Minimal entropy preserves the Lévy property: How and why

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Abstract: Let L be a multidimensional Lévy process under P in its own filtration and consider all probability measures Q turning L into a local martingale. The minimal entropy martingale measure Q^E is the unique Q which minimizes the relative entropy with respect to P. We prove that L is still a Lévy process under Q^E and explain precisely how and why this preservation of the Lévy property occurs.

Key words: Lévy processes, martingale measures, relative entropy, minimal entropy martingale measure, mathematical finance, incomplete markets

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0. Introduction

In the last years, Lévy processes have become very popular for modelling in finance. They provide a lot of flexibility when fitting a model to observed asset prices and yet are very tractable if one needs expressions for derivative prices. One drawback is that the resulting model of a financial market is usually incomplete and thus has multiple martingale measures (and hence non-unique option prices). A popular approach is then to fix one particular martingale measure Q^* for the underlying assets S and to price derivatives by the Q^*-expectation of their discounted payoff. But how should one choose Q^*? Very often, this is done via the minimization of a functional over martingale measures, and the functional is in turn motivated by a dual formulation corresponding to a primal utility maximization problem; see Kallsen (2002) for a list of references. Alternatively, Q^* might be the natural pricing measure arising from a criterion which emphasizes hedging rather than pricing aspects; this produces for instance the minimal or the variance-optimal martingale measures.

In this paper, we consider the pricing-oriented approach and we take the relative entropy of Q with respect to the original measure P as the functional to be minimized. Not only does this allow us to do many computations explicitly; one general argument for that choice is also that the resulting minimal entropy martingale measure is automatically equivalent to P. This is not so for the variance-optimal or more generally the q-optimal martingale measures.

We show that if L is an $\mathbb{R}^d$-valued Lévy process under P and if Q^E minimizes the relative entropy over all Q under which L is a local martingale, then L is again a Lévy process under Q^E. This extends a result by Fujiwara/Miyahara (2003) who simply write down Q^E for $d = 1$ and directly prove its optimality. But more important than the generalization to $d > 1$ is that we also explain precisely how this preservation happens and why Q^E has the structure obtained. Similarly to earlier papers by Foldes (1991a,b) on a different topic, the reasons are very intuitive. But the actual proofs turn out to require quite a lot of work.

The paper is structured as follows. Section 1 formulates the basic problem more precisely, states the two main results and presents the intuitive explanation mentioned above. Section 2 prepares the ground by providing a number of results from general semimartingale theory. Section 3 contains the crucial idea. It shows how one can always reduce relative entropy while preserving the martingale property by a suitable averaging procedure over certain parameters β, Y that characterize Q. This reduces the problem from a minimization over probability measures to a minimization over non-random functions. Section 4 produces a candidate for the optimal function from the first order condition for optimality and proves that the corresponding candidate measure has indeed minimal entropy. The main result from Section 3 is then proved in Section 5 which substantiates a merely plausible reasoning with a rigorous argument. Finally, a number of proofs from Section 2 are collected in the Appendix.

1. Setup and main results

In this section, we introduce some notation, formulate the basic problem and state the two main results. Unexplained terminology used here is standard or explained in the next section.

Let $(\Omega, \mathcal{F}, I\!\!F, P)$ be a filtered probability space with $I\!\!F = (\mathcal{F}_t)_{t\in\mathcal{T}}$ satisfying the usual conditions and either $\mathcal{T} = [0, T_0]$ for some fixed $T_0 \in (0, \infty)$ (finite horizon) or $\mathcal{T} = [0, \infty)$ (infinite horizon). For a probability measure $Q \overset{\text{loc}}{\ll} P$, we denote by

$$I_t(Q|P) := E_Q\Big[\log \tfrac{dQ}{dP}\big|_{\mathcal{F}_t}\Big] \in [0, +\infty]$$

the relative entropy of Q with respect to P on $\mathcal{F}_t$ and call $\big(I_t(Q|P)\big)_{t\in\mathcal{T}}$ the *entropy process* of Q. For an $I\!\!R^d$-valued $I\!\!F$-adapted process $X = (X_t)_{t\in\mathcal{T}}$ and a fixed $d \times d$-matrix U, we introduce the following sets of probability measures on $(\Omega, \mathcal{F})$:

$$\mathcal{Q}^U_a(X) := \Big\{Q \overset{\text{loc}}{\ll} P \,\Big|\, UX \text{ is a local } Q\text{-martingale}\Big\},$$

$$\mathcal{Q}^U_e(X) := \Big\{Q \overset{\text{loc}}{\approx} P \,\Big|\, UX \text{ is a local } Q\text{-martingale}\Big\} \subseteq \mathcal{Q}^U_a(X),$$

$$\mathcal{Q}^U_f(X) := \big\{Q \in \mathcal{Q}^U_a(X) \,\big|\, I_t(Q|P) < \infty \text{ for all } t \in \mathcal{T}\big\},$$

$$\mathcal{Q}^U_\ell(X) := \big\{Q \in \mathcal{Q}^U_a(X) \,\big|\, X \text{ is a Lévy process under } Q\big\}.$$

$\mathcal{Q}^U_\ell(X)$ is mainly used if X is already a Lévy process under P. Note that $Q \in \mathcal{Q}^U_\ell(X)$ means that UX is a local Q-martingale, but X itself is a Q-Lévy process. If U is the identity matrix, we omit the superscript U; hence $\mathcal{Q}^U_s(X) = \mathcal{Q}_s(UX)$ for $s \in \{a, e, f\}$, but not for $s = \ell$.

Elements of $\mathcal{Q}^U_e(X)$ are called *equivalent local martingale measures (ELMMs)* for UX. The *minimal entropy martingale measure (MEMM)* $Q^E(UX)$ is defined by the property that it minimizes the entropy process pointwise in t over all $Q \in \mathcal{Q}^U_a(X)$, i.e., $Q^E(UX)$ is in $\mathcal{Q}^U_a(X)$ and $I_t\big(Q^E(UX)\big|P\big) \le I_t(Q|P)$ for all $Q \in \mathcal{Q}^U_a(X)$ and $t \in \mathcal{T}$. The *minimal entropy Lévy martingale measure* $Q^E_\ell(UX) \in \mathcal{Q}^U_\ell(X)$ is similarly defined by the property that it minimizes the entropy process pointwise in t over all $Q \in \mathcal{Q}^U_\ell(X)$. We want to find $Q^E(UL)$ when L is a Lévy process under P in its own filtration $I\!\!F = I\!\!F^L$.

Remark. In mathematical finance, the above problem naturally arises in the following way. Suppose we have a financial market with d risky assets ("stocks") and one riskless asset ("bank account", B). We express all prices in units of B; this is called discounting with respect to B, and the resulting discounted asset prices are denoted by S. A frequently made modelling assumption is then that $S^i = S^i_0\,\mathcal{E}(L^i)$ for some $I\!\!R^d$-valued Lévy process L, and then S and L have the same ELMMs.

In economic terms, an ELMM can be interpreted as a pricing operator for financial products which is consistent with the a priori given asset prices S; see Harrison/Kreps (1979).

It is also well known that the existence of some ELMM is essentially equivalent to the economically plausible and desirable property that the financial market described by S does not admit arbitrage opportunities ("money pumps"); see Delbaen/Schachermayer (1998) for a precise formulation. Finally, as mentioned in the introduction, minimizing relative entropy is one possible criterion for choosing an ELMM. This explains why we are interested in $Q^E(L)$; the extra U will give some room for more generality.

A result called numeraire invariance provides the (economically intuitive) statement that discounting does not change anything; this is usually taken as justification for choosing $B \equiv 1$ and directly modelling discounted prices. However, this result assumes that the filtration $I\!\!F$ is given a priori. If we wanted to take as $I\!\!F$ the filtration generated by asset prices, it may well make a difference if these are discounted or not as soon as the bank account B is stochastic. Although the use of the filtration generated by prices would be desirable and is for instance advocated in Section 9.6 of Kallianpur/Karandikar (2000), we follow here the standard approach in the literature to work with the filtration generated by the underlying sources of randomness; see for instance the very first pages of Karatzas/Shreve (1998). This explains our choice $I\!\!F = I\!\!F^L$. ◇

As already stated, our goal in this paper is now to identify $Q^E(UL)$ if L is a Lévy process under P for its own filtration $I\!\!F = I\!\!F^L$, and moreover to explain exactly why $Q^E(UL)$ has the particular structure we obtain. The two main results are

Theorem A. *Let L be an $I\!\!R^d$-valued P-Lévy process for $I\!\!F = I\!\!F^L$, and U a fixed $d \times d$-matrix. Suppose that $\mathcal{Q}_e^U(L) \cap \mathcal{Q}_f^U(L) \cap \mathcal{Q}_\ell^U(L) \neq \emptyset$. If $Q^E(UL)$ exists, then L is a Lévy process under $Q^E(UL)$.*

This result explains the first part of the paper's title since it says that the Lévy property of L is *preserved* by passing from P to the minimal entropy martingale measure for UL.

Theorem B. *Let L be an $I\!\!R^d$-valued P-Lévy process for $I\!\!F = I\!\!F^L$ with Lévy characteristics (b, c, K), and U a fixed $d \times d$-matrix. Suppose that there exists $u_* \in \operatorname{range}(U^\top)$ such that*

$$\int_{I\!\!R^d} \left| x e^{u_*^\top x} - h(x) \right| K(dx) < \infty,$$

$$U\Big(b + c u_* + \int_{I\!\!R^d} \left(x e^{u_*^\top x} - h(x) \right) K(dx) \Big) = 0.$$

Then $Q^E(UL)$ exists and coincides with the Esscher martingale measure Q^{u_*} defined by $\frac{dQ^{u_*}}{dP} = \text{const.} \exp\left(u_*^\top L_t\right)$ on $\mathcal{F}_t$ for all $t \in \mathcal{T}$.

This shows *how* the Lévy property of L is preserved, namely by using an Esscher transform of P to construct a martingale measure for UL. The final third of the title, *why* this happens, will become clear from the proofs and constitutes a key insight contributed here.

In comparison to existing literature, perhaps the most characteristic feature of this paper is its combination of intuitive insight with rigorous proofs. This is best understood if we briefly explain how we obtain our results. By using Girsanov's theorem, any $Q \overset{\text{loc}}{\ll} P$ can be described by two parameters β, Y which are in general stochastic processes. The relative entropy $I_t(Q|P)$ is then a convex functional of β and Y, and by Jensen's inequality, it can be reduced if we pass to deterministic time-independent parameters obtained by averaging over ω and t. Moreover, the local martingale property of UL under Q is characterized by a linear constraint between β and Y and so is preserved by this averaging. Hence the MEMM for UL must have deterministic time-independent parameters, which means that L is a Lévy process under it. This explains the intuition behind Theorem A; the rigorous proof, however, must still circumvent integrability problems. We use the assumption that $I\!\!F$ is generated by a Lévy process to identify a measure via its density process by its parameters.

Due to Theorem A, finding $Q^E(UL)$ reduces to a classical optimization problem over deterministic time-independent quantities β, Y. The linear constraint from the local martingale property even eliminates β so that only the non-random function Y needs to be varied. Formally writing down the first order conditions for optimality then produces a candidate Y_*, and Theorem B accomplishes the fairly straightforward task of proving that the corresponding measure Q^{u_*} has indeed minimal entropy. This entire line of reasoning also makes it very transparent *why* minimal entropy preserves the Lévy property.

Remark. Conceptually, our results are similar to Foldes (1991a,b) who considered an investment problem with market returns given by a process R with independent increments. He proved that an optimal portfolio plan can be found in the class of deterministic strategies (and is even time-independent if R has independent and stationary increments). Like here, the main techniques used were computations based on semimartingale characteristics. $\diamond$

From a formal point of view, Theorem B is slightly more general than Theorem 3.1 of Fujiwara/Miyahara (2003) who proved essentially this result for a finite horizon and when L is one-dimensional (U is then the identity matrix). Earlier work on the same problem under additional assumptions is also reviewed in Fujiwara/Miyahara (2003). It seems not quite straightforward to generalize their proof to the multidimensional case, and a number of integrability issues is also not entirely clear from their presentation. We briefly indicate below why including the matrix U is useful for applications. But the main difference to our work is that Fujiwara/Miyahara (2003) simply define Q^{u_*} as in Theorem B and prove directly that this is the MEMM; there is no hint to the reader where this measure comes from.

On the other end of the scale, the paper by Chan (1999) already contains the idea of computing relative entropy as a functional of the parameters β, Y, even if his setting is less general due to exponential moment conditions on L. The crucial difference here is that Chan (1999) argues only heuristically ("it is a little less clear") that a minimization over deterministic parameters is already enough. Making this intuitive idea rigorous in full generality is achieved by our Theorem A and turns out to be more involved; see Section 5.

Two immediate applications that come to mind are the following.

Example 1. *Exponential Lévy processes*: Consider a model where discounted asset prices are strictly positive and given by $S^i = S^i_0 \, \mathcal{E}(L^i)$, $i = 1, \dots, d$, for some $I\!R^d$-valued Lévy process L under P. Since $dS^i = S^i_- \, dL^i$, S is a local Q-martingale if and only if L is, and so $Q^E(S) = Q^E(L)$. Hence the MEMM for S in $I\!F = I\!F^L$ is given by the Esscher measure Q^{u_*} from Theorem B, provided u_* there exists (with $U =$ identity). This generalizes Theorem 3.1 of Fujiwara/Miyahara (2003) to the case $d > 1$. (Actually, Fujiwara/Miyahara (2003) work with $S = S_0 \exp(\widetilde{L})$ for some Lévy process $\widetilde{L}$, but this can be rewritten with L as above.) ◇

Example 2. *Stochastic volatility models driven by Lévy processes*: Let L be a two-dimensional Lévy process under P for $I\!F = I\!F^L$ and model the one-dimensional discounted asset price process S by

$$dS_t = \sigma(t, S_{t-}, L^2_{t-}) \, dL^1_t, \tag{1.1}$$

where $\sigma : [0, T_0] \times (0, \infty) \times I\!R \to I\!R$ is a measurable function such that (1.1) has a strictly positive solution S with the property of being a local Q-martingale if and only if L^1 is. This is a Lévy version of the usual stochastic volatility models where (L^1, L^2) is a diffusion with possibly correlated coordinates; note that L^1 and L^2 may well be dependent. The MEMM $Q^E(S)$ is given by $Q^E(L^1) = Q^E(UL)$, where $U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ gives the projection on the first coordinate, and $Q^E(L^1)$ can be explicitly constructed from Theorem B. See Section 4.4 of Esche (2004) for a more detailed account. ◇

2. Auxiliary results

This section presents some auxiliary results from general semimartingale theory. To facilitate reading, most proofs are relegated to the Appendix. Our basic reference is Jacod/Shiryaev (1987), abbreviated JS. Without special mention, all processes take values in $I\!R^d$.

2.1. Semimartingales, characteristics and Girsanov's theorem

We first fix some notation. For a semimartingale X, we denote by μ^X the random measure associated with the jumps of X and by ν^P the predictable P-compensator of μ^X. If W is a real-valued optional function and μ a random measure, $W * \mu$ is the integral process of W

with respect to μ. *Throughout the entire paper, h is a fixed but arbitrary truncation function.* Our results do not depend on the choice of h; more precisely, we could take a different h' and rewrite everything with h' simply replacing h throughout.

If X is a semimartingale, we denote by (B, C, ν) the triplet of its P-characteristics (relative to the truncation function h). As in Prop. II.2.9 of JS, we can and always do choose a version of the form

$$B = \int b\, dA, \qquad C = \int c\, dA, \qquad \nu(\omega; dt, dx) = dA_t(\omega)\, K_{\omega,t}(dx), \tag{2.1}$$

where A is a real-valued predictable increasing locally integrable process, $b = (b_t^i)$ an $I\!R^d$-valued predictable process, $c = (c_t^{ij})$ a predictable process with values in the set of symmetric nonnegative definite $d \times d$-matrices, and $K_{\omega,t}(dx)$ a transition kernel from $(\Omega \times I\!R_+, \mathcal{P})$ into $(I\!R^d, \mathcal{B}^d)$ which satisfies $K_{\omega,t}(\{0\}) = 0$ and $\int_{I\!R^d} (1 \wedge |x|^2) K_{\omega,t}(dx) \le 1$ for all $t \in \mathcal{T}$. We shall also need the characteristics of a linear transformation of a semimartingale. For a Lévy process X, this is given by Prop. 11.10 of Sato (1999). The argument for the general semimartingale case is routine and therefore omitted.

Proposition 1. *Let X be a semimartingale with characteristics (B, C, ν) and U a $d \times d$-matrix. Then the semimartingale $\tilde{X} = UX$ has the following characteristics $(\tilde{B}, \tilde{C}, \tilde{\nu})$:*

$$\begin{aligned} \tilde{B}_t &= UB_t - \big(Uh(x) - h(Ux)\big) * \nu_t, \\ \tilde{C}_t &= UC_tU^\top, \\ \tilde{\nu}(A_1 \times A_2) &= \nu\big(A_1 \times U^{-1}(A_2 \setminus \{0\})\big) \quad \text{for } A_1 \in \mathcal{B}(\mathcal{T}), A_2 \in \mathcal{B}^d. \end{aligned}$$

We recall Girsanov's theorem from JS, Theorem III.3.24 to introduce some terminology.

Proposition 2. *Let X be a semimartingale with P-characteristics (B^P, C^P, ν^P) and denote by c, A the processes from (2.1). For any probability measure $Q \overset{\text{loc}}{\ll} P$, there exist a predictable function $Y \ge 0$ and a predictable $I\!R^d$-valued process β satisfying*

$$|(Y-1)h| * \nu_t^P + \int_0^t |c_s\beta_s|\, dA_s + \int_0^t \beta_s^\top c_s \beta_s\, dA_s < \infty \quad Q\text{-a.s. for all } t \in \mathcal{T}$$

and such that the Q-characteristics (B^Q, C^Q, ν^Q) of X are given by

$$B_t^Q = B_t^P + \int_0^t c_s\beta_s\, dA_s + \big((Y-1)h\big) * \nu_t^P, \qquad C_t^Q = C_t^P, \qquad \nu^Q(dt, dx) = Y(t,x)\, \nu^P(dt, dx).$$

We call β and Y the Girsanov parameters of Q (with respect to P relative to X).

Remark NV. Intuitively, Y describes how the jump distributions of X change when we pass from P to Q, and β together with Y determines the change in drift. C^P describes the P-quadratic variation of the continuous part of X and is therefore invariant under an absolutely continuous change of measure. Note that the Girsanov parameters are not unique: From the uniqueness of ν^P and ν^Q we only get uniqueness of $Y(\omega;\cdot,\cdot)$ on the support of $\nu^P(\omega)$, and with this and the uniqueness of B^P and B^Q we only get uniqueness of $c\beta$ for fixed c and A. However, we can choose the following *nice versions* of Y and β.

First we take Y such that $Y(\omega;s,x)=1$ identically for $(s,x)\notin \text{supp } \nu^P(\omega)$. Since ν^P does not charge $\{0\}\times I\!R^d$, this implies $Y(\omega;s,0)=1$ identically. Next, β_s is unique only if c_s is regular. If c_s is possibly degenerate, we choose β_s in the following way (and for simplicity, we only consider the case where c is deterministic and time-independent).

Let $\text{rank}\,(c)=r\le d$ and let λ^j be the eigenvalues of c, numbered such that $\lambda^j=0$ exactly for $j>r$. Since c is nonnegative definite, there exist a diagonal matrix $\tilde{c}$ with $\tilde{c}^{jj}=\lambda^j$ and an orthogonal matrix S such that $c=S\tilde{c}S^\top$. If β is any Girsanov parameter, then $c\beta=S\tilde{c}S^\top\beta$ and since $\tilde{c}$ is diagonal with $\tilde{c}^{jj}=0$ for $j>r$, we can set $(S^\top\beta)^j=0$ for $j>r$ without changing $c\beta$. So if we set $\gamma^j=(S^\top\beta)^j$ for $j\le r$ and $\gamma^j=0$ for $j>r$ and then define $\tilde{\beta}=S\gamma$, we get a new predictable process $\tilde{\beta}$ with $c\tilde{\beta}=c\beta$ and $(S^\top\tilde{\beta})^j=0$ for $j>r$. Moreover, $\tilde{\beta}$ with these properties is unique. In fact, $c\tilde{\beta}=c\tilde{\beta}'$ implies $S\tilde{c}S^\top\tilde{\beta}=S\tilde{c}S^\top\tilde{\beta}'$ and thus $\tilde{c}S^\top\tilde{\beta}=\tilde{c}S^\top\tilde{\beta}'$ which yields $(S^\top\tilde{\beta})^j=(S^\top\tilde{\beta}')^j$ for $j\le r$ by the properties of $\tilde{c}$. Finally, since $(S^\top\tilde{\beta})^j=0=(S^\top\tilde{\beta}')^j$ for $j>r$ by assumption, we get $S^\top\tilde{\beta}=S^\top\tilde{\beta}'$ and thus $\tilde{\beta}=\tilde{\beta}'$.

To simplify arguments, *we assume throughout that Y and β are chosen as above*. Our main results do not depend on this choice. ◇

2.2. Lévy processes

Let R be a probability measure on $(\Omega,\mathcal{F})$ and X a stochastic process null at 0 with RCLL paths and adapted to a filtration $I\!F$ satisfying the usual conditions under R. Then X is called an $(R,I\!F)$-*Lévy process* if for all $s\le t$, the random variable X_t-X_s is independent of $\mathcal{F}_s$ under R and has a distribution under R which depends only on $t-s$. (This is called a PIIS by JS in Section II.4.) If there is only a process X with independent and stationary increments under R, we call X an R-*Lévy process*, take as $I\!F$ the R-augmentation of the filtration generated by X and denote this by $I\!F^X$; this satisfies the usual conditions since a Lévy process is a Feller process. For $R=P$, we even sometimes drop the mention of P.

Every $I\!F$-Lévy process is an $I\!F$-semimartingale (JS, Cor. II.4.19), and an $I\!F$-martingale if and only if it is a local $I\!F$-martingale (He/Wang/Yan (1992), Theorem 11.46). X is an $(R,I\!F^X)$-Lévy process if and only if $E_R\left[\exp\left(iu^\top(X_t-X_s)\right)\big|\mathcal{F}^X_s\right]=E_R\left[\exp\left(iu^\top X_{t-s}\right)\right]$ for all $u\in I\!R^d$ and $s\le t$. According to JS, Theorem II.4.15 and Cor. II.4.19, a semimartingale X null at 0 is a Lévy process if and only if its characteristics are deterministic and linear in time, i.e., $B_t=bt$, $C_t=ct$ and $\nu(dt,dx)=K(dx)\,dt$, where $b\in I\!R^d$, c is a symmetric nonnegative definite $d\times d$-matrix and K is a σ-finite measure on $(I\!R^d,\mathcal{B}^d)$ with $K(\{0\})=0$ and $\int_{I\!R^d}(1\wedge|x|^2)\,K(dx)<\infty$. (Note that K is σ-finite because $\varphi(x)=\sqrt{1\wedge|x|^2}$ is >0 K-

a.e. and in $L^2(K)$. We need σ-finiteness later to use Fubini's theorem.) The constant triplet (b, c, K) coincides with the *Lévy characteristics* from the Lévy-Khinchine representation of the infinitely divisible distribution of X_1, and we see that for a Lévy process X, the compensator ν^P of the jump measure μ^X satisfies $\nu^P(\{t\} \times I\!R^d) = 0$ P-a.s. for all $t \in \mathcal{T}$.

2.3. A converse of Girsanov's theorem for Lévy processes

Proposition 2 generally describes a measure $Q \overset{\text{loc}}{\ll} P$ via parameters β, Y, and we want to express the density process Z^Q explicitly in terms of β, Y. This works if X has the weak property of predictable representation (as in He/Wang/Yan (1992), Definition 13.13; in Section III.4 of JS, this is called "all local martingales have the representation property relative to X"). As usual, we denote by $\mathcal{E}(Y)$ the stochastic exponential of a semimartingale Y. Putting together Section II.6, Theorem III.4.34, Theorem III.5.19, Cor. III.5.22 and Prop. III.5.10 from JS and using $\nu^P(\omega; \{t\} \times I\!R) = 0$ for all $t \in \mathcal{T}$ leads to

Proposition 3. *Let L be a P-Lévy process for $I\!F = I\!F^L$. If $Q \overset{\text{loc}}{\approx} P$ with Girsanov parameters β, Y, the density process of Q with respect to P is given by $Z^Q = \mathcal{E}(N^Q)$ with*

$$N_t^Q = \int_0^t \beta_s \, dL_s^c + (Y - 1) * (\mu^L - \nu^P)_t \quad \text{for } t \in \mathcal{T}. \tag{2.2}$$

While Proposition 2 gives a description of measures $Q \overset{\text{loc}}{\ll} P$ in terms of Girsanov parameters β, Y, we also need to go the other way round. We want to start with given quantities β, Y and find a measure $Q \overset{\text{loc}}{\ll} P$ which has β, Y as Girsanov parameters. In the setting of Lévy processes, Proposition 3 makes this look almost straightforward because if we define N^Q from β, Y as in (2.2), the natural candidate for Q should have $Z := \mathcal{E}(N^Q)$ as density process. However, two problems remain: We must verify that the local P-martingale Z is a true P-martingale, and then we need to prove the existence of a probability measure Q with the given martingale Z as density process. The first point needs conditions on β, Y. The second is easily solved for a finite time horizon $T_0 \in (0, \infty)$ by setting $dQ = Z_{T_0} \, dP$, no matter what the underlying space Ω is. For an infinite time horizon, existence of Q still follows if we work on the canonical path space $\Omega := I\!D\big([0, \infty), I\!R^d\big) =: I\!D^d$ with $\mathcal{F} := \mathcal{B}(I\!D^d)$ and appeal to Lemma 18.18 of Kallenberg (2002).

We start this program with a technical result. Its proof is purely analytic and therefore omitted; see Section 2.3 of Esche (2004).

Lemma 4. *The functions $f, g : [0, \infty) \to I\!R$ defined by $f(y) := y \log y - (y - 1)$ (with $0 \log 0 := 0$) and $g(y) := (1 - \sqrt{y})^2$ are convex and satisfy $0 \le g(y) \le f(y)$ for all $y \ge 0$.*

Proposition 5. *Let L be a P-Lévy process for $\mathbb{F} = \mathbb{F}^L$ with Lévy characteristics (b, c, K). If $\bar\beta$ is a predictable process and $\bar Y > 0$ a predictable function such that*

$$E_P\left[\exp\left(\int_0^t \left(\tfrac{1}{2}\bar\beta_s^\top c\,\bar\beta_s + \int_{\mathbb{R}^d} f\big(\bar Y(s,x)\big)\,K(dx)\right) ds\right)\right] < \infty \quad \text{for all } t \in \mathcal{T}, \tag{2.3}$$

then $\bar Y - 1$ is integrable with respect to $\mu^L - \nu^P$, and $\bar Z := \mathcal{E}(\bar N)$ with

$$\bar N_t := \int_0^t \bar\beta_s\,dL_s^c + (\bar Y - 1) * (\mu^L - \nu^P)_t, \quad t \in \mathcal{T} \tag{2.4}$$

is a strictly positive P-martingale on $\mathcal{T}$.

Proof. See Appendix.

If $\bar\beta$ and $\bar Y$ are deterministic and time-independent, we obtain from Proposition 5

Corollary 6. *Let L be a P-Lévy process for $\mathbb{F} = \mathbb{F}^L$ with Lévy characteristics (b, c, K). If $\bar\beta \in \mathbb{R}^d$ and $\bar Y : \mathbb{R}^d \to (0, \infty)$ is a measurable function with*

$$\int_{\mathbb{R}^d} f\big(\bar Y(x)\big)\,K(dx) < \infty, \tag{2.5}$$

then $\bar Z := \mathcal{E}\Big(\bar\beta^\top L^c + (\bar Y - 1) * (\mu^L - \nu^P)\Big)$ is a strictly positive P-martingale on $\mathcal{T}$.

The next result now starts with given quantities $\bar\beta, \bar Y$ and identifies these as Girsanov parameters of a measure $\bar Q$. As pointed out above, there is only one candidate for $\bar Q$, whose existence is ensured as soon as $\mathcal{E}(\bar N)$ is a true P-martingale and we either have a finite time horizon or work on the path space $\mathbb{D}^d$.

Proposition 7. *Let L be a P-Lévy process for $\mathbb{F} = \mathbb{F}^L$ with P-Lévy characteristics (b, c, K). Let $\bar\beta$ be a predictable process and $\bar Y > 0$ a predictable function such that $\bar Y - 1$ is integrable with respect to $\mu^L - \nu^P$, and define $\bar N = \int \bar\beta_s\,dL_s^c + (\bar Y - 1) * (\mu^L - \nu^P)$. If there is a probability measure $\bar Q \overset{\text{loc}}{\approx} P$ with density process $Z^{\bar Q} = \bar Z := \mathcal{E}(\bar N)$, then $\bar\beta$ and $\bar Y$ are the Girsanov parameters of $\bar Q$.*

Proof. See Appendix.

For future reference, we explicitly state the following result. (If we are only interested in constructing the Lévy measure $\bar Q$, an alternative proof could use a combination of Sato (1999), Theorem 8.1 and Cor. 11.6, with JS, Theorem IV.4.39, but would not be much shorter.)

Corollary 8. *Let P be a probability measure on $\Omega = I\!D^d$ with $\mathcal{F} = \mathcal{B}(I\!D^d)$, coordinate process L and $I\!F = I\!F^L$. Suppose that L is a P-Lévy process with P-Lévy characteristics (b, c, K). For any $\bar{\beta} \in I\!R^d$ and any measurable function $\bar{Y} : I\!R^d \to (0, \infty)$ with $\int_{I\!R^d} f\big(\bar{Y}(x)\big)\, K(dx) < \infty$, there exists a probability measure $\bar{Q} \overset{\text{loc}}{\approx} P$ on $(\Omega, \mathcal{F})$ with Girsanov parameters $\bar{\beta}, \bar{Y}$ and such that L is a $\bar{Q}$-Lévy process with $\bar{Q}$-Lévy characteristics*

$$b^{\bar{Q}} = b + c\,\bar{\beta} + \int_{I\!R^d} h(x)\big(\bar{Y}(x) - 1\big)\, K(dx), \qquad c^{\bar{Q}} = c, \qquad K^{\bar{Q}}(dx) = \bar{Y}(x)\, K(dx).$$

For $\mathcal{T} = [0, T_0]$ with $T_0 \in (0, \infty)$, this holds for any probability space $(\Omega, \mathcal{F}, P)$ and any P-Lévy process L if $I\!F = I\!F^L$ and $\mathcal{F} = \mathcal{F}_{T_0}$.

Proof. Combining Corollary 6 and Lemma 18.18 of Kallenberg (2002) gives a measure $\bar{Q} \overset{\text{loc}}{\approx} P$ with Girsanov parameters $\bar{\beta}, \bar{Y}$ by Proposition 7. The $\bar{Q}$-characteristics of L are then given by Proposition 2, and since they are deterministic and linear in time, L is a $\bar{Q}$-Lévy process. **q.e.d.**

2.4. Martingale measures for Lévy processes

As seen above, a measure $Q \overset{\text{loc}}{\ll} P$ can be described via two quantities β, Y that determine the characteristics of X under Q from those under P. By JS, Prop. II.2.29, X is a local Q-martingale if and only if $B^Q + (x - h(x)) * \mu^X$ is. Since this gives a relation between β and Y, a martingale measure Q for X should be determined by a single quantity Y, and for a Q-Lévy process, this should further reduce to a deterministic time-independent function.

To make these ideas more precise, let L be a P-Lévy process and U a fixed $d \times d$-matrix. For a given measure $Q \overset{\text{loc}}{\ll} P$ with Girsanov parameters β, Y, we want to give conditions on β, Y for UL to be a local Q-martingale. We denote by $\nu^P(dt, dx) = K(dx)\, dt$ the P-compensator of the jump measure μ^L of L. For technical reasons, we need to characterize the Q-integrability of large jumps of UL in a different manner, and this is achieved by the following result.

Lemma 9. *Let L be a P-Lévy process, U a fixed $d \times d$-matrix and $Q \overset{\text{loc}}{\ll} P$ with Girsanov parameters β, Y. If $E_Q[f(Y) * \nu^P_t] < \infty$ for all $t \in \mathcal{T}$, we have for all $t \in \mathcal{T}$*

(2.6) $|Ux - h(Ux)| * \nu^Q_t < \infty$ Q-a.s. *if and only if* $|U(xY - h)| * \nu^P_t < \infty$ Q-a.s.,

(2.7) $|Ux - h(Ux)| * \nu^Q_t \in \mathcal{L}^1(Q)$ *if and only if* $|U(xY - h)| * \nu^P_t \in \mathcal{L}^1(Q)$.

Proof. See Appendix.

Remarks. 1) We shall see in Lemma 12 that $E_Q[f(Y) * \nu^P_t] < \infty$ for all $t \in \mathcal{T}$ holds in particular if $Q \overset{\text{loc}}{\approx} P$ has a finite entropy process.

2) By JS, Prop. II.1.28, $|Ux-h(Ux)|*\nu^Q$ is Q-integrable if and only if $|Ux-h(Ux)|*\mu^L = |x-h(x)|*\mu^{UL}$ is, and the latter means that the large jumps of UL are Q-integrable. For $Q \overset{\text{loc}}{\approx} P$ with finite entropy process, this is by Lemma 9 equivalent to Q-integrability of $|U(xY-h)|*\nu^P$ which turns out to be a technically more convenient condition. ◇

Proposition 10. *Let L be a P-Lévy process for $\mathbb{F} = \mathbb{F}^L$ with P-Lévy characteristics (b,c,K), and U a fixed $d \times d$-matrix. Let $Q \overset{\text{loc}}{\ll} P$ with Girsanov parameters β, Y and such that $E_Q[f(Y)*\nu_t^P] < \infty$ for all $t \in \mathcal{T}$. Then UL is a local Q-martingale if and only if we have Q-a.s. for all $t \in \mathcal{T}$ both $|U(xY-h)|*\nu_t^P < \infty$ and*

$$U\Big(b + c\beta_t + \int_{\mathbb{R}^d} \big(xY(t,x) - h(x)\big)\, K(dx)\Big) = 0. \tag{2.8}$$

Condition (2.8) is called the martingale condition for UL.

Proof. See Appendix.

Remarks. **1)** The martingale condition is independent of the choice of the truncation function. In fact, if we replace h by some h', then b is replaced by $b' = b - \int_{\mathbb{R}^d} \big(h(x) - h'(x)\big)\, K(dx)$ (see JS, Prop. II.2.24) and (2.8) holds with (b', c, K) relative to h'.

2) If U is regular, (2.8) is equivalent to

$$b + c\,\beta_t + \int_{\mathbb{R}^d} \big(xY(t,x) - h(x)\big)\, K(dx) = 0 \quad Q\text{-a.s. for all } t \in \mathcal{T}. \tag{2.9}$$

This is the martingale condition as it appears in Bühlmann/Delbaen/Embrechts/Shiryaev (1996), Chan (1999), Fujiwara/Miyahara (2003) or Section VII.3 of Shiryaev (1999), among others. Note that (2.9) requires that the appearing integral is well-defined; this needs

$$\int_{\mathbb{R}^d} |xY(t,x) - h(x)|\, K(dx) < \infty \quad Q\text{-a.s. for } t \in \mathcal{T}$$

which is equivalent to $|U(xY-h)|*\nu_t^P < \infty$ Q-a.s. for $t \in \mathcal{T}$. Actually, not all authors are equally careful or explicit about verifying this condition. However, this does matter; see the comment following "Pseudo-Proposition 14" below. ◇

3. Reducing relative entropy

In this section, we show how the entropy process of any $Q \overset{\text{loc}}{\approx} P$ in a Lévy filtration can be reduced by averaging Girsanov parameters. Since this preserves the linear constraint imposed by the local martingale property, the MEMM, if it exists, must preserve the Lévy property. For reasons of integrability, this is not exactly true, but it does give the correct intuition.

To minimize repetitions, *we assume throughout this section that L is a P-Lévy process for $\mathbb{F} = \mathbb{F}^L$ with P-Lévy characteristics (b, c, K), and U is a fixed $d \times d$-matrix*. We start by computing the entropy process of a given Q in terms of its Girsanov parameters.

Lemma 11. *Fix a probability measure $Q \overset{\text{loc}}{\approx} P$ with Girsanov parameters β, Y and finite entropy process $\big(I_t(Q|P)\big)_{t\in\mathcal{T}}$, and denote by $Z = Z^Q = \mathcal{E}(N)$ its density process with respect to P. The canonical decomposition of the P-submartingale $Z \log Z$ is $Z \log Z = M + A$ with*

$$M = \int Z_-(1 + \log Z_-)\, dN + \big(Z_- f(Y)\big) * (\mu^L - \nu^P),$$
$$A = \tfrac{1}{2} \int Z_-\, d\langle N^c\rangle + \big(Z_- f(Y)\big) * \nu^P =: A' + A''.$$

Moreover, A'_t and A''_t are P-integrable for all $t \in \mathcal{T}$.

Proof. It is straightforward to check that $Z \log Z$ is a P-submartingale because the entropy process of Q is finite-valued. By the product rule, we have

$$d(Z \log Z) = Z_-\, d(\log Z) + (\log Z_-)\, dZ + d[Z, \log Z] \tag{3.1}$$

and the explicit expression for $Z = \mathcal{E}(N)$ yields

$$\log Z_t = N_t - \tfrac{1}{2}\langle N^c\rangle_t + \sum_{s\le t} \big(\log(1 + \Delta N_s) - \Delta N_s\big) =: N_t - \tfrac{1}{2}\langle N^c\rangle_t + D_t, \tag{3.2}$$

where the sum is absolutely convergent for all $t \in \mathcal{T}$. In fact, $|\Delta N_s| > \frac{1}{2}$ only for finitely many $s \le t$, and for $|x|^2 \le \frac{1}{2}$ we have $|\log(1 + x) - x| \le \text{const.}\, |x|^2$ so that

$$\sum_{s\le t} |\log(1 + \Delta N_s) - \Delta N_s| I_{\{|\Delta N_s|\le \frac{1}{2}\}} \le \text{const.} \sum_{s\le t} |\Delta N_s|^2 \le \text{const.}\, [N]_t < \infty.$$

To compute the $d[Z, \log Z]$-term in (3.1), we use $dZ = Z_-\, dN$ and (3.2) to get

$$d[Z, \log Z] = Z_-\, d[N, \log Z] = Z_- \left(d[N] - \tfrac{1}{2}\, d[N, \langle N^c\rangle] + d[N, D]\right). \tag{3.3}$$

Since $\langle N^c\rangle$ is continuous, $[N, \langle N^c\rangle]$ vanishes, and since D is of finite variation, we have

$$[N, D]_t = \sum_{s\le t} \Delta N_s \Delta D_s = \sum_{s\le t} \Delta N_s \big(\log(1 + \Delta N_s) - \Delta N_s\big). \tag{3.4}$$

This sum is absolutely convergent since $\sum_{s\le t} |\Delta N_s \Delta D_s| = \int_0^t \big|d[N, D]_s\big| \le ([N]_t)^{\frac{1}{2}}([D]_t)^{\frac{1}{2}}$ by the Kunita-Watanabe inequality, and since $\sum_{s\le t}(\Delta N_s)^2 \le [N]_t$ converges as well, we may decompose the sum in (3.4) and get

$$[N, D] = \sum_s \Delta N_s \log(1 + \Delta N_s) - \sum_s (\Delta N_s)^2.$$

This yields

$$[N, \log Z] = [N] - \sum_s (\Delta N_s)^2 + \sum_s \Delta N_s \log(1 + \Delta N_s) = \langle N^c \rangle + \sum_s \Delta N_s \log(1 + \Delta N_s),$$

or in terms of (3.3)

$$[Z, \log Z] = \int Z_- \, d\langle N^c \rangle + \sum_s Z_{s-} \Delta N_s \log(1 + \Delta N_s).$$

Putting all this together and using $dZ = Z_- \, dN$, we finally get a decomposition

$$\begin{aligned} Z \log Z &= \int Z_-(1 + \log Z_-) \, dN + \tfrac{1}{2} \int Z_- \, d\langle N^c \rangle + \sum_s Z_{s-} \big(\Delta D_s + \Delta N_s \log(1 + \Delta N_s) \big) \\ &= \int Z_-(1 + \log Z_-) \, dN + \tfrac{1}{2} \int Z_- \, d\langle N^c \rangle + \sum_s Z_{s-} f(1 + \Delta N_s) \qquad (3.5) \\ &=: M' + A' + V, \end{aligned}$$

where M' is a local P-martingale, A' is continuous and increasing, and V is increasing since $\Delta N_s > -1$ and $f \geq 0$. However, V is not predictable so that (3.5) is not yet the canonical decomposition. But $V = Z \log Z - M' - A'$ is locally P-integrable since all terms on the right-hand side are. Moreover, $\Delta N_s = \big(Y(s, \Delta L_s) - 1 \big) I_{\{\Delta L_s \neq 0\}}$ and $Y(s, 0) = 1$ yields $f(1 + \Delta N_s) = f\big(Y(s, \Delta L_s) \big) I_{\{\Delta L_s \neq 0\}}$ and therefore

$$V = \big(Z_- f(Y) \big) * \mu^L = |Z_- f(Y)| * \mu^L,$$

since $Z_- f(Y) \geq 0$. Because V is locally P-integrable, we obtain from JS, Prop. II.1.28 that

$$\big(Z_- f(Y) \big) * \mu^L = \big(Z_- f(Y) \big) * (\mu^L - \nu^P) + \big(Z_- f(Y) \big) * \nu^P,$$

so $\big(Z_- f(Y) \big) * \nu^P$ is the P-compensator of V and we end up with

$$Z \log Z = \Big(M' + \big(Z_- f(Y) \big) * (\mu^L - \nu^P) \Big) + \Big(A' + \big(Z_- f(Y) \big) * \nu^P \Big).$$

This is now in fact the canonical decomposition since the first term is a local P-martingale and the second is predictable and of finite variation.

As A' and A'' are both nonnegative, the final assertion follows if we prove that A_t is P-integrable for each $t \in \mathcal{T}$. But $Z \log Z$ is a P-submartingale with $Z_t \log Z_t \in \mathcal{L}^1(P)$ since $I_t(Q|P) < \infty$, and so the family $\{ Z_\tau \log Z_\tau \,|\, \tau \leq t \text{ is a stopping time} \}$ is uniformly integrable because $-e^{-1} \leq Z_\tau \log Z_\tau \leq E_P[Z_t \log Z_t | \mathcal{F}_\tau]$. Thus $(Z \log Z)^t$ is a P-submartingale of class (D) and so the increasing process in its unique Doob-Meyer decomposition is P-integrable. By uniqueness, $(Z \log Z)^t = M^t + A^t$ and therefore $E_P[A_t] = E_P[A^t_\infty] < \infty$. **q.e.d.**

The next result provides us with a number of important integrability properties.

Lemma 12. *For $Q \overset{\text{loc}}{\approx} P$ with finite entropy process and Girsanov parameters β, Y, the following random variables are Q-integrable for all $t \in \mathcal{T}$:*

a) $\int_0^t \beta_s^\top c\, \beta_s\, ds$; **b)** $\int_0^t |\beta_s|\, ds$; **c)** $f(Y) * \nu_t^P$; **d)** $\int_0^t Y(s,x)\, ds$ *for* $x \in \operatorname{supp} K$.

Moreover, the entropy process of Q with respect to P is explicitly given by

$$I_t(Q|P) = \tfrac{1}{2} E_Q\Big[\int_0^t (\beta_s)^\top c\, \beta_s\, ds\Big] + E_Q\Big[f(Y) * \nu_t^P\Big]. \tag{3.6}$$

Proof. a) The quadratic variation $\langle N^c \rangle_t = \int_0^t \beta_s^\top c\, \beta_s\, ds$ is the same under P and Q. Hence Lemma I.3.12 of JS and Lemma 11 give

$$E_Q\left[\langle N^c\rangle_t\right] = E_P\left[Z_t \langle N^c\rangle_t\right] = E_P\Big[\int_0^t Z_{s-}\, d\langle N^c\rangle_s\Big] = 2E_P[A'_t] < \infty.$$

b) Let $r = \operatorname{rank}(c)$ and λ^j be the eigenvalues of c, numbered such that $\lambda^j = 0$ exactly for $j > r$. Choose β as in Remark NV so that $\beta_s = S\gamma_s$ with $\gamma_s^j = 0$ for $j > r$. Then $\beta_s^\top c\, \beta_s = \gamma_s^\top \tilde{c}\, \gamma_s = \sum_{j=1}^r \lambda^j |\gamma_s^j|^2$ and $\beta_s^i = \sum_{j=1}^r S^{ij}\gamma_s^j$ so that $\int_0^t |\beta_s^i|\, ds \le \sum_{j=1}^r |S^{ij}| \int_0^t |\gamma_s^j|\, ds$.

Hence it suffices to show that $\int_0^t |\gamma_s^j|\, ds$ is Q-integrable, and this follows from part a) since

$$\begin{aligned}
\Big(E_Q\Big[\tfrac{1}{t}\int_0^t |\gamma_s^j|\, ds\Big]\Big)^2 &\le E_Q\Big[\tfrac{1}{t}\int_0^t |\gamma_s^j|^2\, ds\Big] \\
&\le \text{const.}\, E_Q\Big[\int_0^t \sum_{j=1}^r \lambda^j |\gamma_s^j|^2\, ds\Big] \\
&= \text{const.}\, E_Q\Big[\int_0^t \beta_s^\top c\, \beta_s\, ds\Big].
\end{aligned}$$

c) As in part a), Lemma I.3.12 of JS and Lemma 11 yield

$$\begin{aligned}
E_Q\left[f(Y) * \nu_t^P\right] &= E_P\Big[Z_t \int_0^t \int_{\mathbb{R}^d} f\big(Y(s,x)\big)\, K(dx)\, ds\Big] \\
&= E_P\Big[\int_0^t Z_{s-} \int_{\mathbb{R}^d} f\big(Y(s,x)\big)\, K(dx)\, ds\Big] \\
&= E_P\left[\big(Z_- f(Y)\big) * \nu_t^P\right] \\
&= 2E_P[A''_t] < \infty.
\end{aligned}$$

d) Since $E_Q\Big[\int_{\mathbb{R}^d \times [0,t]} f(Y)\, K(dx)\, ds\Big] = E_Q\left[f(Y) * \nu_t^P\right] < \infty$ by part c), we obtain $E_Q\Big[\int_0^t f\big(Y(s,x)\big)\, ds\Big] < \infty$ for $x \in \operatorname{supp} K$ by Fubini's theorem. Because f is convex, Jensen's inequality yields

$$f\Big(E_Q\Big[\tfrac{1}{t}\int_0^t Y(s,x)\,ds\Big]\Big) \le E_Q\Big[\tfrac{1}{t}\int_0^t f\big(Y(s,x)\big)\,ds\Big] < \infty \quad \text{for } x \in \operatorname{supp} K,$$

and as $f(y) < \infty$ implies $y < \infty$, the assertion follows.

To obtain (3.6), note that $Z \log Z = M + A$ and M^t is a uniformly integrable P-martingale by the last argument in the proof of Lemma 11. So parts a) and c) give

$$I_t(Q|P) = E_P[Z_t \log Z_t] = E_P[A_t] = \tfrac{1}{2} E_Q\Big[\int_0^t (\beta_s)^\top c\,\beta_s\,ds\Big] + E_Q\left[f(Y) * \nu_t^P\right].$$

q.e.d.

Now we can prove that relative entropy is reduced by averaging Girsanov parameters.

Theorem 13. *Suppose that* $Q \overset{\text{loc}}{\approx} P$ *with* $I_T(Q|P) < \infty$ *for some* $T \in (0,\infty)$, *and define*

$$\beta^\ell = \tfrac{1}{T}\, E_Q\Big[\int_0^T \beta_s^Q\,ds\Big],$$

$$Y^\ell(x) = \tfrac{1}{T}\, E_Q\Big[\int_0^T Y^Q(s,x)\,ds\Big] \quad \textit{for } x \in \operatorname{supp} K.$$

a) *There exists a probability measure* $Q^\ell \approx P$ *on* $\mathcal{F}_{T_0}$ *with Girsanov parameters* β^ℓ *and* Y^ℓ, *which satisfies* $I_{T_0}(Q^\ell|P) \le I_{T_0}(Q|P)$, *and such that the restriction of* L *to the interval* $[0,T_0]$ *is a* Q^ℓ*-Lévy process.*

b) *Let* $\Omega = I\!D^d$ *with* $\mathcal{F} = \mathcal{B}(I\!D^d)$, *coordinate process* L *and* $I\!F = I\!F^L$. *Then there exists a probability measure* $Q^\ell \overset{\text{loc}}{\approx} P$ *on* $(\Omega,\mathcal{F})$ *with Girsanov parameters* β^ℓ *and* Y^ℓ, *which satisfies* $I_T(Q^\ell|P) \le I_T(Q|P)$, *and such that* L *is a* Q^ℓ*-Lévy process on* $[0,\infty)$.

c) *For* Q^ℓ *constructed as above,* $I_T(Q^\ell|P) = I_T(Q|P)$ *if and only if both* $\beta_\cdot^Q = \beta^\ell$ $P \otimes \lambda$*-a.e. on* $\Omega \times [0,T]$ *and* $Y^Q(\cdot,x) = Y^\ell(x)$ $P \otimes \lambda$*-a.e. on* $\Omega \times [0,T]$, *for all* $x \in \operatorname{supp} K$, *i.e., if and only if* L *is a* Q*-Lévy process on* $[0,T]$.

Proof. By Lemma 12, β^ℓ and Y^ℓ are well-defined, and Corollary 8 yields the existence of Q^ℓ with Girsanov parameters β^ℓ, Y^ℓ and the Q^ℓ-Lévy property for L because

$$(3.7)\quad \int_{I\!R^d} f\big(Y^\ell(x)\big)\,K(dx) \le \int_{I\!R^d} E_Q\Big[\tfrac{1}{T}\int_0^T f\big(Y^Q(s,x)\big)\,ds\Big]\,K(dx) = \tfrac{1}{T} E_Q\left[f(Y^Q) * \nu_T^P\right] < \infty$$

by the definition of Y^ℓ, Jensen's inequality, Fubini's theorem for nonnegative functions and part c) of Lemma 12. Moreover, (3.6) gives

$$I_T(R|P) = \tfrac{1}{2} E_R\Big[\int_0^T (\beta_s^R)^\top c\,\beta_s^R\,ds\Big] + E_R\left[f(Y^R) * \nu_T^P\right] \quad \text{for } R \in \{Q, Q^\ell\},$$

and we claim that

$$E_Q\Big[\int_0^T (\beta_s^Q)^\top c\,\beta_s^Q\,ds\Big] \ge E_{Q^\ell}\Big[\int_0^T (\beta^\ell)^\top c\,\beta^\ell\,ds\Big] = T(\beta^\ell)^\top c\,\beta^\ell, \tag{3.8}$$

$$E_Q\left[f(Y^Q) * \nu_T^P\right] \ge E_{Q^\ell}\left[f(Y^\ell) * \nu_T^P\right] = T\int_{I\!R^d} f\big(Y^\ell(x)\big)\,K(dx), \tag{3.9}$$

with equality if and only if $\beta_\cdot^Q = \beta^\ell$ $P\otimes\lambda$-a.e. on $\Omega\times[0,T]$ and $Y^Q(\cdot,x) = Y^\ell(\cdot,x)$ $P\otimes\lambda$-a.e. on $\Omega\times[0,T]$, for all $x\in\operatorname{supp}K$. For brevity, we omit to say "on $\Omega\times[0,T]$" below.

Now (3.9) is simply (3.7); since f is strictly convex, equality holds if and only if we have $Y^Q(\cdot,x) = Y^\ell(\cdot,x)$ $P\otimes\lambda$-a.e., for all $x\in\operatorname{supp}K$. For the proof of (3.8), we use the notation of Remark NV and define $\tilde\gamma_s = \sqrt{\tilde c}\,S^\top\beta_s^Q$ so that $(\beta_s^Q)^\top c\,\beta_s^Q = |\tilde\gamma_s|^2$. Jensen's inequality then gives $\frac{1}{T}\int_0^T |\tilde\gamma_s|^2\,ds \ge \Big|\frac{1}{T}\int_0^T \tilde\gamma_s\,ds\Big|^2$ and therefore

$$E_Q\Big[\frac{1}{T}\int_0^T (\beta_s^Q)^\top c\,\beta_s^Q\,ds\Big] \ge E_Q\bigg[\Big|\frac{1}{T}\int_0^T \tilde\gamma_s\,ds\Big|^2\bigg] \ge \bigg|\frac{1}{T}E_Q\Big[\int_0^T \tilde\gamma_s\,ds\Big]\bigg|^2;$$

equality holds if and only if $\tilde\gamma$ (or, equivalently, β^Q) is constant $P\otimes\lambda$-a.e. But

$$\frac{1}{T}E_Q\Big[\int_0^T \tilde\gamma_s\,ds\Big] = \sqrt{\tilde c}\,S^\top\frac{1}{T}E_Q\Big[\int_0^T \beta_s^Q\,ds\Big] = \sqrt{\tilde c}\,S^\top\beta^\ell$$

by the definitions of $\tilde\gamma$ and β^ℓ and therefore $E_Q\Big[\int_0^T (\beta_s^Q)^\top c\,\beta_s^Q\,ds\Big] \ge T(\beta^\ell)^\top c\,\beta^\ell$, with equality if and only if $\beta_\cdot^Q = \beta^\ell$ $P\otimes\lambda$-a.e. This proves b) and c). To obtain a), we argue with $T = T_0$ if $I_{T_0}(Q^\ell|P)<\infty$; otherwise, we use (3.6) to get $I_{T_0}(Q^\ell|P) = \frac{T_0}{T}I_T(Q^\ell|P) < \infty$. **q.e.d.**

Using the description of the local Q-martingale property of UL in Proposition 10 yields

"Pseudo-Proposition 14". *Suppose Q^ℓ is constructed from Q as in Theorem 13. If UL is a local martingale under Q, it is still a local martingale under Q^ℓ.*

"Pseudo-Proof". By construction, the Girsanov parameters of Q^ℓ are obtained by averaging those of Q. But the local martingale property of UL is characterized by the linear constraint (2.8) between Girsanov parameters, and this is preserved by averaging. **"q.e.d."**

We have put "Pseudo-Proposition 14" and its "pseudo-proof" in quotation marks because they are not necessarily true as they stand. More precisely, we need Fubini's theorem to prove that (2.8) is preserved by averaging, and this requires the additional assumption (on Q) that $E_Q\big[|U(xY-h)| * \nu_T^P\big]<\infty$. Hence the subsequent "pseudo-proof" of the next result is also flawed. Nevertheless, Theorem A itself is true, and we shall provide a proper proof in Section 5. The current presentation has been chosen to highlight the key idea behind the argument.

Theorem A. *Let L be a P-Lévy process for $I\!F = I\!F^L$, and U a fixed $d \times d$-matrix. Suppose that $\mathcal{Q}_e^U(L) \cap \mathcal{Q}_f^U(L) \cap \mathcal{Q}_\ell^U(L) \neq \emptyset$. If $Q^E(UL)$ exists, then L is a Lévy process under $Q^E(UL)$.*

"Pseudo-Proof". For brevity, write Q^E for $Q^E(UL)$. If the assertion were not true, we could use Theorem 13 to construct $(Q^E)^\ell$ which would be a local martingale measure for UL by "Pseudo-Proposition 14" and satisfy $I_T\big((Q^E)^\ell \big| P\big) < I_T(Q^E|P)$ for some $T \in (0,\infty)$ by part c) of Theorem 13, in contradiction to the optimality of Q^E. **"q.e.d."**

4. Identifying the minimal entropy martingale measure

In this section, we give a very explicit representation for the MEMM $Q^E(UL)$, and one important point is to make transparent where this comes from. We have seen in Theorem A that $Q^E(UL)$, if it exists, preserves the Lévy property of L. Instead of minimizing relative entropy over all ELMMs for UL, it should thus be sufficient to minimize only over those which in addition preserve the Lévy property of L. (That this is indeed enough is proved in Corollary 20 in Section 5.) We use this intuition to derive by partly formal arguments a candidate for $Q^E(UL)$, and then we prove that this candidate gives indeed the optimal measure. For simplicity, we give the derivation for the case $L = UL$ where U is the identity matrix, and for brevity, we often write Q^E for $Q^E(UL)$ and $\mathcal{Q}_s^U$ for $\mathcal{Q}_s^U(L)$, where $s \in \{a, e, f, \ell\}$.

To find a candidate for Q^E, we start with any Q in $\mathcal{Q}_e^U \cap \mathcal{Q}_f^U \cap \mathcal{Q}_\ell^U$ because this is where Q^E should lie. Since Q is in $\mathcal{Q}_\ell^U$, it has deterministic time-independent Girsanov parameters $\beta \in I\!R^d$ and $Y : I\!R^d \to (0,\infty)$. As $Q \in \mathcal{Q}_f^U$, (3.6) gives an explicit expression for $I_t(Q|P)$ in terms of β, Y, and as $Q \in \mathcal{Q}_e^U$, the martingale condition (2.8) or (2.9) relates β and Y by

$$c\beta = -b - \int_{I\!R^d} \big(xY(x) - h(x)\big)\,K(dx) =: -b - k(Y). \tag{4.1}$$

If we take c regular for simplicity, we can solve (4.1) for β and plug into (3.6) to get

$$I_t(Q|P) = \Big(\tfrac{1}{2}\big(b + k(Y)\big)^\top c^{-1}\big(b + k(Y)\big) + \int_{I\!R^d} f\big(Y(x)\big)\,K(dx)\Big)\,t =: \bar{I}(Y)\,t. \tag{4.2}$$

As explained intuitively in Subsection 2.4, we have now parametrized Q by a function Y and want to minimize the functional $\bar{I}(Y)$. If Y_* is optimal, we obtain for any Y and all $\varepsilon > 0$

$$\begin{aligned}
0 &\le \bar{I}\big(Y_* + \varepsilon(Y - Y_*)\big) - \bar{I}(Y_*) \\
&= \int \Big(f\big(Y_* + \varepsilon(Y - Y_*)\big) - f(Y_*)\Big)\,dK + \varepsilon\Big(\int x(Y - Y_*)\,dK\Big)^\top c^{-1}\Big(b + \int (xY_* - h)\,dK\Big) \\
&\quad + \tfrac{1}{2}\varepsilon^2\Big(\int x(Y - Y_*)\,dK\Big)^\top c^{-1}\Big(\int x(Y - Y_*)\,dK\Big)
\end{aligned}$$

by using (4.2) and the expression for $k(Y)$. Now divide by ε and let ε tend to 0 to get

$$0 \leq \int f'(Y_*)(Y - Y_*)\, dK + \big(\int x(Y - Y_*)\, dK\big)^\top c^{-1}\big(b + \int (xY_* - h)\, dK\big) \quad \text{for all } Y.$$

The particular choice $Y := (1 \pm \delta)Y_*$ with $\delta > 0$ leads to

$$0 = (\int xY_*\, dK)^\top c^{-1}\big(b + \int (xY_* - h)\, dK\big) + \int f'(Y_*)Y_*\, dK = \int (-\beta_*^\top x + \log Y_*)Y_*\, dK,$$

where $\beta_* = \beta_*(Y_*) = -c^{-1}\big(b + \int (xY_* - h)\, dK\big)$ is the optimal β from the martingale condition (4.1) and we have used $f'(y) = \log y$. As $Y_* > 0$, we thus should have $\log Y_*(x) - \beta_*^\top x = 0$ or

$$Y_*(x) = e^{\beta_*^\top x} \quad \text{(at least on the support of } K\text{)}.$$

Hence the optimal measure Q^* should have Girsanov parameters $\beta_* = u_*$ and $Y_*(x) = e^{u_*^\top x}$ for some $u_* \in I\!R^d$ which must be determined from the martingale condition

$$b + cu_* + \int_{I\!R^d} \big(xe^{u_*^\top x} - h(x)\big)\, K(dx) = 0.$$

This recipe gives our candidate for Q^E. To make it even more explicit, we define as in Corollary 6 $Z^* := \mathcal{E}\Big(u_*^\top L^c + (Y_* - 1) * (\mu^L - \nu^P)\Big)$ and find by formal calculations that $Z_t^* = \exp\big(u_*^\top L_t^c + (u_*^\top x) * \mu_t^L - \text{const.}\, t\big)$ which suggests that the density process of Q^E should be of the form $Z_t^{Q^E} = \text{const.}(t)\, e^{u_*^\top L_t}$. Hence we expect Q^E to be a so-called Esscher measure.

To explain this more carefully, we start with a P-Lévy process L with P-Lévy characteristics (b, c, K) and fix a $d \times d$-matrix U. We define

$$\mathcal{A} := \big\{u \in I\!R^d \,\big|\, E_P\big[e^{u^\top L_1}\big] < \infty\big\}$$

and recall from Theorem 25.17 of Sato (1999) that

$$\Psi(u) := b^\top u + \tfrac{1}{2} u^\top c\, u + \int_{I\!R^d} \big(e^{u^\top x} - 1 - (u^\top x) I_{\{|x| \leq 1\}}\big)\, K(dx)$$

is well-defined on $\mathcal{A}$ and that $E_P\big[e^{u^\top L_t}\big] = e^{t\Psi(u)}$ for $u \in \mathcal{A}$. Due to the Lévy structure of L under P, it is easy to see that $Z_t^u := \exp\big(u^\top L_t - t\Psi(u)\big)$ is a strictly positive P-martingale and therefore the density process of a measure $Q^u \overset{\text{loc}}{\approx} P$ on $\big(I\!D^d, \mathcal{B}(I\!D^d)\big)$ by Lemma 18.18 of Kallenberg (2002). Any such Q^u is called *Esscher measure (for L with parameter u)*. If Q^u is in addition a martingale measure for UL, we call Q^u *Esscher martingale measure for UL*.

The next result collects some simple properties of Esscher measures.

Lemma 15. *Fix $u \in \mathcal{A}$ and let Q^u be an Esscher measure with parameter u. Then:*

a) *L is a Lévy process under Q^u.*

b) *The Girsanov parameters of Q^u are given by $\beta^u = u$, $Y^u(x) = e^{u^\top x}$.*

c) *If Q^u is in addition an Esscher martingale measure for UL and $u \in \operatorname{range}(U^\top)$, the entropy process of Q^u is finite-valued and given by $I_t(Q^u|P) = -t\Psi(u)$ for all $t \in \mathcal{T}$.*

Proof. a) See Shiryaev (1999), Theorem VII.3c.1.

b) If β, Y are the Girsanov parameters of Q^u, part a) implies that β is a constant and $Y = Y(x)$ is a deterministic function. Proposition 3 yields

$$Z_t^u = \mathcal{E}(N^u)_t = \mathcal{E}\Big(\beta^\top L^c + (Y-1) * (\mu^L - \nu^P)\Big)_t$$

and the explicit formula for the stochastic exponential gives

$$\begin{aligned} \log Z_t^u &= \beta^\top L_t^c - \tfrac{1}{2}\beta^\top c\,\beta\,t + (Y-1) * (\mu^L - \nu^P)_t + \sum_{s \le t} \big(\log(1 + \Delta N_s^u) - \Delta N_s^u\big) \\ &= u^\top L_t - t\Psi(u) \end{aligned}$$

by the definition of Z^u. Comparing the continuous local martingale parts of the two representations yields $\beta = u$, and since $\Delta N_t^u = Y(\Delta L_t) - 1$, comparing the jumps implies $u^\top \Delta L_t = \log Y(\Delta L_t)$ so that we get $Y(x) = e^{u^\top x}$ on the support of K.

c) Write $u = U^\top \tilde{u}$. By part a) and Proposition 1, UL is both a Lévy process and a local martingale under Q^u and hence a true Q^u-martingale. Because $u \in \mathcal{A}$, this gives

$$I_t(Q^u|P) = E_{Q^u}\big[\log Z_t^u\big] = E_{Q^u}\big[\tilde{u}^\top U L_t - t\Psi(u)\big] = -t\Psi(u) < \infty.$$

q.e.d.

To prove that our candidate is indeed optimal, we use the following Lévy version of Prop. 3.2 in Grandits/Rheinländer (2002). It tells us that the Esscher martingale measure for UL is optimal in $\mathcal{Q}_\ell^U$ if it exists. Note that we do not assume that Q^E exists.

Lemma 16. *Let L be a P-Lévy process. If there exists an Esscher martingale measure Q^u for UL with $u \in \operatorname{range}(U^\top)$, then $I_t(Q^u|P) \le I_t(R|P)$ for all $R \in \mathcal{Q}_\ell^U$ and for all $t \in \mathcal{T}$, or in other words, $Q^u = Q_\ell^E(UL)$.*

Proof. From part c) of Lemma 15, we know that $I_t(Q^u|P) = -t\Psi(u) < \infty$. Write $u = U^\top \tilde{u}$ and fix $R \in \mathcal{Q}_\ell^U$. Then UL is under R a Lévy process and a local martingale, hence a true martingale, and because relative entropy is nonnegative, we get as in the proof of Lemma 15

$$I_t(R|P) = I_t(R|Q^u) + E_R\big[\log Z_t^u\big] = I_t(R|Q^u) - t\Psi(u) \ge -t\Psi(u) = I_t(Q^u|P).$$

q.e.d.

Now we can prove that the heuristically derived recipe for our candidate produces indeed the minimal entropy martingale measure $Q^E(UL)$.

Theorem B. *Let L be a P-Lévy process for $I\!F = I\!F^L$ with Lévy characteristics (b, c, K), and U a fixed $d \times d$-matrix. Suppose that there exists $u_* \in \operatorname{range}(U^\top)$ such that*

$$\int_{I\!R^d} \left| x e^{u_*^\top x} - h(x) \right| K(dx) < \infty, \tag{4.3}$$

$$U\Big(b + c u_* + \int_{I\!R^d} \big(x e^{u_*^\top x} - h(x) \big) K(dx) \Big) = 0. \tag{4.4}$$

Then both the Esscher measure Q^{u_*} and the minimal entropy martingale measure $Q^E(UL)$ exist and coincide.

Proof. Existence of Q^{u_*} follows if we show that $u_* \in \mathcal{A}$, and by Theorem 25.17 of Sato (1999), this holds if and only if $\int_{\{|x|>1\}} e^{u_*^\top x} K(dx) < \infty$. But with $h_0(x) := |x| I_{\{|x| \le 1\}}$, we easily get $|h_0(x) - h(x)| \le \text{const.}\,(1 \wedge |x|^2)$ and therefore

$$\begin{aligned}
\int_{\{|x|>1\}} e^{u_*^\top x} K(dx) &\le \int_{\{|x|>1\}} |x| e^{u_*^\top x} K(dx) + \int_{\{|x|\le 1\}} \left| x \big(e^{u_*^\top x} - 1 \big) \right| K(dx) \\
&= \int_{I\!R^d} \left| x e^{u_*^\top x} - h_0(x) \right| K(dx) \\
&\le \int_{I\!R^d} \left| x e^{u_*^\top x} - h(x) \right| K(dx) + \int_{I\!R^d} |h_0(x) - h(x)| K(dx) < \infty
\end{aligned}$$

by (4.3) and the properties of K. By part b) of Lemma 15, the Girsanov parameters of Q^{u_*} are $\beta_* = u_*$ and $Y_*(x) = e^{u_*^\top x}$. Hence (4.3) and (4.4) are the conditions from Proposition 10 for UL to be a local Q^{u_*}-martingale so that $Q^{u_*} \in \mathcal{Q}_e^U \cap \mathcal{Q}_f^U \cap \mathcal{Q}_\ell^U$ by part c) of Lemma 15. Lemma 16 implies that Q^{u_*} has minimal entropy among all $Q \in \mathcal{Q}_e^U \cap \mathcal{Q}_f^U \cap \mathcal{Q}_\ell^U$ so that $Q_\ell^E(UL)$ exists and coincides with Q^{u_*}. But then Corollary 20 below implies that $Q^E(UL)$ exists as well and $Q^E(UL) = Q_\ell^E(UL) = Q^{u_*}$. **q.e.d.**

Remark. The derivation of our candidate for $Q^E(UL)$ suggests in particular that finding the MEMM for a Lévy process can be reduced to a *deterministic optimization problem*. In fact, consider for $\beta \in I\!R$ and measurable functions $Y : I\!R^d \to (0, \infty)$ the functional

$$\hat{I}(\beta, Y) := \tfrac{1}{2} \beta^\top c \, \beta + \int_{I\!R^d} f\big(Y(x)\big) K(dx)$$

which by (3.6) equals $I_1(Q|P)$ for the measure $Q \overset{\text{loc}}{\approx} P$ with Girsanov parameters β, Y. Denote by $\mathcal{H}$ the class of all pairs (β, Y) satisfying

$$\int_{I\!R^d} f\big(Y(x)\big) K(dx) < \infty, \tag{4.5}$$

$$\int_{\mathbb{R}^d} |xY(x) - h(x)|\, K(dx) < \infty, \tag{4.6}$$

$$U\Big(b + c\beta + \int_{\mathbb{R}^d} \big(xY(x) - h(x)\big)\, K(dx)\Big) = 0. \tag{4.7}$$

By Corollary 8, (4.5) is the condition for the existence of Q with $I_1(Q|P) < \infty$, whereas (4.6) and (4.7) come from the martingale condition in Proposition 10. If we set $Y^u(x) := e^{u^\top x}$ for $u \in \mathbb{R}^d$, purely analytic arguments show that if there is some $u_* \in \operatorname{range}(U^\top)$ with $(u_*, Y^{u_*}) \in \mathcal{H}$, then $\hat{I}(u_*, Y^{u_*}) \le \hat{I}(\beta, Y)$ for all $(\beta, Y) \in \mathcal{H}$. The crucial point is to prove

$$\begin{aligned} 0 &\le \tfrac{1}{2}(\beta - u_*)^\top c\,(\beta - u_*) + \int_{\mathbb{R}^d} Y^{u_*}(x) f\Big(\tfrac{Y(x)}{Y^{u_*}(x)}\Big)\, K(dx) \\ &= \tfrac{1}{2}\beta^\top c\,\beta + \int_{\mathbb{R}^d} f\big(Y(x)\big)\, K(dx) - \Big(\tfrac{1}{2}u_*^\top c\, u_* + \int_{\mathbb{R}^d} f\big(Y^{u_*}(x)\big)\, K(dx)\Big), \end{aligned}$$

where the first inequality is obvious and the second corresponds to the probabilistic argument in the proof of Lemma 16. For details, we refer to Section 4.3 of Esche (2004). $\diamond$

5. A proper proof of Theorem A

In this section, we give a rigorous proof of Theorem A. *Throughout the section, L is a P-Lévy process for $\mathbb{F} = \mathbb{F}^L$ with Lévy characteristics (b, c, K), and U is a fixed $d \times d$-matrix.* The basic idea is the assertion of “Pseudo-Proposition 14” that the local martingale property of UL under Q is preserved under an averaging of Girsanov parameters. However, we can rigorously prove this only if the big jumps of UL are Q-integrable. To make this precise, we define for a semimartingale X a new set of martingale measures by

$$\mathcal{Q}^U_{\text{int}}(X) := \big\{ Q \in \mathcal{Q}^U_a(X) \,\big|\, E_Q\big[|U(xY - h)| * \nu^P_t\big] < \infty \text{ for all } t \in \mathcal{T} \big\}$$

and write $\mathcal{Q}^U_s = \mathcal{Q}^U_s(L)$ for $s \in \{a, e, f, \text{int}, \ell\}$. As pointed out after Lemma 9, Q being in $\mathcal{Q}^U_{\text{int}}$ is equivalent to Q-integrability of $|x - h(x)| * \mu^{UL}$, the sum over large jumps of UL, if Q has a finite entropy process. But for proof purposes, the above formulation is more convenient.

Remark. Any $Q \in \mathcal{Q}^U_e \cap \mathcal{Q}^U_f \cap \mathcal{Q}^U_\ell$ is also in $\mathcal{Q}^U_{\text{int}}$. In fact, part c) of Lemma 12 and the proof of Proposition 10 show that $|U(xY - h)| * \nu^P$ is finite-valued; this uses only $Q \in \mathcal{Q}^U_e \cap \mathcal{Q}^U_f$. If also $Q \in \mathcal{Q}^U_\ell$, then Y is deterministic, hence so is $|U(xY - h)| * \nu^P$, and then finiteness is the same as Q-integrability. $\diamond$

The correct version of “Pseudo-Proposition 14” is now

Proposition 17. *Let $Q \in \mathcal{Q}_e^U \cap \mathcal{Q}_f^U \cap \mathcal{Q}_{\text{int}}^U$ with Girsanov parameters β, Y. Then Q^ℓ constructed from Q in Theorem 13 is in $\mathcal{Q}_e^U \cap \mathcal{Q}_f^U \cap \mathcal{Q}_\ell^U$ so that UL is still a local Q^ℓ-martingale.*

Proof. Let β^ℓ, Y^ℓ be the Girsanov parameters of Q^ℓ. Theorem 13 gives $Q^\ell \overset{\text{loc}}{\approx} P$, that L is a Q^ℓ-Lévy process and $I_T(Q^\ell|P) < \infty$ for some $T \in (0, \infty)$. Since β^ℓ, Y^ℓ are deterministic and time-independent, this implies $I_t(Q^\ell|P) = t\, I_1(Q^\ell|P) < \infty$ for all $t \in \mathcal{T}$ and it only remains to show that UL is a local Q^ℓ-martingale. By Proposition 10, we need to verify that $\int_{\mathbb{R}^d} \left|U\left(xY^\ell(x) - h(x)\right)\right| K(dx) < \infty$ and that β^ℓ, Y^ℓ satisfy the martingale condition (2.8).

Using the definition of Y^ℓ, Jensen's inequality and Fubini's theorem yields

$$\begin{aligned}\int_{\mathbb{R}^d} \left|U\left(xY^\ell(x) - h(x)\right)\right| K(dx) &= \int_{\mathbb{R}^d} \left|E_Q\left[\tfrac{1}{T}\int_0^T U\big(xY(s,x) - h(x)\big)\, ds\right]\right| K(dx) \\ &\le \tfrac{1}{T} E_Q\left[|U(xY - h)| * \nu_T^P\right] < \infty\end{aligned}$$

since $Q \in \mathcal{Q}_{\text{int}}^U$. This allows us now to use Fubini's theorem for $U\big(xY(s,x) - h(x)\big)$ and combine this with (2.8) for β, Y to conclude

$$\begin{aligned}&U\Big(b + c\beta^\ell + \int_{\mathbb{R}^d} \big(xY^\ell(x) - h(x)\big)\, K(dx)\Big) \\ &= \tfrac{1}{T} E_Q\Big[\int_0^T U\Big(b + c\beta_s + \int_{\mathbb{R}^d} \big(xY(s,x) - h(x)\big)\, K(dx)\Big)\, ds\Big] \\ &= 0\end{aligned}$$

so that β^ℓ, Y^ℓ satisfy the martingale condition for UL as well. **q.e.d.**

If Q is not in $\mathcal{Q}_{\text{int}}^U$, we do not know if Q^ℓ from Theorem 13 preserves the local martingale property of UL. The key idea for using Proposition 17 in a proper proof of Theorem A is thus to argue that the martingale measures in $\mathcal{Q}_e^U \cap \mathcal{Q}_f^U \cap \mathcal{Q}_{\text{int}}^U$ are dense in the set of all martingale measures $\mathcal{Q}_e^U \cap \mathcal{Q}_f^U$ in a suitable sense. This is achieved by

Proposition 18. *Let $Q \in \mathcal{Q}_e^U \cap \mathcal{Q}_f^U$ and suppose $\mathcal{Q}_e^U \cap \mathcal{Q}_f^U \cap \mathcal{Q}_\ell^U \neq \emptyset$. Then there exists a sequence $(Q^n)_{n \in \mathbb{N}}$ in $\mathcal{Q}_e^U \cap \mathcal{Q}_f^U \cap \mathcal{Q}_{\text{int}}^U$ with $\lim_{n\to\infty} I_t(Q^n|P) = I_t(Q|P)$ for all $t \in \mathcal{T}$.*

Proof. Choose $\bar{Q} \in \mathcal{Q}_e^U \cap \mathcal{Q}_f^U \cap \mathcal{Q}_\ell^U$ so that $\bar{Q} \in \mathcal{Q}_{\text{int}}^U$ by the remark before Proposition 17. Denote by β, Y the Girsanov parameters of Q and write the density process as $Z^Q = \mathcal{E}(N)$ with $N = \int \beta\, dL^c + (Y - 1) * (\mu^L - \nu^P)$ by Proposition 3. Analogous quantities with a bar $\bar{}$ refer to $\bar{Q}$. Because UL is a local Q-martingale, $|U(xY - h)| * \nu^P$ is continuous and finite-valued by Proposition 10, hence locally Q-integrable with localizing sequence $(\tau_n)_{n\in\mathbb{N}}$. To construct Q^n which agrees with Q up to τ_n and with $\bar{Q}$ afterwards, we define for $n \in \mathbb{N}$

$$\beta^n_s = \beta_s I_{[\![0,\tau_n]\!]} + \bar{\beta} I_{]\!]\tau_n,\infty[\![},$$
$$Y^n(s,x) = Y(s,x) I_{[\![0,\tau_n]\!]} + \bar{Y}(x) I_{]\!]\tau_n,\infty[\![},$$

and set $N^n = \int \beta^n \, dL^c + (Y^n - 1) * (\mu^L - \nu^P)$ and $Z^n = \mathcal{E}(N^n)$. It is straightforward to check that $N^n = N^{\tau_n} + \bar{N} - \bar{N}^{\tau_n}$ and $Z^n = Z I_{[\![0,\tau_n]\!]} + \frac{Z_{\tau_n}}{\bar{Z}_{\tau_n}} \bar{Z} I_{]\!]\tau_n,\infty[\![}$. Hence Z^n is a strictly positive martingale starting at 1 and there exists $Q^n \overset{\text{loc}}{\approx} P$ with density process Z^n. (For $\mathcal{T} = [0,\infty)$, we work on the path space $\mathbb{D}^d$ as usual.) It follows from Proposition 7 that β^n, Y^n are the Girsanov parameters of Q^n, and $Q^n = Q$ on $\mathcal{F}_{\tau_n}$ since $Z^n_{\tau_n} = Z_{\tau_n}$. We claim that Q^n is a local martingale measure for UL with $Q^n \in \mathcal{Q}^U_{\text{int}}$. In fact, the definition of Y^n yields

$$|U(xY^n - h)| * \nu^P_t \le |U(xY - h)| * \nu^P_t + |U(x\bar{Y} - h)| * \nu^P_t < \infty \quad Q^n\text{-a.s. for all } t \in \mathcal{T}$$

by Proposition 10 since $Q, \bar{Q} \in \mathcal{Q}^U_e$, and β^n, Y^n satisfy (2.8) by construction so that Q^n is in $\mathcal{Q}^U_e$ as well. Moreover, using $Q^n = Q$ on $\mathcal{F}_{\tau_n}$, the fact that $\bar{Y}$ is deterministic and time-independent, and $\int_0^t I_{]\!]\tau_n,\infty[\![} \, ds \le t$ yields

$$E_{Q^n}\left[|U(xY^n - h)| * \nu^P_t \right] \le E_Q\left[|U(xY - h)| * \nu^P_{\tau_n} \right] + t \int_{\mathbb{R}^d} \left| U\big(x\bar{Y}(x) - h(x)\big) \right| K(dx) < \infty$$

by the choice of τ_n and since $\bar{Q} \in \mathcal{Q}^U_{\text{int}}$. Hence Q^n is in $\mathcal{Q}^U_e \cap \mathcal{Q}^U_{\text{int}}$ as claimed above.

It remains to show that each Q^n is in $\mathcal{Q}^U_f$ and the convergence of $I_t(Q^n|P)$ to $I_t(Q|P)$. From Lemma 12, we know that

$$(5.1) \qquad I_t(R|P) = \tfrac{1}{2} E_R\Big[\int_0^t (\beta^R_s)^\top c \, \beta^R_s \, ds \Big] + E_R\left[f(Y^R) * \nu^P_t \right] \quad \text{for } R \in \{Q, Q^n\},$$

and because $Q^n = Q$ on $\mathcal{F}_{\tau_n}$ and τ_n is $\mathcal{F}_{\tau_n}$-measurable, we get from the definition of β^n that

$$E_{Q^n}\Big[\int_0^t (\beta^n_s)^\top c \, \beta^n_s \, ds \Big] = E_Q\Big[\int_0^t \beta^\top_s c \, \beta_s I_{[\![0,\tau_n]\!]}(s) \, ds \Big] + \bar{\beta}^\top c \, \bar{\beta} E_Q[(t-\tau_n)^+] \to E_Q\Big[\int_0^t \beta^\top_s c \, \beta_s \, ds \Big]$$

by monotone convergence since $\tau_n \uparrow \infty$ Q-a.s. In the same way, the definition of Y^n yields

$$\begin{aligned} E_{Q^n}\left[f(Y^n) * \nu^P_t \right] &= E_Q\Big[\int_0^t \int_{\mathbb{R}^d} f\big(Y(s,x)\big) I_{[\![0,\tau_n]\!]}(s) \, K(dx) \, ds \Big] \\ &\quad + \int_{\mathbb{R}^d} f\big(\bar{Y}(x)\big) K(dx) \, E_Q[(t-\tau_n)^+] \\ &\longrightarrow E_Q\Big[\int_0^t \int_{\mathbb{R}^d} f\big(Y(s,x)\big) K(dx) \, ds \Big] \\ &= E_Q\left[f(Y) * \nu^P_t \right] \end{aligned}$$

by monotone convergence, and in view of (5.1), this completes the proof. **q.e.d.**

The next result shows that for the approximating martingale measures in Proposition 18, we also get convergence of entropies for the corresponding "Lévyfied" measures.

Proposition 19. *In the setting of Proposition 18, let Q^ℓ and $Q^{n,\ell} = (Q^n)^\ell$ be constructed as in Theorem 13 for some $T \in (0,\infty)$. Then $\lim_{n\to\infty} I_t(Q^{n,\ell}|P) = I_t(Q^\ell|P)$ for all $t \in \mathcal{T}$.*

Proof. Since $Q^\ell, Q^{n,\ell}$ have deterministic and time-independent Girsanov parameters,

$$I_t(R|P) = \Big(\tfrac{1}{2}(\beta^R)^\top c\,\beta^R + \int_{I\!R^d} f\big(Y^R(x)\big)\,K(dx)\Big)\,t \quad \text{for } R \in \{Q^\ell, Q^{n,\ell}\}$$

by Lemma 12 and so it is enough to prove that $\beta^{n,\ell} \to \beta^\ell$ and $\int_{I\!R^d} f\big(Y^{n,\ell}(x)\big)\,K(dx) \to \int_{I\!R^d} f\big(Y^\ell(x)\big)\,K(dx)$.

Denote by β, Y and β^n, Y^n the Girsanov parameters of Q and Q^n. By the construction of $\beta^{n,\ell}$ and β^n, and since $Q^n = Q$ on $\mathcal{F}_{\tau_n}$ and τ_n is $\mathcal{F}_{\tau_n}$-measurable, we have

$$\beta^{n,\ell} = E_{Q^n}\Big[\tfrac{1}{T}\int_0^T \beta^n_s\,ds\Big] = E_Q\Big[\tfrac{1}{T}\int_0^T \beta_s I_{[\![0,\tau_n]\!]}(s)\,ds\Big] + \tfrac{1}{T}\bar\beta E_Q[(T-\tau_n)^+] \to E_Q\Big[\tfrac{1}{T}\int_0^T \beta_s\,ds\Big] = \beta^\ell$$

by monotone convergence for the second and dominated convergence for the first term, because $\Big|\int_0^T \beta_s I_{[\![0,\tau_n]\!]}(s)\,ds\Big| \le \int_0^T |\beta_s|\,ds \in \mathcal{L}^1(Q)$ by part b) of Lemma 12. In the same way, we obtain $f\big(Y^{n,\ell}(x)\big) \to f\big(Y^\ell(x)\big)$ for all $x \in \operatorname{supp} K$ by using part d) of Lemma 12 and continuity of f. To find a K-integrable dominating function for $f\big(Y^{n,\ell}(x)\big)$, we use the definition of $Y^{n,\ell}$, Jensen's inequality for the convex function $f \ge 0$, the definition of Y^n, and again that $Q^n = Q$ on $\mathcal{F}_{\tau_n}$ and $\mathcal{F}_{\tau_n}$-measurability of τ_n to obtain

$$\begin{aligned} f\big(Y^{n,\ell}(x)\big) &\le E_{Q^n}\Big[\tfrac{1}{T}\int_0^T f\big(Y^n(s,x)\big)\,ds\Big] \\ &= E_Q\Big[\tfrac{1}{T}\int_0^T f\big(Y(s,x)\big) I_{[\![0,\tau_n]\!]}(s)\,ds\Big] + \tfrac{1}{T} f\big(\bar Y(x)\big) E_Q[(T-\tau_n)^+] \\ &\le E_Q\Big[\tfrac{1}{T}\int_0^T f\big(Y(s,x)\big)\,ds\Big] + f\big(\bar Y(x)\big). \end{aligned}$$

But Q and $\bar Q$ both have finite relative entropy and since $f \ge 0$, we can use Fubini's theorem and part c) of Lemma 12 to get

$$\int_{I\!R^d} E_Q\Big[\int_0^T f\big(Y(s,x)\big)\,ds\Big]\,K(dx) = E_Q\Big[\int_{I\!R^d}\int_0^T f\big(Y(s,x)\big)\,ds\,K(dx)\Big] = E_Q\big[f(Y) * \nu^P_T\big] < \infty.$$

In the same way, we get K-integrability of $f\big(\bar{Y}(x)\big)$. Hence dominated convergence yields $\int_{\mathbb{R}^d} f\big(Y^{n,\ell}(x)\big)\, K(dx) \to \int_{\mathbb{R}^d} f\big(Y^\ell(x)\big)\, K(dx)$, and this completes the proof. **q.e.d.**

Now we can finally prove Theorem A which we recall for the convenience of the reader.

Theorem A. *Let L be a P-Lévy process for $\mathbb{F} = \mathbb{F}^L$, and U a fixed $d \times d$-matrix. Suppose that $\mathcal{Q}_e^U(L) \cap \mathcal{Q}_f^U(L) \cap \mathcal{Q}_\ell^U(L) \neq \emptyset$. If $Q^E(UL)$ exists, then L is a Lévy process under $Q^E(UL)$.*

Proof. For brevity, we write Q^E for $Q^E(UL)$. If L is not a Lévy process under Q^E, there exists $T \in (0, \infty)$ such that L is not a Q^E-Lévy process on $[0, T]$. For the measure $Q^{E,\ell} = (Q^E)^\ell$ obtained from Theorem 13, we then have $I_T(Q^{E,\ell}|P) < I_T(Q^E|P)$. However, this is not yet a contradiction to the optimality of Q^E; we do not know whether UL is a local martingale under $Q^{E,\ell}$ since Q^E is perhaps not in $\mathcal{Q}_{\text{int}}^U$. But if $(Q^{E,n})_{n\in\mathbb{N}}$ is the sequence in $\mathcal{Q}_e^U \cap \mathcal{Q}_f^U \cap \mathcal{Q}_{\text{int}}^U$ for Q^E from Proposition 18 and $Q^{E,n,\ell} = (Q^{E,n})^\ell$ are the corresponding Lévy martingale measures for UL obtained from Theorem 13, Proposition 19 yields

$$\lim_{n\to\infty} I_T(Q^{E,n,\ell}|P) = I_T(Q^{E,\ell}|P) < I_T(Q^E|P).$$

So for n sufficiently large we have $I_T(Q^{E,n,\ell}|P) < I_T(Q^E|P)$ and $Q^{E,n,\ell} \in \mathcal{Q}_e^U \cap \mathcal{Q}_f^U$ by Proposition 17 which is the desired contradiction. **q.e.d.**

In view of Theorem A, it seems clear that we should be able to find $Q^E(UL)$ by minimizing relative entropy only over *Lévy* martingale measures. This is indeed true:

Corollary 20. *Let L be a P-Lévy process for $\mathbb{F} = \mathbb{F}^L$, and U a fixed $d \times d$-matrix. Suppose that $\mathcal{Q}_e^U(L) \cap \mathcal{Q}_f^U(L) \cap \mathcal{Q}_\ell^U(L) \neq \emptyset$. If $Q_\ell^E(UL)$ exists, then $Q^E(UL)$ exists as well and coincides with $Q_\ell^E(UL)$. In particular, we have $Q_\ell^E(UL) \overset{\text{loc}}{\approx} P$.*

Proof. We again omit writing (L) and (UL) for brevity. If Q^E exists, it is in $\mathcal{Q}_e^U \cap \mathcal{Q}_f^U \cap \mathcal{Q}_\ell^U$ by Theorem A. Then we must have $Q^E = Q_\ell^E$, and it also follows from Theorem 2.2 of Frittelli (2000) that $Q^E \overset{\text{loc}}{\approx} P$.

Suppose Q^E does not exist. Then there is some $T \in (0, \infty)$ and some $Q \in \mathcal{Q}_f^U$ with $I_T(Q|P) < I_T(Q_\ell^E|P)$. Since $\mathcal{Q}_e^U \cap \mathcal{Q}_f^U \neq \emptyset$ and $I_T(\cdot\,|P)$ is convex, we may assume that $Q \in \mathcal{Q}_e^U$ as well $\big($otherwise replace Q by $(1-\varepsilon)Q + \varepsilon Q'$ for some $Q' \in \mathcal{Q}_e^U \cap \mathcal{Q}_f^U\big)$. Construct Q^ℓ from Q via Theorem 13, the sequence $(Q^n)_{n\in\mathbb{N}}$ for Q via Proposition 18 and then $Q^{n,\ell} := (Q^n)^\ell$ from Q^n via Theorem 13. Then Proposition 19 yields

$$\lim_{n\to\infty} I_T(Q^{n,\ell}|P) = I_T(Q^\ell|P) \le I_T(Q|P) < I_T(Q_\ell^E|P)$$

and thus $I_T(Q^{n,\ell}|P) < I_T(Q^E_\ell|P)$ for large n. But since $Q^{n,\ell} \in \mathcal{Q}^U_e \cap \mathcal{Q}^U_f \cap \mathcal{Q}^U_\ell$, this contradicts the optimality of Q^E_ℓ, and so Q^E does exist. **q.e.d.**

Remark. Theorem A implies that in order to determine $Q^E(UL)$ it suffices to find a martingale measure which is optimal in $\mathcal{Q}^U_f \cap \mathcal{Q}^U_\ell$, and Corollary 20 shows that this measure must be locally equivalent to P. Hence we have to look for the optimal measure in $\mathcal{Q}^U_e \cap \mathcal{Q}^U_f \cap \mathcal{Q}^U_\ell$ so that the assumption $\mathcal{Q}^U_e \cap \mathcal{Q}^U_f \cap \mathcal{Q}^U_\ell \neq \emptyset$ is entirely natural. ⋄

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6. Appendix

This section collects a number of proofs omitted from the body of the paper for better reading.

Proof of Proposition 5. By Lemma 4 and since $f(\bar{Y}) * \nu^P \geq 0$, we have for all $t \in \mathcal{T}$

$$g(\bar{Y}) * \nu^P_t \leq f(\bar{Y}) * \nu^P_t \leq \exp\left(\int_0^t \left(\tfrac{1}{2}\bar{\beta}^\top_s c\, \bar{\beta}_s + \int_{\mathbb{R}^d} f\big(\bar{Y}(s,x)\big)\, K(dx)\right) ds\right). \tag{6.1}$$

So $\big(1-\sqrt{\bar{Y}}\big)^2 * \nu^P$ is locally P-integrable by (2.3), and JS, Theorem II.1.33 gives the integrability of $\bar{Y}-1$ with respect to $\mu^L - \nu^P$. By (2.3), $\int \bar{\beta}^\top_s c\, \bar{\beta}_s\, ds$ is also locally P-integrable so that $\bar{\beta}$ is integrable with respect to L^c and $\bar{N}$ is well-defined. Since $\bar{N}$ is a local P-martingale and (2.4) is its decomposition into continuous and purely discontinuous parts, $\Delta \bar{N}_t = \big(\bar{Y}(t,\Delta L_t) - 1\big) I_{\{\Delta L_t \neq 0\}} > -1$ P-a.s. since $\bar{Y} > 0$. Hence $\bar{Z} = \mathcal{E}(\bar{N})$ is a strictly positive local P-martingale, and a true P-martingale if $E_P[\mathcal{E}(\bar{N})_\tau] = 1$ for every bounded stopping time τ. But if $\tau \leq t_0$ for some deterministic $t_0 \in (0,\infty)$, then $\mathcal{E}(\bar{N})_\tau = \mathcal{E}(\bar{N}^{t_0})_\tau$ and $M := \bar{N}^{t_0}$ is again a local P-martingale null at 0 with $\Delta M > -1$. So if we define A by

$$A_t := \tfrac{1}{2}\langle M^c\rangle_t + \sum_{s \leq t} \big((1+\Delta M_s)\log(1+\Delta M_s) - \Delta M_s\big) \quad \text{for } t \leq t_0$$

and show that A admits a predictable P-compensator B with $E_P[\exp(B_{t_0})] < \infty$, then Theorem III.1 of Lepingle/Mémin (1978) implies that $\mathcal{E}(M)$ is a uniformly integrable P-martingale and therefore $E_P[\mathcal{E}(\bar{N})_\tau] = E_P\left[\mathcal{E}(M)_\tau\right] = 1$, which will end the proof.

To find the P-compensator B of A, note that $\langle \bar N^c\rangle = \langle \int \bar\beta\, dL^c\rangle = \int \bar\beta_s^\top c\,\bar\beta_s\, ds$ so that

$$A_t = \tfrac{1}{2}\int_0^t \bar\beta_s^\top c\,\bar\beta_s\, ds + \sum_{s\le t}\big(\bar Y(s,\Delta L_s)\,\log \bar Y(s,\Delta L_s) - \bar Y(s,\Delta L_s) + 1\big) I_{\{\Delta L_s \neq 0\}}$$

$$= \tfrac{1}{2}\int_0^t \bar\beta_s^\top c\,\bar\beta_s\, ds + f(\bar Y) * \mu_t^L \quad \text{for } t \le t_0.$$

Now $|f(\bar Y)| * \nu_t^P = f(\bar Y) * \nu_t^P$ is P-integrable for all $t \in \mathcal{T}$ by (6.1) and (2.3), and so we get from JS, Prop. II.1.28 that $f(\bar Y)$ is integrable with respect to $\mu^L - \nu^P$ and that $f(\bar Y) * (\mu^L - \nu^P) = f(\bar Y) * \mu^L - f(\bar Y) * \nu^P$. Hence $B_t = \frac{1}{2}\int_0^t \bar\beta_s^\top c\,\bar\beta_s\, ds + f(\bar Y) * \nu_t^P$ is the P-compensator of A, and we have $E[\exp(B_{t_0})] < \infty$ by assumption (2.3). **q.e.d.**

Proof of Proposition 7. By assumption, the density process $Z^{\bar Q} = \mathcal{E}(\bar N)$ is a strictly positive P-martingale. On the other hand, Proposition 2 gives us a predictable function $\hat Y \ge 0$ and a predictable process $\hat\beta$ with $\int_0^t \hat\beta_s^\top c\,\hat\beta_s\, ds < \infty$ and $|(\hat Y - 1)h| * \nu_t^P < \infty$ P-a.s. for all $t \in \mathcal{T}$, and we know that $Z^{\bar Q} = \mathcal{E}(N^{\bar Q})$ with $N^{\bar Q} = \int \hat\beta_s\, dL_s^c + (\hat Y - 1) * (\mu^L - \nu^P)$ by Proposition 3. So since $\mathcal{E}(N^{\bar Q}) = \mathcal{E}(\bar N) > 0$, we have $N^{\bar Q} = \bar N$ or, equivalently,

$$V_t^1 := \int_0^t (\bar\beta_s - \hat\beta_s)\, dL_s^c = (\hat Y - \bar Y) * (\mu^L - \nu^P)_t =: V_t^2, \quad t \in \mathcal{T}.$$

As V^1 is a continuous and V^2 a purely discontinuous local P-martingale, we get $V^1 \equiv 0 \equiv V^2$, and this implies $\hat\beta = \bar\beta$ and $\hat Y = \bar Y$. In fact, $\langle V^1\rangle = \int (\bar\beta_s - \hat\beta_s)^\top c\,(\bar\beta_s - \hat\beta_s)\, ds \equiv 0$ yields

$$(\bar\beta_s - \hat\beta_s)^\top S\,\tilde c\, S^\top (\bar\beta_s - \hat\beta_s) = 0 \quad P\text{-a.s. for all } s \in \mathcal{T},$$

and because $\bar\beta$ and $\hat\beta$ are chosen as in Remark NV, this implies $\big(S^\top(\bar\beta_s - \hat\beta_s)\big)^j = 0$ for $j \le \operatorname{rank}(c)$ and $(S^\top\hat\beta_s)^j = 0 = (S^\top\bar\beta_s)^j$ for $j > \operatorname{rank}(c)$. Hence we get $S^\top(\bar\beta_s - \hat\beta_s) = 0$ and thus $\bar\beta_s = \hat\beta_s$ P-a.s. for all $s \in \mathcal{T}$. Moreover, applying JS, Theorem II.1.33 to the square-integrable P-martingale V^2 yields

$$0 = \langle V^2\rangle_t = (\hat Y - \bar Y)^2 * \nu_t^P = \int_0^t \int_{\mathbb{R}^d} \big(\hat Y(s,x) - \bar Y(s,x)\big)^2 K(dx)\, ds \quad P\text{-a.s. for all } t \in \mathcal{T}$$

so that $\hat Y(s,x) = \bar Y(s,x)$ ν^P-a.e., P-a.s. Thus $\bar\beta$ and $\bar Y$ are the Girsanov parameters of $\bar Q$.
q.e.d.

Proof of Lemma 9. We claim that we have for every truncation function h the estimates

$$|Uh(x) - h(Ux)| * \nu_t^Q \le \text{const.}\,\big(t + f(Y) * \nu_t^P\big), \tag{6.2}$$

$$|(Y-1)h| * \nu_t^P \le \text{const.}\,\big(t + f(Y) * \nu_t^P\big). \tag{6.3}$$

Moreover, the triangle inequality gives

$$|Ux - h(Ux)|Y \le |U(xY - h)| + |Uh(x)(1-Y)| + |Uh(x) - h(Ux)|Y,$$
$$|U(xY - h)| \le |Ux - h(Ux)|Y + |Uh(x)(1-Y)| + |Uh(x) - h(Ux)|Y$$

and using (6.2), (6.3) and $\nu^Q = Y\nu^P$, we obtain

$$|Ux - h(Ux)| * \nu_t^Q \le |U(xY - h)| * \nu_t^P + \text{const.}\big(t + f(Y) * \nu_t^P\big),$$
$$|U(xY - h)| * \nu_t^P \le |Ux - h(Ux)| * \nu_t^Q + \text{const.}\big(t + f(Y) * \nu_t^P\big).$$

So (2.6) and (2.7) follow directly from (6.2) and (6.3). Since $\int_{\mathbb{R}^d} (1 \wedge |x|^2)\, K(dx) < \infty$ and $\nu^Q(dt, dx) = Y(t,x)\,\nu^P(dt, dx) = Y(t,x)\,K(dx)\,dt$, we obtain (6.2) and (6.3) if we show that

$$|Uh(x) - h(Ux)|Y \le \text{const.}\big(1 \wedge |x|^2 + f(Y)\big),$$
$$|(Y-1)h| \le \text{const.}\big(1 \wedge |x|^2 + f(Y)\big).$$

Since this is just analysis, we omit the proof; see Appendix C of Esche (2004). **q.e.d.**

Proof of Proposition 10. Let $(\tilde{B}, \tilde{C}, \tilde{\nu})$ be the Q-characteristics of UL. Due to JS, Prop. II.2.29, UL is a local Q-martingale if and only if $M := \tilde{B} + \big(x - h(x)\big) * \mu^{UL}$ is, and using Proposition 2 to get the Q-characteristics of L and Proposition 1 to pass to UL gives

$$M_t = Ubt + \int_0^t Uc\,\beta_s\,ds + \Big(Uh(x)(Y-1) - \big(Uh(x) - h(Ux)\big)Y\Big) * \nu_t^P + \big(x - h(x)\big) * \mu_t^{UL}$$
$$= \int_0^t \Big(Ub + Uc\,\beta_s + \int_{\mathbb{R}^d} \big(h(Ux)Y(s,x) - Uh(x)\big)\,K(dx)\Big)\,ds + \big(Ux - h(Ux)\big) * \mu_t^L.$$

If $|Ux - h(Ux)| * \nu^Q$ is locally Q-integrable, we can use JS, Prop. II.1.28 and $\nu^Q(ds, dx) = Y(s,x)\,K(dx)\,ds$ to write

$$M_t = \int_0^t \Big(Ub + Uc\,\beta_s + \int_{\mathbb{R}^d} U\big(xY(s,x) - h(x)\big)\,K(dx)\Big)\,ds + \big(Ux - h(Ux)\big) * (\mu^L - \nu^Q)_t$$
$$=: \int_0^t \tilde{m}_s\,ds + \big(Ux - h(Ux)\big) * (\mu^L - \nu^Q)_t.$$

Note that $\tilde{m}_t = 0$ Q-a.s. for all $t \in \mathcal{T}$ is just the martingale condition (2.8).

If UL is a local Q-martingale, $(|x|^2 \wedge |x|) * \mu^{UL}$ is locally Q-integrable by JS, Prop. II.1.28 and Prop. II.2.29. Since $|x - h(x)| \le \text{const.}\,(|x|^2 \wedge |x|)$, we conclude that $|Ux - h(Ux)| * \mu^L = |x - h(x)| * \mu^{UL}$ is locally Q-integrable. By JS, Prop. II.1.28, $|Ux - h(Ux)| * \nu^Q$ is locally

Q-integrable as well and thus finite-valued so that $|U(xY-h)| * \nu_t^P < \infty$ Q-a.s. for all $t \in \mathcal{T}$ by Lemma 9. Moreover, M and $\big(Ux - h(Ux)\big) * (\mu^L - \nu^Q)$ are local Q-martingales which implies that $\tilde{m}_t = 0$ Q-a.s. for all $t \in \mathcal{T}$.

Conversely, $|U(xY-h)| * \nu_t^P < \infty$ Q-a.s. for all $t \in \mathcal{T}$ implies that $|Ux - h(Ux)| * \nu^Q$ is locally Q-integrable because it is continuous $\big($this uses $\nu^Q(ds, dx) = Y(s,x)\, K(dx)\, ds\big)$ and finite-valued by Lemma 9. Thus $M = \int \tilde{m}_s\, ds + (Ux - h(Ux)) * (\mu^L - \nu^Q)$, and if we also have (2.8), the first term vanishes and M, hence also UL, is a local Q-martingale. **q.e.d.**

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