheta}) \equiv \varepsilon$, and $\bar{\vartheta}$ is also self-financing because

$$\Delta V_{n+1}(\bar{\vartheta}) - \bar{\vartheta}_{n+1} \cdot \Delta S_{n+1} = 0 - \bar{\vartheta}_{n+1}^1 \Delta S_{n+1}^1 - \bar{\vartheta}_{n+1}^2 \Delta S_{n+1}^2 \equiv 0$$

due to (6.18) and (6.17). So $\bar{\vartheta}$ has all the claimed properties, and this ends the example.

Example 6.11. *DSE for η need not imply NFLVR$_\infty$, not even for a classic model of the form $S = (1, X)$.* Similarly as in Example 6.10, let $Y = (Y_n)_{n \in \mathbb{N}_0}$ be the (unique) martingale valued in $(0,1)$ with $Y_0 = \frac{1}{2}$ and $Y_n \in \{\frac{1}{2} 2^{-n}, 1 - \frac{1}{2} 2^{-n}\}$. This converges P-a.s. to Y_∞ which takes the values 0 and 1 each with probability $\frac{1}{2}$. Set $Y' := 1 - Y$ and define

$$S := (1, X) := \left(1, \frac{Y'}{Y}\right) = \left(1, \frac{1-Y}{Y}\right).$$

Then $\mathbb{1} \cdot S = \frac{1}{Y}$ and so $S^{\mathbb{1}} = (Y, 1-Y)$ is a bounded P-martingale with $\mathbb{1} \cdot S^{\mathbb{1}} \equiv 1 \in \mathcal{S}_{++}^{\mathrm{unif}}$. So S satisfies (e5) in Theorem 2.14 with $D = \mathbb{1} \cdot S$ and $\eta \equiv \mathbb{1}$, and this implies that S satisfies DSE for $\mathbb{1}$. However, we clearly have $X \geq 0$ and

$$\lim_{n \to \infty} X_n = \lim_{n \to \infty} \frac{1 - Y_n}{Y_n} = +\infty \qquad \text{on } \{\lim_{n\to\infty} Y_n = 0\}.$$

As $P[Y_\infty = 0] = \frac{1}{2} > 0$, $S = (1, X)$ does not satisfy NUPBR$_\infty$ and thus also not NFLVR$_\infty$.

A Appendix

This section contains some technical proofs and auxiliary results.

Proof of Lemma 3.1. If ϑ is not svm for S/D, there are $f \in L_+^0 \setminus \{0\}$ and for any $\varepsilon > 0$ some $\hat{\vartheta}^\varepsilon \in \Theta_+^{\mathrm{sf}}$ with $V_0(\hat{\vartheta}^\varepsilon, S/D) \leq V_0(\vartheta, S/D) + \varepsilon$, hence $V_0(\hat{\vartheta}^\varepsilon) \leq V_0(\vartheta) + \varepsilon D_0$, and

$$\text{(A.1)} \qquad \liminf_{t \to \infty} V_t(\hat{\vartheta}^\varepsilon - \vartheta, S/D) \geq f \geq 0 \qquad P\text{-a.s.}$$

As $D \in \mathcal{S}_{++}^{\mathrm{unif}}$ has $\inf_{t \geq 0} D_t > 0$ P-a.s., $f' := f \liminf_{t \to \infty} D_t$ is in $L_+^0 \setminus \{0\}$. Because $D \in \mathcal{S}_{++}^{\mathrm{unif}}$ also has $\sup_{t \geq 0} D_t < \infty$ P-a.s., (A.1) implies by Lemma A.1 that P-a.s.,

$$\liminf_{t \to \infty} V_t(\hat{\vartheta}^\varepsilon - \vartheta) = \liminf_{t \to \infty} \Big(V_t(\hat{\vartheta}^\varepsilon - \vartheta, S/D) D_t\Big) \geq \liminf_{t \to \infty} V_t(\hat{\vartheta}^\varepsilon - \vartheta, S/D) \liminf_{t \to \infty} D_t \geq f'.$$

This shows that ϑ is not svm for S either.

For the second part, if $\alpha\vartheta$ is not svm for S, we can find $f \in L_+^0 \setminus \{0\}$ and for every $\varepsilon > 0$ some $\hat{\vartheta}^\varepsilon \in \Theta_+^{\mathrm{sf}}$ with $V_0(\hat{\vartheta}^\varepsilon) \leq V_0(\alpha\vartheta) + \varepsilon$ and $\liminf_{t \to \infty} V_t(\hat{\vartheta}^\varepsilon - \alpha\vartheta) \geq f$ P-a.s. There are two cases. If $\alpha > 0$, then $\tilde{\vartheta} := \hat{\vartheta}^\varepsilon/\alpha \in \Theta_+^{\mathrm{sf}}$ satisfies $V_0(\tilde{\vartheta}) = V_0(\hat{\vartheta}^\varepsilon)/\alpha \le V_0(\vartheta) + \varepsilon/\alpha$ and $\liminf_{t\to\infty} V_t(\tilde{\vartheta}-\vartheta) = \liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon - \alpha\vartheta)/\alpha \ge f/\alpha$ P-a.s. So ϑ is not svm for S as f/α is in $L_+^0 \setminus \{0\}$. If $\alpha = 0$, $\tilde{\vartheta} := \vartheta + \hat{\vartheta}^\varepsilon \in \Theta_+^{\mathrm{sf}}$ has $V_0(\tilde{\vartheta}) \le V_0(\vartheta) + V_0(\alpha\vartheta) + \varepsilon = V_0(\vartheta) + \varepsilon$ and $\liminf_{t\to\infty} V_t(\tilde{\vartheta} - \vartheta) = \liminf_{t\to\infty} V_t(\hat{\vartheta}^\varepsilon - \alpha\vartheta) \ge f$ P-a.s.; so again ϑ is not svm for S. $\square$

Proof of Lemma 3.3. For brevity, we introduce the set $\Gamma := \{V_\tau(\vartheta^1) < V_\tau(\vartheta^2)\} \in \mathcal{F}_\tau$ and set $\varphi := \vartheta^1 \oslash_\tau^\xi \vartheta^2$. We use $V(\xi, S^\xi) = \xi \cdot S^\xi \equiv 1$, which also gives $\xi \bullet S^\xi \equiv 0$. Then using the definition of φ, the general fact that $XI_{[\![0,\tau]\!]} = X^\tau - X_\tau I_{]\!]\tau,\infty[\![}$, the fact that ϑ^1, ϑ^2 are self-financing and again the definition of φ yields

$$\begin{aligned}
&V(\varphi, S^\xi)\\
&= I_{[\![0,\tau]\!]} V(\vartheta^1, S^\xi) + I_{]\!]\tau,\infty[\![}\Big(I_\Gamma V(\vartheta^1, S^\xi) + I_{\Gamma^c} V(\vartheta^2, S^\xi) + I_{\Gamma^c} V_\tau(\vartheta^1 - \vartheta^2, S^\xi)\Big)\\
&= \Big(V(\vartheta^1, S^\xi)\Big)^\tau + I_{]\!]\tau,\infty[\![}\Big(I_\Gamma\Big(V(\vartheta^1, S^\xi) - V_\tau(\vartheta^1, S^\xi)\Big) + I_{\Gamma^c}\Big(V(\vartheta^2, S^\xi) - V_\tau(\vartheta^2, S^\xi)\Big)\Big)\\
&= V_0(\vartheta^1, S^\xi) + (\vartheta^1 I_{[\![0,\tau]\!]}) \bullet S^\xi + \Big(I_{]\!]\tau,\infty[\![}\Big(I_\Gamma \vartheta^1 + I_{\Gamma^c}\vartheta^2 + I_{\Gamma^c} V_\tau(\vartheta^1 - \vartheta^2, S^\xi)\xi\Big)\Big) \bullet S^\xi\\
&= V_0(\varphi, S^\xi) + \varphi \bullet S^\xi.
\end{aligned}$$

This shows that φ is self-financing. If both ϑ^1, ϑ^2 are in Θ_+^{sf}, the second line above is nonnegative so that also φ is in Θ_+^{sf}. $\square$

For any function $z : [0,\infty) \to \mathbb{R}^N$, set $\underline{z}(\infty) := \liminf_{t\to\infty} z(t)$, with the lim inf taken coordinatewise. If the limit exists, again coordinatewise, we write $z(\infty) := \lim_{t\to\infty} z(t)$. In $\mathbb{R}_+$, the product of ∞ and 0 is 0.

Lemma A.1. *Suppose the functions $x, y : [0,\infty) \to \mathbb{R}^N$ satisfy*

(a) *$y \ge 0$ is bounded (uniformly in $t \ge 0$) by some $C < \infty$.*

(b) *$\underline{x}^i(\infty) \ge 0$ for $i = 1, \dots, N$.*

Then

$$\underline{(x \cdot y)}(\infty) \ge \underline{x}(\infty) \cdot \underline{y}(\infty). \tag{A.2}$$

Proof. Fix $\varepsilon > 0$. Decompose $\{1, \dots, N\}$ into indices ℓ with $\underline{x}^\ell(\infty) = \infty$ and indices m with $\underline{x}^m(\infty) < \infty$. For any ℓ and $t \ge T = T(\ell)$, we have $x^\ell(t) \ge 0$ and $y^\ell(t) \ge \frac{1}{2}\underline{y}^\ell(\infty)$, and for any m, we get $x^m(t) \ge \underline{x}^m(\infty) - \varepsilon$ for $t \ge T = T(m, \varepsilon)$ and $0 \le y^m(t) \le C$ for all t. This implies $x^m(t)y^m(t) \ge (\underline{x}^m(\infty) - \varepsilon)y^m(t) \ge \underline{x}^m(\infty)y^m(t) - \varepsilon C$ and therefore

$$(x \cdot y)(t) = \sum_\ell x^\ell(t)y^\ell(t) + \sum_m x^m(t)y^m(t) \ge \frac{1}{2}\sum_\ell x^\ell(t)\underline{y}^\ell(\infty) + \sum_m \Big(\underline{x}^m(\infty)y^m(t) - \varepsilon C\Big).$$

Let $t \to \infty$ and use on the right-hand side the superadditivity of lim inf, $y \ge 0$ and the fact that $\underline{x}^m(\infty) \in [0,\infty)$ for all m, to obtain

$$\underline{(x \cdot y)}(\infty) \ge \frac{1}{2}\sum_\ell \underline{x}^\ell(\infty)\underline{y}^\ell(\infty) + \sum_m \underline{x}^m(\infty)\underline{y}^m(\infty) - N\varepsilon C.$$

If there is an ℓ with $\underline{y}^\ell(\infty) > 0$, the right-hand side is $+\infty$ and (A.2) holds trivially. So we can assume for the rest of the proof that $\underline{y}^\ell(\infty) = 0$ for all ℓ; then $\underline{x}^\ell(\infty)\underline{y}^\ell(\infty) = 0$ for all ℓ by our convention, and we end up with

$$\underline{(x \cdot y)}(\infty) \geq \sum_m \underline{x}^m(\infty)\underline{y}^m(\infty) - N\varepsilon C = \sum_{i=1}^N \underline{x}^i(\infty)\underline{y}^i(\infty) - N\varepsilon C.$$

Letting $\varepsilon \searrow 0$ then again gives (A.2) and completes the proof. $\square$

The next auxiliary result is extracted from the proof of [18, Proposition 1].

Lemma A.2. *A convex set $C \subseteq L^0_+$ is bounded in L^0 if and only if C contains no sequence $(V^n)_{n\in\mathbb{N}}$ satisfying $V^n \geq n\xi$ P-a.s. for all $n \in \mathbb{N}$ and for some $\xi \in L^0_+ \setminus \{0\}$.*

Proof. The "only if" part is clear. For the "if" part, suppose C is not bounded in L^0 and let $\Omega_u \in \mathcal{F}$ be as in [2, Lemma 2.3]. (In the terminology of [2], C is hereditarily unbounded in probability on Ω_u.) Note that $P[\Omega_u] > 0$ because $P[\Omega_u] = 0$ would imply that C is bounded in L^0. Then [2, Lemma 2.3, part 4)] implies with $\varepsilon := 2^{-n}$ that for each $n \in \mathbb{N}$, there is some $V^n \in C$ such that

$$P[\{V^n \leq n\} \cap \Omega_u] \leq P[\{V^n \leq 2^n\} \cap \Omega_u] \leq 2^{-n}.$$

Take $N \in \mathbb{N}$ with $\sum_{n=N}^\infty 2^{-n} \leq P[\Omega_u]/2$. For $n \geq N$, set $A_n := \{V^n > n\} \cap \Omega_u \in \mathcal{F}$ and define $A := \bigcap_{n\geq N} A_n \in \mathcal{F}$ so that $V^n \geq nI_{A_n} \geq nI_A$ due to $V^n \in C \subseteq L^0_+$. Then

$$P[A] \geq P[\Omega_u] - \sum_{n=N}^\infty P[A_n^c \cap \Omega_u] \geq P[\Omega_u]/2 > 0$$

shows that $\xi := I_A \in L^0_+ \setminus \{0\}$, and we have $V^n \geq n\xi$ P-a.s. for all $n \in \mathbb{N}$. But this contradicts the assumption for the "if" part, and so we are done. $\square$

Lemma A.3. *Suppose $S \geq 0$ and there exists a $\xi \in \Theta^{\mathrm{sf}}_{++}$ with $V(\xi) = \xi \cdot S \in \mathcal{S}^{\mathrm{unif}}_{++}$ and $\xi \geq 0$. If S satisfies DSV for $\mathbb{1}$, it also satisfies $\inf_{t\geq 0}(\mathbb{1} \cdot S_t) > 0$ P-a.s.*

Proof. Suppose to the contrary that $\liminf_{t\to\infty}(\mathbb{1}\cdot S_t) = 0$ on some $A \in \mathcal{F}_\infty$ with $P[A] > 0$. For each $n \in \mathbb{N}$, $\tau^n := \inf\{t \geq 0 : \mathbb{1} \cdot S_t \leq 1/n\}$ is a stopping time and $\tau^n < \infty$ and $\mathbb{1} \cdot S_{\tau^n} \leq 1/n$ on A, P-a.s. Take ξ as above and define $\vartheta^n := (\xi/n) \oslash^{\mathbb{1}}_{\tau^n} 0$. Then ϑ^n is in Θ^{sf}_+ by Lemma 3.3, $V_0(\vartheta^n, S^{\mathbb{1}}) = (\xi_0 \cdot S^{\mathbb{1}}_0)/n$, and we have on A that for $t \geq \tau_n$,

$$\vartheta^n_t \cdot S^{\mathbb{1}}_t = \vartheta^n_{\tau_n} \cdot S^{\mathbb{1}}_{\tau_n} = \frac{1}{n}\frac{\xi_{\tau_n} \cdot S_{\tau_n}}{\mathbb{1} \cdot S_{\tau_n}} \geq \xi_{\tau_n} \cdot S_{\tau_n} \qquad P\text{-a.s.}$$

So $\xi \cdot S \in \mathcal{S}^{\mathrm{unif}}_{++}$ implies $\liminf_{t\to\infty}(\vartheta^n_t \cdot S^{\mathbb{1}}_t) \geq I_A \inf_{t\geq 0}(\xi_t \cdot S_t) \in L^0_+ \setminus \{0\}$, and thus 0 is not svm for $S^{\mathbb{1}}$. By Theorem 3.4, 1) for $S^{\mathbb{1}}$ and because dynamic share viability is discounting-invariant, 0 is then not ssm for $\mathbb{1}$ which contradicts DSV for $\mathbb{1}$. $\square$

Acknowledgements

We gratefully acknowledge financial support by the Swiss Finance Institute (SFI) and the ETH Foundation via the Stochastic Finance Group at ETH Zurich.

The first author thanks Martin Herdegen, Sebastian Herrmann, Chong Liu, Ariel Neufeld, Sara Svaluto-Ferro and Danijel Zivoi for discussions, questions and general support. Special thanks go to Matteo Burzoni and David Prömel for extensively discussing the meaning and possible extensions of results, and for giving valuable advice.

We are grateful to the referees of an earlier version who provided substantial criticism and feedback. This has helped us to reshape, shorten and improve the paper considerably.

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