The minimal martingale measure

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Abstract: Suppose discounted asset prices in a financial market are given by a $P$-semi-martingale of the form $S = S_0 + M + A$. The minimal martingale measure for $S$ is characterised by the properties that it turns $S$ into a local martingale and preserves the martingale property for any local $P$-martingale strongly $P$-orthogonal to $M$. It plays a key role in finding locally risk-minimising strategies, and it comes up in various other contexts as well. Importantly, its density process can be written explicitly in terms of $M$ and $A$, so that one can use it very generally and broadly. In some specific settings, it also has other optimality properties.

Key words: martingale measure, local risk-minimisation, structure condition, Föllmer–Schweizer decomposition, hedging, option pricing, quadratic hedging criteria


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Let $S = (S_t)$ be a stochastic process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ that models the discounted prices of primary traded assets in a financial market. An equivalent local martingale measure (ELMM) for $S$ is a probability measure $Q$ equivalent to the original (historical) measure $P$ such that $S$ is a local $Q$-martingale; see [equivalent martingale measure and ramifications]. If $S$ is a nonnegative $P$-semimartingale, the fundamental theorem of asset pricing says that an ELMM $Q$ for $S$ exists if and only if $S$ satisfies the no-arbitrage condition (NFLVR), i.e. admits no free lunch with vanishing risk; see [fundamental theorem of asset pricing]. By Girsanov’s theorem, $S$ is then under $P$ a semimartingale with a decomposition $S = S_0 + M + A$ into a local $P$-martingale $M$ and an adapted process $A$ of finite variation. If $S$ is special under $P$, then $A$ can be chosen predictable and the resulting canonical decomposition of $S$ is unique. We say that $S$ satisfies the structure condition (SC) if $M$ is locally $P$-square-integrable and $A$ has the form $A = \int d(M)\lambda$ for a predictable process $\lambda$ such that the increasing process $\int \lambda' d(M)\lambda$ is finite-valued. In an Itô process model where $S$ is given by a stochastic differential equation $dS_t = S_t((\mu_t - r_t) dt + \sigma_t dW_t)$, the latter process is given by $\int ((\mu_t - r_t)/\sigma_t)^2 dt$, the integrated squared instantaneous Sharpe ratio of $S$; see [Sharpe ratio].

**Definition.** Suppose $S$ satisfies (SC). An ELMM $\hat{P}$ for $S$ with $P$-square-integrable density $d\hat{P}/dP$ is called minimal martingale measure (for $S$) if $\hat{P} = P$ on $\mathcal{F}_0$ and if every local $P$-martingale $L$ which is locally $P$-square-integrable and strongly $P$-orthogonal to $M$ is also a local $\hat{P}$-martingale. We call $\hat{P}$ orthogonality-preserving if $L$ is also strongly $\hat{P}$-orthogonal to $S$.

The basic idea for the minimal martingale measure (MMM) first appeared in [46] in a more specific model, where it was used as an auxiliary technical tool in the context of local risk-minimisation. (See also [hedging, general concepts] for an overview of key ideas on hedging and [mean-variance hedging and portfolio selection] for an alternative quadratic approach.) More precisely, the so-called locally risk-minimising strategy for a given contingent claim $H$ was obtained there (under some specific assumptions) as the integrand from the classical Galtchouk–Kunita–Watanabe decomposition of $H$ under $\hat{P}$. However, the introduction of $\hat{P}$ in [46] and also in [47] was still somewhat ad hoc. The above definition was given in [18] where also the main results presented here can be found. In particular, [18] showed that for continuous $S$, the Galtchouk–Kunita–Watanabe decomposition of $H$ under the minimal martingale measure $\hat{P}$ provides (under very mild integrability conditions) the so-called Föllmer–Schweizer decomposition of $H$ under the original measure $P$, and this in turn immediately gives the locally risk-minimising strategy for $H$. We emphasise that this is no longer true in general if $S$ has jumps. The MMM subsequently found various other applications and uses and has become fairly popular, especially in models with continuous
price processes.

Suppose now that $S$ satisfies the structure condition (SC). For every ELMM $Q$ for $S$ with $dQ/dP \in L^2(P)$, the density process then takes the form

$$Z^Q := \frac{dQ}{dP}\big|_\mathcal{F} = Z_0^Q \mathcal{E} \left( - \int \lambda \, dM + L^Q \right)$$

with some locally $P$-square-integrable local $P$-martingale $L^Q$. If the MMM $\hat{P}$ exists, then it has $\hat{Z}_0 = 1$ and $L^{\hat{P}} \equiv 0$, and its density process is thus given by the stochastic exponential (see [stochastic exponentials])

$$\hat{Z} = \mathcal{E} \left( - \int \lambda \, dM \right)$$

$$= \exp \left( - \int \lambda \, dM - \frac{1}{2} \int \lambda' \, d[M] \lambda \right) \prod (1 - \lambda' \Delta M) \exp \left( \lambda' \Delta M + \frac{1}{2} (\lambda' \Delta M)^2 \right).$$

The advantage of this explicit representation is that it allows to determine the minimal martingale measure $\hat{P}$ and its density process $\hat{Z}$ directly from the ingredients $M$ and $\lambda$ of the canonical decomposition of $S$. Conversely, one can start with the above expression for $\hat{Z}$ to define a candidate for the density process of the MMM. This gives existence of the MMM under the following conditions:

(i) $\hat{Z}$ is strictly positive; this happens if and only if $\lambda' \Delta M < 1$, i.e. all the jumps of $\int \lambda \, dM$ are strictly below 1.

(ii) The local $P$-martingale $\hat{Z}$ is a true $P$-martingale.

(iii) $\hat{Z}$ is $P$-square-integrable.

Condition (i) automatically holds (on any finite time interval) if $S$, hence also $M$, is continuous; it typically fails in models where $S$ has jumps. Conditions (ii) and (iii) can fail even if (i) holds and even if there exists some ELMM for $S$ with $P$-square-integrable density; see [45] or [15] for a counterexample.

The above explicit formula for $\hat{Z}$ shows that $\hat{P}$ is minimal in the sense that its density process contains the smallest number of symbols among all ELMMs $Q$. More seriously, the original idea was that $\hat{P}$ should turn $S$ into a (local) martingale while having a minimal impact on the overall martingale structure of our setting. This is captured and made precise by the definition. If $S$ is continuous, one can show that $\hat{P}$ is even orthogonality-preserving; see [18] for this, and note that this usually fails if $S$ has jumps.

To some extent, the naming of the “minimal” martingale measure is misleading since $\hat{P}$ was not originally defined as the minimiser of a particular functional on ELMMs. However, if $S$ is continuous, Föllmer and Schweizer [18] have proved that $\hat{P}$ minimises

$$Q \mapsto H(Q|P) - E_Q \left[ \int_0^\infty \lambda'_u \, d(M)_u \Delta u \right]$$

2
over all ELMMs $Q$ for $S$; see also [49]. Moreover, Schweizer [50] has shown that if $S$ is continuous, then $\hat{P}$ minimises the reverse relative entropy $H(P|Q)$ over all ELMMs $Q$ for $S$; this no longer holds if $S$ has jumps. Under more restrictive assumptions, other minimality properties for $\hat{P}$ have been obtained by several authors. But a general result under the sole assumption (SC) is not available so far.

There is a large amount of literature related to the MMM. In fact, a Google Scholar search for “minimal martingale measure” (enclosed in quotation marks) produced in April 2008 a list of well over 400 hits. As a first category, this contains papers where the MMM is studied per se, or used as in the original approach of local risk-minimisation. In terms of topics, the following areas of related work can be found in that category:

- properties, characterisation results and generalisations for the MMM: [1], [4], [9], [11], [14], [19], [33], [36], [37], [49], [51].
- convergence results for option prices (computed under the MMM): [25], [32], [42], [44].
- applications to hedging: [7], [39], [47], [48]. See also [hedging, general concepts].
- uses for option pricing: [8], [13], [55], to name only a very a few; comparison results for option prices are given in [22], [24], [34]. See also [risk neutral pricing].
- problems and counterexamples: [15], [16], [43], [45], [52].
- equilibrium justifications for using the MMM: [26], [40].

A second category of papers contains those where the MMM has (sometimes unexpectedly) come up in connection with various other problems and topics in mathematical finance. Examples include

- classical utility maximisation and utility indifference valuation ([3], [20], [21], [23], [35], [41], [53], [54]); the MMM here often appears because the special structure of a given model implies that it has a particular optimality property. See also [expected utility maximization], [expected utility maximization], [utility indifference valuation] and [minimal entropy martingale measure].
- the numeraire portfolio and growth-optimal investment ([2], [12]); this is related to the minimisation of the reverse relative entropy $H(P|\cdot)$ over ELMMs. See also [Kelly problem].
- the concept of value preservation ([28], [29], [30]); here the link seems to come up because value preservation is like local risk-minimisation a local optimality criterion.
- good deal bounds in incomplete markets ([5], [6]); the MMM naturally shows up here because good deal bounds are formulated via instantaneous quadratic restrictions on the pricing kernel (ELMM) to be chosen. See also [good-deal bounds], [Sharpe ratio] and [pricing kernels].
- local utility maximisation ([27]); again, the link here is due to the local nature of the criterion that is used.
- risk-sensitive control ([17], [31], [38]); this is an area where the connection to the MMM
seems not yet well understood. See also [risk-sensitive asset management].

References


approaches to optimal portfolios”, Mathematical Finance 10, 227–241


