

Exponential utility indifference valuation in two Brownian settings with stochastic correlation

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Abstract

We study the exponential utility indifference valuation of a contingent claim B in an incomplete market driven by two Brownian motions. The claim depends on a nontradable asset stochastically correlated with the traded asset available for hedging. We use martingale arguments to provide upper and lower bounds, in terms of bounds on the correlation, for the value V^B of the exponential utility maximization problem with the claim B as random endowment. This yields an explicit formula for the indifference value b of B at any time, even with a fairly general stochastic correlation. Earlier results with constant correlation are recovered and extended. The reason why all this works is that after a transformation to the minimal martingale measure, the value V^B enjoys a monotonicity property in the correlation between tradable and nontradable asset.

Key words: exponential utility, indifference valuation, nontradable asset, stochastic volatility, random endowment, distortion power, correlated Brownian motions

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1 Introduction

It is widely accepted that valuing contingent claims in incomplete markets needs subjective criteria in addition to absence of arbitrage. Utility indifference provides one such criterion, which has an economic justification because the chosen utility function reflects the individual investor's risk preferences. The basic idea of this approach is that the investor should achieve the same expected utility in the two cases where (1) he is free of obligation, or (2) he has to pay out the contingent claim B but has an initial capital increased by the amount of the *indifference (seller) value* of B . A precise mathematical formulation leads to a pair of optimization problems over investment strategies. By using dual methods, which involve finding an optimal probability measure instead of an optimal strategy, one can derive some general properties for the exponential utility indifference value of a claim; see for instance Delbaen et al. [3] or Mania and Schweizer [15], or Becherer [1] for an approach via BSDEs in a model with jumps. However, it is difficult to derive a closed-form formula for the indifference value even in an explicit model.

Our basic model presented in Section 2 consists of a risk-free bank account and a stock S driven by a Brownian motion W . The contingent claim B to be valued depends on another Brownian motion Y , which has stochastic instantaneous correlation ρ with W . Indifference valuation is done via exponential utility. In the literature, which we compare in Section 4 with our results, there are two main approaches to obtain explicit formulas for the value of the resulting optimization problem. In a Markovian setting, Henderson [7], Henderson and Hobson [9, 10] and Musiela and Zariphopoulou [18], among others, start with the Hamilton-Jacobi-Bellman nonlinear PDE for the value function of the underlying stochastic control problem. This PDE is then linearized by a power transformation with a constant exponent called *distortion power*. This method works only if one has a Markovian model and if ρ is constant. In an alternative approach, Tehranchi [21] first proves a Hölder-type inequality which he then applies to solve the portfolio optimization problem. The distortion power there arises as an exponent from the Hölder-type inequality. Tehranchi finds an explicit expression for the indifference value at time 0 if ρ is constant. While this method needs no Markovian assumption and can treat claims which are general (bounded) functionals of the process Y , it is still restricted to situations with constant correlation.

Since (exponential) utility indifference valuation hinges on (exponential) utility maximization with a random endowment, we start by tackling the latter. With the goal of deriving explicit results in our Brownian setting, Section 2.2 motivates the introduction of an auxiliary abstract optimization problem in a martingale framework. Our main theoretical result is Theo-

rem 1 in Section 3; it gives an explicit formula for the value of this abstract problem. The proof uses martingale arguments to give upper and lower bounds on that value, in terms of bounds on ρ . Crucially, these bounds have the same structure, which enables us to derive a closed-form expression by interpolation. In particular, this allows us to handle a *random* correlation ρ .

Section 4 contains two applications of Theorem 1. In the first, case (I), we extend the model of Tehranchi [21] to a fairly general stochastic correlation; the typical example is a model with stochastic volatility which is correlated with the stock in a nondeterministic way. In the second, case (II), the asset driving the claim B is traded in principle, but nontradable for our investor. A typical example here is the valuation of (European) executive stock options. In both cases, we obtain closed-form expressions for the exponential utility indifference value of the claim B at all times $t \in [0, T]$. The key feature of our formulas is that *the explicit form of the indifference value is preserved at any time t* , except that the distortion power, which is shown to exist but not explicitly determined, may now be random and depend on the contingent claim B to be valued. To the best of our knowledge, this is the first explicit result on exponential utility indifference valuation in a setting with nonconstant and nondeterministic correlation. As another novelty, our general framework allows us to distinguish (via measurability conditions) between the settings of case (I) and case (II); this is impossible when ρ and the instantaneous Sharpe ratio λ of S are constant, as in most of the existing literature. Section 4.2 discusses this and other issues in more detail.

In Section 5, we provide both intuitive and rigorous explanations for our results. We show that the value of the abstract optimization problem is monotonic in $|\rho|$. Because this value can be computed explicitly for constant ρ and is continuous in the ρ -argument, interpolation implies that the basic structure is preserved for a random ρ . This explains why we can obtain our nice and explicit results. However, the precise interpretation of the above monotonicity is delicate, since it only holds when the (ρ -dependent) probability measure $\hat{P}(\rho)$ appearing in the abstract problem is kept fixed. A counterexample shows that the value of the original optimization problem under P may fail to be monotonic in $|\rho|$ if we allow $\hat{P}(\rho)$ to vary with ρ , and we explain how keeping $\hat{P}(\rho)$ fixed is linked to standard financial reasoning.

For concreteness and ease of exposition, all our results are given for two correlated Brownian motions W and Y . The final Section 6 briefly shows how everything can be generalized to a multidimensional Itô process setting.

2 Preliminaries

2.1 Model setup

We work on a finite time interval $[0, T]$ for a fixed $T > 0$ and a complete filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, P)$. The filtration $\mathbb{G} = (\mathcal{G}_s)_{0 \leq s \leq T}$ satisfies the usual conditions, has \mathcal{G}_0 trivial, and $Y = (Y_s)_{0 \leq s \leq T}$ and Y^\perp are two independent (\mathbb{G}, P) -Brownian motions. Unless otherwise mentioned, all processes and filtrations are indexed by $s \in [0, T]$, and we fix $t \in [0, T]$. For any process X , $\mathbb{F}^X = (\mathcal{F}_s^X)$ denotes the P -augmented filtration generated by X . For any filtration $\mathbb{F} \subseteq \mathbb{G}$, a process X is called \mathbb{F} -predictable if it is measurable with respect to the \mathbb{F} -predictable σ -field on $[0, T] \times \Omega$, completed by the nullsets of $(\text{Lebesgue measure}) \otimes P$. To simplify computations, we use the notation $\mathcal{E}(N)_{s,y} := \exp(N_y - N_s - \frac{1}{2}(\langle N \rangle_y - \langle N \rangle_s))$, $0 \leq s \leq y \leq T$ for a continuous \mathbb{G} -semimartingale N . Notions such as L^∞ or ‘almost surely’ (a.s.) always refer to P (or any probability measure equivalent to P).

The stochastic framework of our model consists of two Brownian motions W and Y with random instantaneous correlation ρ . To construct this, let $\rho = (\rho_s)$ be a \mathbb{G} -predictable process valued in $[-1, 1]$ such that $|\rho|$ is bounded away from one (uniformly in s, ω) and define

$$W_s := \int_0^s \rho_y dY_y + \int_0^s \sqrt{1 - \rho_y^2} dY_y^\perp, \quad 0 \leq s \leq T. \quad (2.1)$$

In our financial market, two assets are available for investing and going short: a risk-free *bank account* and a *stock* S . The instantaneous yield of the bank account is described by a deterministic spot interest rate function $r : [0, T] \rightarrow [0, \infty)$ which is bounded and Borel-measurable. For ease of notation, we directly pass to discounted quantities which means that we take $r \equiv 0$. (See Section 4 for more comments on this.) The (discounted) dynamics of the stock is given by

$$dS_s = \mu_s S_s ds + \sigma_s S_s dW_s, \quad 0 \leq s \leq T, \quad S_0 > 0, \quad (2.2)$$

where the drift μ and the volatility σ are \mathbb{G} -predictable processes. We assume for simplicity that μ is bounded and σ is bounded away from zero and infinity. Hence the *instantaneous Sharpe ratio* $\lambda := \frac{\mu}{\sigma}$ is also bounded. We write

$$\frac{dS_s}{S_s} = \mu_s ds + \sigma_s dW_s = \sigma_s d\hat{W}_s, \quad 0 \leq s \leq T$$

and note that by Girsanov’s theorem, the processes

$$\hat{W}_s := W_s + \int_0^s \lambda_y dy \quad \text{and} \quad \hat{Y}_s := Y_s + \int_0^s \rho_y \lambda_y dy, \quad 0 \leq s \leq T \quad (2.3)$$

are Brownian motions under the probability $\hat{P} \approx P$ on (Ω, \mathcal{G}_T) given by

$$\frac{d\hat{P}}{dP} := \mathcal{E} \left(- \int_0^T \lambda dW \right)_{0,T}. \quad (2.4)$$

In the terminology of Föllmer and Schweizer [5], \hat{P} is the *minimal martingale measure* for S .

Let B be a bounded \mathcal{G}_T -measurable random variable, interpreted as a contingent claim or payoff due at time T . To value B , we assume that our investor has an exponential utility function $U(x) = -\exp(-\gamma x)$, $x \in \mathbb{R}$, for a fixed $\gamma > 0$. He starts at time $t \in [0, T]$ with initial capital x_t and runs a self-financing strategy $\pi = (\pi_s)_{t \leq s \leq T}$ so that his wealth at time $s \in [t, T]$ is

$$X_s^{x_t, \pi} = x_t + \int_t^s \frac{\pi_y}{S_y} dS_y = x_t + \int_t^s \pi_y \sigma_y d\hat{W}_y,$$

where π represents the amount invested in S . The set \mathcal{A}_t of *admissible strategies* on $[t, T]$ consists of all \mathbb{G} -predictable processes $\pi = (\pi_s)_{t \leq s \leq T}$ which satisfy $\int_t^T \pi_s^2 ds < \infty$ a.s. and are such that

$$\left(\exp \left(-\gamma \int_t^s \pi_y \sigma_y d\hat{W}_y \right) \right)_{t \leq s \leq T} \text{ is of class } (D) \text{ on } (\Omega, \mathcal{G}_T, \mathbb{G}, P), \quad (2.5)$$

abbreviated by ‘of P -class (D) ’. We define V^B (and analogously V^0) for $t \in [0, T]$ and x_t bounded \mathcal{G}_t -measurable by

$$\begin{aligned} V_t^B(x_t) &:= \operatorname{ess\,sup}_{\pi \in \mathcal{A}_t} E_P [U(X_T^{x_t, \pi} - B) | \mathcal{G}_t] \\ &= -e^{-\gamma x_t} \operatorname{ess\,inf}_{\pi \in \mathcal{A}_t} E_P \left[\exp \left(-\gamma \int_t^T \pi_s \sigma_s d\hat{W}_s + \gamma B \right) \middle| \mathcal{G}_t \right], \end{aligned} \quad (2.6)$$

using that (bounded) \mathcal{G}_t -measurable factors can be pulled out. Thus $V_t^B(x_t)$ is the maximal expected utility the investor can achieve by starting at time t with initial capital x_t , using some admissible strategy π , and paying out B at time T .

Viewed over time t , $V^B(0)$ defined (up to a minus sign) by the essential infimum in (2.6) is the *dynamic value process* for the stochastic control problem associated to exponential utility maximization. One can show by standard arguments that $V^B(0)$ has an RCLL version and then study its dynamic properties as a process; see for instance [1], [15] or [17]. However, our goal in this paper is rather to provide explicit or structural formulas for $V_t^B(0)$ with a fixed t .

Remark. Condition (2.5) is technically useful, but also has the following desirable implication. From an economic point of view, one should only allow strategies which are close in some sense to investments with finite credit lines, as Schachermayer [19] emphasizes after his Definition 1.3. In our model, any $\pi \in \mathcal{A}_t$ can be approximated in the following way. Consider a sequence $(\tau_n)_{n \in \mathbb{N}}$ of \mathbb{G} -stopping times increasing to T stationarily and define a self-financing strategy $\pi^{(n)} = \pi I_{\llbracket t, \tau_n \rrbracket}$ by trading according to π until τ_n and then putting all the capital into the bank account. This gives a terminal portfolio value $x_t + \int_t^{\tau_n} \pi_s \sigma_s d\hat{W}_s$, leading to the individual utility $-\exp(-\gamma x_t - \gamma \int_t^{\tau_n} \pi_s \sigma_s d\hat{W}_s)$, which converges in $L^1(P)$ to the utility of the final value $X_T^{x_t, \pi}$ of the strategy π due to (2.5). If we specifically choose

$$\tau_n := \inf \{s \in [t, T] \mid X_s^{x_t, \pi} - x_t \leq -n\} \wedge T, \quad n \in \mathbb{N},$$

each of the approximating $\pi^{(n)}$ represents an investment with finite credit line. A similar approximation is used in Proposition 1 to find an upper bound for V^B , and the same class of strategies has been used in Hu et al. [12]. \diamond

The *indifference (seller) value* $b_t(x_t)$ at time t for B is implicitly defined by

$$V_t^0(x_t) = V_t^B(x_t + b_t(x_t)).$$

This says that the investor is indifferent between solely trading with initial capital x_t , versus trading with initial capital $x_t + b_t(x_t)$ but paying out B at T . Our final goal is to find an explicit formula for $b_t(x_t)$. By (2.6),

$$b_t = b_t(x_t) = \frac{1}{\gamma} \log \frac{V_t^B(0)}{V_t^0(0)} \quad (2.7)$$

does not depend on x_t . This also shows that we are done once we have $V_t^B(0)$ explicitly, and so our focus henceforth lies on the optimization problem (2.6).

2.2 Motivation

Our goal is to find an explicit expression for

$$-V_t^B(0) = \operatorname{ess\,inf}_{\pi \in \mathcal{A}_t} E_P \left[\exp \left(-\gamma \int_t^T \pi_s \sigma_s d\hat{W}_s + \gamma B \right) \middle| \mathcal{G}_t \right]. \quad (2.8)$$

Section 3 studies and solves an abstract martingale version of this problem, and we first explain how that formulation naturally arises out of (2.8). Since we only want to provide motivation, we ignore here all technical issues like integrability etc.

Suppose first that $B \equiv 0$ and S is a (local) P -martingale; equivalently, $\mu = \lambda = 0$ and \hat{W} is a P -Brownian motion. Then the stochastic integral in (2.8) is a P -martingale, we minimize the expectation of a convex function of this, and so Jensen's inequality immediately tells us that the optimizer is $\pi^* \equiv 0$ and that $V_t^B(0) = -1$.

In the general case where S is a P -semimartingale, the idea is now to reduce (2.8) to the martingale case by writing

$$-V_t^B(0) = \operatorname{ess\,inf}_{\pi \in \mathcal{A}_t} E_{P'} \left[\frac{Z'_t}{Z'_T} \exp \left(-\gamma \int_t^T \pi_s \sigma_s d\hat{W}_s + \gamma B \right) \middle| \mathcal{G}_t \right], \quad (2.9)$$

where Z' is the P -density process of some fixed measure P' (not depending on π) under which S or \hat{W} is a local martingale. To choose a good P' , one might be tempted by the duality results of [3] to take the minimal entropy martingale measure Q^E , because its density Z_T^E is up to a constant the exponential of a stochastic integral of S . However, this is not true for the density Z_t^E on \mathcal{G}_t , and it is in general also very difficult to find Q^E explicitly in any given model. Because we want explicit formulas, we need Z'_t/Z'_T as explicitly as possible. Now any equivalent local martingale measure P' has a P -density process of the form $Z' = \mathcal{E}(-\int \lambda dW) \mathcal{E}(N)$ for some local P -martingale N orthogonal to W , and inserting this expression for Z' in (2.9) gives after a straightforward calculation

$$\begin{aligned} & E_P \left[\exp \left(-\gamma \int_t^T \pi_s \sigma_s d\hat{W}_s + \gamma B \right) \middle| \mathcal{G}_t \right] \\ &= E_{\hat{P}} \left[\exp \left(-\int_t^T (\gamma \pi_s \sigma_s - \lambda_s) d\hat{W}_s + \gamma B - \frac{1}{2} \int_t^T \lambda_s^2 ds \right) \middle| \mathcal{G}_t \right]. \end{aligned} \quad (2.10)$$

The minimal martingale measure \hat{P} from (2.4) appears naturally in this way, and it has the enormous benefit that its density $\hat{Z} = \mathcal{E}(-\int \lambda dW)$ is completely explicit. Combining (2.10) with (2.8) gives

$$-V_t^B(0) = \operatorname{ess\,inf}_{\pi \in \mathcal{A}_t} E_{\hat{P}} \left[\exp \left(-\int_t^T (\gamma \pi_s \sigma_s - \lambda_s) d\hat{W}_s + \bar{B} \right) \middle| \mathcal{G}_t \right], \quad (2.11)$$

and we can recognize this as a ‘‘martingale version’’ of (2.8) with an artificial random endowment

$$\bar{B} := \gamma B - \frac{1}{2} \int_t^T \lambda_s^2 ds. \quad (2.12)$$

Note that in the genuine semimartingale case $\lambda \not\equiv 0$, the quantity \bar{B} appears even if the claim B is zero. Hence there is no simplification from assuming $B \equiv 0$, and so we do not discuss this case separately.

3 The main (but abstract) result

3.1 An explicit formula for an optimization problem

This section contains the main mathematical contribution of this paper. We derive an explicit formula for the value of the optimization problem

$$-\hat{V}_t^{\hat{B}} := \operatorname{ess\,inf}_{\hat{\pi} \in \hat{\mathcal{A}}_t} E_{\hat{P}} \left[\exp \left(- \int_t^T \hat{\pi}_s d\hat{W}_s + \hat{B} \right) \middle| \mathcal{G}_t \right] =: \operatorname{ess\,inf}_{\hat{\pi} \in \hat{\mathcal{A}}_t} \hat{\varphi}_t(\hat{\pi}), \quad (3.1)$$

where $\hat{\mathcal{A}}_t$ consists of all \mathbb{G} -predictable $\hat{\pi} = (\hat{\pi}_s)_{t \leq s \leq T}$ satisfying $\int_t^T \hat{\pi}_s^2 ds < \infty$ a.s. and such that $\left(\exp \left(- \int_t^s \hat{\pi}_y d\hat{W}_y \right) \right)_{t \leq s \leq T}$ is of \hat{P} -class (D) . \hat{B} is a bounded $\hat{\mathcal{H}}_T$ -measurable random variable, where $\hat{\mathbb{H}} = (\hat{\mathcal{H}}_s) \subseteq \mathbb{G}$ is a filtration such that the \hat{P} -Brownian motion \hat{Y} from (2.3) has the representation property in $\hat{\mathbb{H}}$. This means that any $(\hat{\mathbb{H}}, \hat{P})$ -martingale L is of the form $L = L_0 + \int \zeta d\hat{Y}$ for an $\hat{\mathbb{H}}$ -predictable ζ with $\int_0^T \zeta_s^2 ds < \infty$ a.s. The assumption $\hat{B} \in L^\infty(\hat{\mathcal{H}}_T)$ is slightly weaker than $\hat{B} \in L^\infty(\mathcal{F}_T^{\hat{Y}})$, and the two different applications in Section 4 will make it clear why this is useful. It is worth pointing out that all the subsequent arguments only involve the filtration $\hat{\mathbb{H}}$; this is the reason why we can formulate our model with a general filtration $\mathbb{G} \supseteq \mathbb{F}^{Y, Y^\perp}$ such that Y and Y^\perp are (\mathbb{G}, P) -Brownian motions.

While the idea of considering a problem like (3.1) has been motivated in Section 2.2 from (2.8), it is not clear at this stage how \hat{B} and especially $\hat{\mathbb{H}}$ arise. This will become clearer in Section 4 from the applications. However, we already point out that \hat{B} and the artificial claim $\bar{B} = \gamma B - \frac{1}{2} \int_t^T \lambda_s^2 ds$ from (2.12) can well be different.

Theorem 1. *Under the above assumptions, set*

$$\underline{\delta}_t := \inf_{s \in [t, T]} \frac{1}{\|1 - \rho_s^2\|_{L^\infty}} \quad \text{and} \quad \bar{\delta}_t := \sup_{s \in [t, T]} \left\| \frac{1}{1 - \rho_s^2} \right\|_{L^\infty}. \quad (3.2)$$

Then there exists a \mathcal{G}_t -measurable random variable $\delta_t^{\hat{B}}$ with values in $[\underline{\delta}_t, \bar{\delta}_t]$ such that

$$-\hat{V}_t^{\hat{B}}(\omega) = \left(E_{\hat{P}} \left[\exp(\hat{B})^{\frac{1}{\delta}} \middle| \hat{\mathcal{H}}_t \right] (\omega) \right)^\delta \Big|_{\delta = \delta_t^{\hat{B}}(\omega)} \quad (3.3)$$

for almost all $\omega \in \Omega$.

The right-hand side of (3.3) is understood as follows: We compute for fixed δ (a version of) the $(\hat{\mathcal{H}}_t, \hat{P})$ -conditional expectation of $\exp(\hat{B})^{\frac{1}{\delta}}$, evaluate that (version) in the given ω and then insert for δ the value $\delta_t^{\hat{B}}(\omega)$.

Before we actually prove Theorem 1, we provide here an outline of the proof. The key idea is to find a family of processes $Z^{(\hat{\pi})}$ with

$$Z_T^{(\hat{\pi})} = \exp\left(-\int_t^T \hat{\pi}_s d\hat{W}_s + \hat{B}\right) \quad (3.4)$$

and such that $Z^{(\hat{\pi})}$ is a (\mathbb{G}, \hat{P}) -submartingale for every $\hat{\pi} \in \hat{\mathcal{A}}_t$, and a (\mathbb{G}, \hat{P}) -martingale for some $\hat{\pi} = \hat{\pi}^* \in \hat{\mathcal{A}}_t$. If we can do this, the same argument as in Section 2.2 easily shows that the essential infimum in (3.1) is attained for $\hat{\pi}^*$.

To find such a family $Z^{(\hat{\pi})}$, we need a good representation for $e^{\hat{B}}$, and the multiplicative form of (3.4) might suggest that we write $e^{\hat{B}}$ as the final value of some stochastic exponential martingale. But unless we believe that $\hat{\pi}^* \equiv 0$ happens to be optimal, $e^{\hat{B}} = Z_T^{(0)}$ should be the final value of a (\mathbb{G}, \hat{P}) -submartingale rather than a (\mathbb{G}, \hat{P}) -martingale. Again in view of the multiplicative structure, the simplest way to transform a positive martingale into a submartingale is to raise it to a power bigger than one. Fixing a constant $\delta \geq 1$ to be specified later and using $\hat{B} \in L^\infty(\hat{\mathcal{H}}_T)$, we thus write

$$\exp(\hat{B}) = \exp(\hat{B}/\delta)^\delta = (c_t \mathcal{E}(L)_{t,T})^\delta, \quad c_t := E_{\hat{P}}[\exp(\hat{B}/\delta) | \hat{\mathcal{H}}_t] \quad (3.5)$$

for a $BMO(\hat{\mathbb{H}}, \hat{P})$ -martingale L . (More precisely, the positive $(\hat{\mathbb{H}}, \hat{P})$ -martingale with final value $\exp(\hat{B}/\delta)$ is uniformly bounded away from zero and infinity, and thus its stochastic logarithm L is in BMO .) By the representation property of \hat{Y} in $\hat{\mathbb{H}}$, L is of the form

$$L = \int \zeta d\hat{Y} \quad \text{for an } \hat{\mathbb{H}}\text{-predictable } \zeta \text{ with } E_{\hat{P}}\left[\int_0^T \zeta_s^2 ds\right] < \infty. \quad (3.6)$$

So L is a $BMO(\mathbb{G}, \hat{P})$ -martingale, too, and combining (3.4) and (3.5) gives

$$\begin{aligned} Z_T^{(\hat{\pi})} &= c_t^\delta (\mathcal{E}(L)_{t,T})^\delta \exp\left(-\int_t^T \hat{\pi}_s d\hat{W}_s\right) \\ &= c_t^\delta \mathcal{E}(\delta L)_{t,T} \mathcal{E}\left(-\int_t^T \hat{\pi} d\hat{W}\right)_{t,T} \exp\left(\frac{1}{2} \int_t^T ((\delta^2 - \delta)\zeta_s^2 + \hat{\pi}_s^2) ds\right). \end{aligned}$$

Using Yor's formula, (3.5) and $d\langle \hat{Y}, \hat{W} \rangle_s = \rho_s ds$ yields

$$\begin{aligned} \mathcal{E}(\delta L) \mathcal{E}\left(-\int \hat{\pi} d\hat{W}\right) &= \mathcal{E}\left(\delta L - \int \hat{\pi} d\hat{W} - \left\langle \delta L, \int \hat{\pi} d\hat{W} \right\rangle\right) \\ &= M^{(\hat{\pi})} \exp\left(-\int \delta \zeta_s \hat{\pi}_s \rho_s ds\right) \end{aligned}$$

with the local (\mathbb{G}, \hat{P}) -martingale

$$M^{(\hat{\pi})} := \mathcal{E} \left(\delta L - \int \hat{\pi} d\hat{W} \right) = \mathcal{E} \left(\int \delta \zeta d\hat{Y} - \int \hat{\pi} d\hat{W} \right), \quad (3.7)$$

and putting everything together and completing squares leads us to define

$$Z_s^{(\hat{\pi})} := c_t^\delta M_{t,s}^{(\hat{\pi})} \exp \left(\frac{1}{2} \int_t^s \left((\hat{\pi}_y - \delta \zeta_y \rho_y)^2 + \zeta_y^2 \delta (\delta (1 - \rho_y^2) - 1) \right) dy \right) \quad (3.8)$$

for $t \leq s \leq T$. This gives (3.4) by construction, and if ρ is constant, choosing $\delta := \frac{1}{1-\rho^2}$ ensures that the integrand in (3.8) is always nonnegative and vanishes for $\hat{\pi}^* = \delta \zeta \rho$. Hence $Z^{(\hat{\pi})}$ is then on $[t, T]$ a local (\mathbb{G}, \hat{P}) -submartingale for every $\hat{\pi}$ and a local (\mathbb{G}, \hat{P}) -martingale for $\hat{\pi}^*$. Apart from integrability issues, we thus have achieved our goal in that case.

In general, ρ is not constant. Then we choose one δ for the submartingale property of $Z^{(\hat{\pi})}$ for all $\hat{\pi}$, and another δ for the martingale property of $Z^{(\hat{\pi}^*)}$. This gives an upper and a lower bound for $\hat{V}_t^{\hat{B}}$, and Theorem 1 is obtained by interpolation. The detailed proof is given in the next section.

Remark. The attentive reader may have noticed that we only give results on the *value* of the optimization problem, and may object that for hedging or investing purposes, one would also like to know the optimal *strategy* explicitly. While this is a valid point, it is a well-known unfortunate fact that this problem is notoriously difficult even in quite specific (e.g., Markovian) settings. We hope to address this question in subsequent work, as it goes beyond the scope of the present paper. \diamond

3.2 Proof of Theorem 1

The argument for Theorem 1 follows the outline given in Section 3.1. We assume throughout that the assumptions of Theorem 1 hold and first derive an upper bound for $\hat{V}_t^{\hat{B}}$. Recall $\hat{\varphi}_t(\hat{\pi})$ from (3.1) and $\bar{\delta}_t$ from (3.2).

Proposition 1. *For all $\hat{\pi} \in \hat{\mathcal{A}}_t$, we have*

$$\hat{\varphi}_t(\hat{\pi}) \geq E_{\hat{P}} \left[\exp(\hat{B}/\bar{\delta}_t) | \hat{\mathcal{H}}_t \right]^{\bar{\delta}_t} \quad a.s.$$

Proof. We use the reasonings and notations from Section 3.1 with $\delta := \bar{\delta}_t$.

1) Suppose first that $\int_t^T \hat{\pi}_s^2 ds$ is uniformly bounded. Then $\int \hat{\pi} d\hat{W}$ is a $BMO(\mathbb{G}, \hat{P})$ -martingale like $L = \int \zeta d\hat{Y}$ from (3.6), and hence by Theorem 2.3 of Kazamaki [14], $M^{(\hat{\pi})} = \mathcal{E}(\delta L - \int \hat{\pi} d\hat{W})$ from (3.7) is a (\mathbb{G}, \hat{P}) -martingale. The choice $\delta = \bar{\delta}_t$ implies that the integrand in (3.8) is nonnegative and thus $Z^{(\hat{\pi})}$ is a (\mathbb{G}, \hat{P}) -submartingale; in fact, integrability follows

via (3.4) because $Z_T^{(\hat{\pi})} = \exp(-\int_t^T \hat{\pi}_s d\hat{W}_s + \hat{B})$ is in $L^1(\hat{P})$ since $\int \hat{\pi} d\hat{W}$ is in BMO . Thus (3.1), (3.4), (3.8) and (3.5) yield

$$\hat{\varphi}_t(\hat{\pi}) = E_{\hat{P}}[Z_T^{(\hat{\pi})} | \mathcal{G}_t] \geq Z_t^{(\hat{\pi})} = E_{\hat{P}}[\exp(\hat{B}/\bar{\delta}_t) | \hat{\mathcal{H}}_t]^{\bar{\delta}_t} \quad \text{a.s.}$$

2) In general, we define a localizing sequence by

$$\tau_n := \inf \left\{ s \in [t, T] \text{ such that } \int_t^s \hat{\pi}_y^2 dy \geq n \right\} \wedge T, \quad n \in \mathbb{N}$$

and set $\hat{\pi}^{(n)} := \hat{\pi} I_{\llbracket t, \tau_n \rrbracket} \in \hat{\mathcal{A}}_t$. Applying step 1) to $\hat{\pi}^{(n)}$ then gives

$$\hat{\varphi}_t(\hat{\pi}^{(n)}) \geq E_{\hat{P}}[\exp(\hat{B}/\bar{\delta}_t) | \hat{\mathcal{H}}_t]^{\bar{\delta}_t} \quad \text{a.s.} \quad (3.9)$$

Because $\left(\exp(-\int_t^s \hat{\pi}_y d\hat{W}_y) \right)_{t \leq s \leq T}$ is of \hat{P} -class (D) and \hat{B} is bounded, the sequence $\left(Z_T^{(\hat{\pi}^{(n)})} \right)_{n \in \mathbb{N}} = \left(\exp(-\int_t^T \hat{\pi}_s^{(n)} d\hat{W}_s + \hat{B}) \right)_{n \in \mathbb{N}}$ is \hat{P} -uniformly integrable and converges a.s. to $Z_T^{(\hat{\pi})}$. Hence the conditioned random variables $\hat{\varphi}_t(\hat{\pi}^{(n)}) = E_{\hat{P}}[Z_T^{(\hat{\pi}^{(n)})} | \mathcal{G}_t]$, $n \in \mathbb{N}$, converge to $\hat{\varphi}_t(\hat{\pi})$ in $L^1(\hat{P})$ and therefore also a.s. along a subsequence. This concludes the proof in view of (3.9). \square

The next result entails a lower bound for $\hat{V}_t^{\hat{B}}$. Recall $\underline{\delta}_t$ from (3.2).

Proposition 2. Define $\hat{\pi}^* = (\hat{\pi}_s^*)_{t \leq s \leq T}$ by

$$\hat{\pi}_s^* := \left(\rho_s \underline{\delta}_t + \sqrt{\rho_s^2 \underline{\delta}_t^2 + \underline{\delta}_t - \underline{\delta}_t^2} \right) \zeta_s, \quad t \leq s \leq T, \quad (3.10)$$

where ζ is now defined in (3.5), (3.6) with $\delta := \underline{\delta}_t$. Then we have

$$\hat{\varphi}_t(\hat{\pi}^*) = E_{\hat{P}}[\exp(\hat{B}/\underline{\delta}_t) | \hat{\mathcal{H}}_t]^{\underline{\delta}_t} \quad \text{a.s.}, \quad \text{and} \quad \hat{\pi}^* \in \hat{\mathcal{A}}_t. \quad (3.11)$$

To be more precise, $\hat{\pi}_s^*$ is for any $s \in [t, T]$ defined by (3.10) on the set $\tilde{\Omega}_s = \left\{ \omega \in \Omega \mid \frac{1}{1-\rho_s^2(\omega)} \geq \underline{\delta}_t \right\}$, which has $(P$ - and \hat{P} -)probability one. For $\omega \notin \tilde{\Omega}_s$, we set $\hat{\pi}_s^*(\omega) := 0$. By the definition of $\underline{\delta}_t$, the expression under the square root in (3.10) is nonnegative, and $\hat{\pi}^*$ is \mathbb{G} -predictable.

Proof. We use the notations of Section 3.1 with $\delta := \underline{\delta}_t$, and first show the equality in (3.11). Because $\hat{\pi}^*$ in (3.10) is chosen to make the integrand in (3.8) vanish, we get from (3.8)

$$Z_T^{(\hat{\pi}^*)} = c_t^{\underline{\delta}_t} M_{t,T}^{(\hat{\pi}^*)} = E_{\hat{P}}[\exp(\hat{B}/\underline{\delta}_t) | \hat{\mathcal{H}}_t]^{\underline{\delta}_t} \mathcal{E}(N^{(\hat{\pi}^*)})_{t,T} \quad (3.12)$$

with $N^{(\hat{\pi}^*)} := \int \underline{\delta}_t \zeta d\hat{Y} - \int \hat{\pi}^* d\hat{W}$. An easy computation using (3.10) yields

$$\langle N^{(\hat{\pi}^*)} \rangle_s - \langle N^{(\hat{\pi}^*)} \rangle_t = \underline{\delta}_t \int_t^s \zeta_y^2 dy = \underline{\delta}_t (\langle L \rangle_s - \langle L \rangle_t), \quad t \leq s \leq T \quad (3.13)$$

and so $N^{(\hat{\pi}^*)}$ is like $\int \zeta d\hat{Y} = L$ a $BMO(\mathbb{G}, \hat{P})$ -martingale; see below (3.6). We conclude by Theorem 2.3 of Kazamaki [14] that $M^{(\hat{\pi}^*)} = \mathcal{E}(N^{(\hat{\pi}^*)})$ is a (\mathbb{G}, \hat{P}) -martingale, and so (3.1), (3.4) and (3.12) yield the equality in (3.11).

To prove $\hat{\pi}^* \in \hat{\mathcal{A}}_t$, we first note that $E_{\hat{P}} \left[\int_t^T \zeta_s^2 ds \right] < \infty$ from (3.6) implies $E_{\hat{P}} \left[\int_t^T |\hat{\pi}_s^*|^2 ds \right] < \infty$ by (3.10). To show that $(\exp(-\int_t^s \hat{\pi}_y^* d\hat{W}_y))_{t \leq s \leq T}$ is of \hat{P} -class (D) , we observe that (3.6) and (3.13) yield

$$\exp\left(-\int_t^s \hat{\pi}_y^* d\hat{W}_y\right) = \exp\left(N_s^{(\hat{\pi}^*)} - N_t^{(\hat{\pi}^*)} - \underline{\delta}_t \int_t^s \zeta_y d\hat{Y}_y\right) = M_{t,s}^{(\hat{\pi}^*)} \mathcal{E}(L)_{t,s}^{-\underline{\delta}_t}$$

for $s \in [t, T]$, and the process $\mathcal{E}(L)^{-\underline{\delta}_t}$ is bounded because (3.5) gives

$$\mathcal{E}(L)_{t,s} = \frac{E_{\hat{P}}[\exp(\hat{B}/\underline{\delta}_t) | \hat{\mathcal{H}}_s]}{E_{\hat{P}}[\exp(\hat{B}/\underline{\delta}_t) | \hat{\mathcal{H}}_t]}, \quad t \leq s \leq T$$

and \hat{B} is bounded. Moreover, $M^{(\hat{\pi}^*)}$ as a (\mathbb{G}, \hat{P}) -martingale is of \hat{P} -class (D) , and hence so is $(\exp(-\int_t^s \hat{\pi}_y^* d\hat{W}_y))_{t \leq s \leq T}$. Thus $\hat{\pi}^*$ is in $\hat{\mathcal{A}}_t$. \square

Remark. The choice of $\hat{\pi}^*$ in (3.10) deserves a comment. As we have seen in the proof of Proposition 2, it ensures that the integrand

$$(\underline{\delta}_t \rho_s \zeta_s - \hat{\pi}_s^*)^2 + \zeta_s^2 \underline{\delta}_t (\underline{\delta}_t (1 - \rho_s^2) - 1) \quad (3.14)$$

in (3.8) (with $\delta = \underline{\delta}_t$ and $\hat{\pi} = \hat{\pi}^*$) vanishes identically. But for fixed $s \in [t, T]$ and $\omega \in \Omega$, (3.14) is a quadratic function in $\hat{\pi}_s^*(\omega)$, and requiring it to be zero for each s does not determine the process $\hat{\pi}^*$ uniquely. In fact, Proposition 2 remains true if $\hat{\pi}^*$ is replaced by $\hat{\pi}^\eta$ with

$$\hat{\pi}_s^\eta := \left(\rho_s \underline{\delta}_t + \eta_s \sqrt{\rho_s^2 \underline{\delta}_t^2 + \underline{\delta}_t - \underline{\delta}_t^2} \right) \zeta_s, \quad t \leq s \leq T$$

for any \mathbb{G} -predictable process η on $[t, T]$ with values in $\{-1, 1\}$.

Suppose now we replace $\hat{\pi}^*$ by $\hat{\pi}^{**}$ with $\hat{\pi}_s^{**} := \underline{\delta}_t \rho_s \zeta_s$, $s \in [t, T]$, which minimizes (3.14) pointwise and makes it nonpositive. Then we get

$$\hat{\varphi}_t(\hat{\pi}^{**}) = E_{\hat{P}} \left[Z_T^{(\hat{\pi}^{**})} \middle| \mathcal{G}_t \right] \leq E_{\hat{P}} \left[E_{\hat{P}} \left[\exp(\hat{B}/\underline{\delta}_t) | \hat{\mathcal{H}}_t \right]^{\underline{\delta}_t} M_{t,T}^{(\hat{\pi}^{**})} \middle| \mathcal{G}_t \right] = \hat{\varphi}_t(\hat{\pi}^*),$$

using that $M^{(\hat{\pi}^{**})}$ is like $M^{(\hat{\pi}^*)}$ a (\mathbb{G}, \hat{P}) -martingale. Similar arguments as for $\hat{\pi}^*$ also yield $\hat{\pi}^{**} \in \hat{\mathcal{A}}_t$, and we even obtain $\hat{\varphi}_t(\hat{\pi}^{**}) < \hat{\varphi}_t(\hat{\pi}^*)$ on a set $A \in \mathcal{G}_t$ with $\hat{P}[A] > 0$ if $\int_t^T \zeta_s^2 (\underline{\delta}_t(1 - \rho_s^2) - 1) ds$ is non-zero with positive probability. This shows that the lower bound

$$-\hat{\varphi}_t(\hat{\pi}^*) \leq -\operatorname{ess\,inf}_{\hat{\pi} \in \hat{\mathcal{A}}_t} \hat{\varphi}_t(\hat{\pi}) = \hat{V}_t^{\hat{B}} \quad \text{a.s.} \quad (3.15)$$

entailed by Proposition 2 need not be sharp. Nevertheless, we work with $\hat{\pi}^*$ and not with $\hat{\pi}^{**}$, because $\hat{\varphi}_t(\hat{\pi}^*)$ has the nice representation (3.11) which allows us to obtain an explicit expression for $\hat{V}_t^{\hat{B}}$; see the interpolation argument below. The sharper bound given via $\hat{\pi}^{**}$ is not explicit enough to give this result. We remark that if ρ is constant, $\hat{\pi}^{**}$ and $\hat{\pi}^*$ coincide and (3.15) holds with equality; compare Propositions 1 and 2. \diamond

As announced, we now prove Theorem 1 by an interpolation argument.

Proof of Theorem 1. Define $f(\cdot, \cdot) : [\underline{\delta}_t, \bar{\delta}_t] \times \Omega \rightarrow \mathbb{R}$ by

$$f(\delta, \omega) := \left(E_{\hat{P}} \left[\exp(\hat{B})^{\frac{1}{\delta}} \middle| \hat{\mathcal{H}}_t \right] (\omega) \right)^\delta, \quad (\delta, \omega) \in [\underline{\delta}_t, \bar{\delta}_t] \times \Omega.$$

Because \hat{B} is bounded, dominated convergence and Jensen's inequality imply that f admits a version which is continuous and nonincreasing in δ for each fixed $\omega \in \Omega$. We use this version in the sequel. From Propositions 1 and 2 we already know that

$$f(\bar{\delta}_t, \omega) \leq -\hat{V}_t^{\hat{B}}(\omega) \leq f(\underline{\delta}_t, \omega) \quad \text{for a.a. } \omega \in \Omega.$$

By the intermediate value theorem, the set

$$\Delta(\omega) := \left\{ \delta \in [\underline{\delta}_t, \bar{\delta}_t] \mid f(\delta, \omega) = -\hat{V}_t^{\hat{B}}(\omega) \right\}$$

is thus nonempty for almost all $\omega \in \Omega$. Define $\delta_t^{\hat{B}} : \Omega \rightarrow [\underline{\delta}_t, \bar{\delta}_t]$ by

$$\delta_t^{\hat{B}}(\omega) := \sup \Delta(\omega), \quad \omega \in \Omega, \quad (3.16)$$

setting $\delta_t^{\hat{B}}(\cdot) := (\underline{\delta}_t + \bar{\delta}_t)/2$ on the nullset $\{\omega \in \Omega \mid \Delta(\omega) = \emptyset\}$. By the continuity of f in its first argument, $\Delta(\omega)$ is closed in \mathbb{R} for all $\omega \in \Omega$, and we obtain for almost all $\omega \in \Omega$ that

$$f(\delta_t^{\hat{B}}(\omega), \omega) = -\hat{V}_t^{\hat{B}}(\omega). \quad (3.17)$$

It remains to prove that the mapping $\omega \mapsto \delta_t^{\hat{B}}(\omega)$ is \mathcal{G}_t -measurable. Because f is nonincreasing and due to (3.16) and (3.17), we have for any $a \in [\underline{\delta}_t, \bar{\delta}_t]$ that up to a null set,

$$\begin{aligned} \{\omega \in \Omega \mid \delta_t^{\hat{B}}(\omega) < a\} &= \{\omega \in \Omega \mid f(\delta_t^{\hat{B}}(\omega), \omega) > f(a, \omega)\} \\ &= \{\omega \in \Omega \mid -\hat{V}_t^{\hat{B}}(\omega) > f(a, \omega)\} \\ &= \bigcup_{q \in \mathbb{Q}} \{\omega \mid -\hat{V}_t^{\hat{B}}(\omega) > q\} \cap \{\omega \mid q > f(a, \omega)\}. \end{aligned}$$

The last set is in \mathcal{G}_t since $\hat{V}_t^{\hat{B}}$ and $f(a, \cdot)$ for a fixed $a \in [\underline{\delta}_t, \bar{\delta}_t]$ are \mathcal{G}_t -measurable. Since \mathcal{G}_t is complete, we have $\{\omega \in \Omega \mid \delta_t^{\hat{B}}(\omega) < a\} \in \mathcal{G}_t$ for every $a \in \mathbb{R}$, and so $\delta_t^{\hat{B}}(\cdot)$ is \mathcal{G}_t -measurable. This ends the proof. \square

4 Applications in two settings

4.1 Explicit formulas for the indifference value

Our goal in this section is to find explicit formulas for $V_t^B(0)$ in (2.6) or (2.11) in two different settings. This will be achieved by applying Theorem 1 and will also yield explicit results for the indifference value b_t via (2.7). We recall W and \hat{Y} from (2.1) and (2.3) and write for brevity

$$\mathbb{F} = (\mathcal{F}_s) \text{ for } \mathbb{F}^W, \quad \mathbb{Y} = (\mathcal{Y}_s) \text{ for } \mathbb{F}^Y, \quad \hat{\mathbb{Y}} = (\hat{\mathcal{Y}}_s) \text{ for } \mathbb{F}^{\hat{Y}}.$$

If $\rho\lambda$ is \mathbb{Y} -predictable, then \hat{Y} from (2.3) is \mathbb{Y} -adapted and hence $\hat{\mathbb{Y}} \subseteq \mathbb{Y}$. In general, however, none of the above three filtrations contains any other.

Theorem 1 gives us the freedom to specify the artificial endowment \hat{B} , but also the task of finding a filtration $\hat{\mathbb{H}}$ such that \hat{B} is $\hat{\mathcal{H}}_T$ -measurable and \hat{Y} has the representation property in $\hat{\mathbb{H}}$. Comparing (2.11) with (3.1) suggests to choose $\hat{B} = \bar{B} = \gamma B - \frac{1}{2} \int_t^T \lambda_s^2 ds$. In a first application, we do this, set $\hat{\mathbb{H}} = \mathbb{Y}$ and assume that B is \mathcal{Y}_T -measurable and λ is \mathbb{Y} -predictable, to ensure that \hat{B} is $\hat{\mathcal{H}}_T$ -measurable. We shall later see in the proof of Theorem 2 that we also need to assume that ρ is \mathbb{Y} -predictable to guarantee that \hat{Y} has the representation property in \mathbb{Y} .

For our second application, we choose $\hat{B} = \gamma B$ and assume that $\frac{1}{2} \int_t^T \lambda_s^2 ds$ is replicable by trading in S . This is satisfied if λ is \mathbb{F} -predictable, as we shall see in the proof of Theorem 3. In this case, we moreover set $\hat{\mathbb{H}} = \hat{\mathbb{Y}}$ and assume that B is $\hat{\mathcal{Y}}_T$ -measurable.

Cases. *In more detail, we consider one of the following two situations:*

- (I) $B \in L^\infty(\mathcal{Y}_T)$, λ is \mathbb{Y} -predictable, and ρ is \mathbb{Y} -predictable;
- (II) $B \in L^\infty(\hat{\mathcal{Y}}_T)$, λ is $\mathbb{F}^{S, \hat{Y}}$ -predictable, and λ is \mathbb{F} -predictable.

The assumption in case (II) that λ is $\mathbb{F}^{S, \hat{Y}}$ -predictable is quite natural since S and \hat{Y} are the quantities observable for our investor. Moreover, it guarantees by Lemma 1 below that $\mathbb{F}^{Y, Y^\perp} \subseteq \mathbb{F}^{S, \hat{Y}}$, i.e., the two basic driving Brownian motions Y and Y^\perp are observable from S and \hat{Y} . In particular, if we take $\mathbb{G} = \mathbb{F}^{Y, Y^\perp}$, the a priori condition that λ is $\mathbb{F}^{S, \hat{Y}}$ -predictable turns out to be innocent a posteriori.

To motivate our model choice, we discuss for each case a typical example.

Case (I): Here one should think of a *stochastic volatility* model, where μ and σ are \mathbb{Y} -predictable and the contingent claim B depends only on σ (e.g., a variance swap). The stock S is driven by the Brownian motion W , whereas its drift and volatility depend on a second factor Y . Our approach allows us to consider the situation where the correlation between W and Y is not constant, but more realistically a functional of Y .

In this setting, B is naturally \mathcal{Y}_T -measurable and $\lambda = \frac{\mu}{\sigma}$ is \mathbb{Y} -predictable like μ and σ . The only genuine condition is that ρ should be \mathbb{Y} -predictable, which we technically need to guarantee that \hat{Y} has the representation property not only in $\hat{\mathbb{Y}}$, but also in \mathbb{Y} .

Case (II): A good application here comes from *executive stock options*. Consider a manager who receives call options on the stock (driven by \hat{Y}) of her company as part of her performance-related compensation. The manager must not trade the company stock and all its derivatives because of legal restrictions. However, she might be able to trade other, correlated stocks. So S is here a market index, a representative portfolio of other companies in the same line of business, or the stock of a leading company in the same line of business, which serves as a benchmark. We assume that the only source of incompleteness is the fact that the manager is not allowed to directly trade the stock of her company. In particular, we suppose that the market formed by the bank account and S is complete by assuming that μ and σ are both \mathbb{F} - and \mathbb{F}^S -predictable. Then $\mathbb{F} = \mathbb{F}^S$, i.e., the uncertainty (\mathbb{F}) about S equals the information (\mathbb{F}^S) available from S . This follows from (2.2) because σ is bounded away from zero. We then provide a fair value for the executive options in such a situation.

In this setting, $\lambda = \frac{\mu}{\sigma}$ is \mathbb{F} - and \mathbb{F}^S -predictable like μ and σ . The only genuine condition here is that B is $\hat{\mathcal{Y}}_T$ -measurable, and the next remark explains why this is natural. Equivalently, that remark clarifies why we view the nontradable asset here as driven by \hat{Y} and not by Y .

In both cases, the measurability assumptions make precise the underlying idea: The payoff B is driven by Y (or \hat{Y}), whereas hedging can only be done in S which is imperfectly correlated with Y (or \hat{Y}). The examples also illustrate two reasons why direct hedging in the underlying of B may be impossible; either its driver is not traded at all (e.g., a volatility or a consumer price index), or it is traded in principle but not tradable by our investor, due to legal, liquidity, practicability, cost or other reasons.

Remark. To see why $\hat{\mathcal{Y}}_T$ -measurability of B is reasonable in case (II), recall that B is a claim on some asset Z , and write $dY = \rho dW + \sqrt{1 - \rho^2} dW^\perp$ for a (\mathbb{G}, P) -Brownian motion W^\perp P -independent of W . The asset change dZ is driven by two factors: the market development $\frac{dS}{S}$ of the benchmark S , and company specific risks dW^\perp . To determine the genuine driver of Z , we weight the two factors by the correlation process ρ , but first make them comparable by “normalizing” $\frac{dS}{S}$, which means that we use $\frac{1}{\sigma} \frac{dS}{S} = d\hat{W}$ instead of $\frac{dS}{S}$. Thus Z is driven by

$$\rho d\hat{W} + \sqrt{1 - \rho^2} dW^\perp = \rho \lambda ds + \rho dW + \sqrt{1 - \rho^2} dW^\perp = \rho \lambda ds + dY = d\hat{Y},$$

using (2.3). Hence assuming the Z -dependent claim B to be $\hat{\mathcal{Y}}_T$ -measurable is more natural than having it \mathcal{Y}_T -measurable. Note that the filtrations $\hat{\mathbb{Y}}$ and \mathbb{Y} differ in general, but coincide if ρ and λ are deterministic. \diamond

Let us now briefly look at the information available to our investor. We always assume that the tradable stock S is observable. In addition, we assume in both cases (I) and (II) that the driver for the uncertainty behind B (i.e., Y or \hat{Y} , respectively) is also observable. The following result shows that the observable filtration then contains the filtration \mathbb{F}^{Y, Y^\perp} of the underlying Brownian motions, and this justifies why we always use $\mathbb{G} \supseteq \mathbb{F}^{Y, Y^\perp}$ to describe the information on which our strategies $\pi \in \mathcal{A}_t$ must be based.

Lemma 1. *In case (I), $\mathbb{F}^{Y, Y^\perp} \subseteq \mathbb{F}^{S, Y}$, and in case (II), $\mathbb{F}^{Y, Y^\perp} \subseteq \mathbb{F}^{S, \hat{Y}}$.*

Proof. Note that the argument in each case uses only the middle condition on λ . For brevity, we write $Z \in \mathbb{F}^X$ to mean that Z is \mathbb{F}^X -predictable.

Case (I): By (2.2), $\langle S \rangle = \int \sigma^2 S^2 ds$ and $\langle S \rangle \in \mathbb{F}^S$ as continuous pathwise quadratic variation; so $\sigma S = +\sqrt{\sigma^2 S^2} \in \mathbb{F}^S$ and hence also $\sigma \in \mathbb{F}^S$. Next, $\lambda \in \mathbb{Y}$ by assumption, and so $\mu = \sigma \lambda \in \mathbb{F}^{S, Y}$. Because σ is bounded away from 0, we obtain $W = \int \frac{1}{\sigma S} dS - \int \frac{\mu}{\sigma} ds \in \mathbb{F}^{S, Y}$. As a consequence, $\rho \in \mathbb{F}^{S, Y}$ since it is the density of $\langle W, Y \rangle$ with respect to Lebesgue measure and $\langle W, Y \rangle \in \mathbb{F}^{S, Y}$ as a continuous pathwise quadratic covariation. Finally,

$|\rho|$ is bounded away from 1; so solving (2.1) for Y^\perp implies that $Y^\perp \in \mathbb{F}^{S,Y}$ and therefore $\mathbb{F}^{Y,Y^\perp} \subseteq \mathbb{F}^{S,Y}$.

Case (II): Again, $\sigma \in \mathbb{F}^S$. Moreover, (2.2), (2.1) and the definition (2.3) of \hat{Y} give $\langle S, \hat{Y} \rangle = \int \sigma S \rho ds$ so that $\rho \in \mathbb{F}^{S,\hat{Y}}$. Because $\lambda \in \mathbb{F}^{S,\hat{Y}}$ by assumption, we get $\mu = \sigma \lambda \in \mathbb{F}^{S,\hat{Y}}$, and now we can argue like in case (I) to deduce that $Y^\perp \in \mathbb{F}^{S,\hat{Y}}$. Moreover, $Y = \hat{Y} - \int \rho \lambda ds \in \mathbb{F}^{S,\hat{Y}}$ and hence $\mathbb{F}^{Y,Y^\perp} \subseteq \mathbb{F}^{S,\hat{Y}}$. \square

The two following theorems give explicit formulas for the value V^B and the indifference value b in cases (I) and (II). To facilitate comparisons with the literature, we state them for a spot interest rate on the bank account given by a bounded deterministic Borel-measurable function $r : [0, T] \rightarrow [0, \infty)$. Our results and arguments given for $r \equiv 0$ easily extend to this case; allowing r to be stochastic, however, would be a different issue.

Theorem 2. *Consider the setting and the assumptions from Section 2 and recall $\underline{\delta}_t, \bar{\delta}_t$ from (3.2). In case (I), define $\hat{B} := \gamma B - \frac{1}{2} \int_t^T \frac{(\mu_s - r(s))^2}{\sigma_s^2} ds$ and $\hat{0} := -\frac{1}{2} \int_t^T \frac{(\mu_s - r(s))^2}{\sigma_s^2} ds$. Then there exist \mathcal{G}_t -measurable random variables $\delta_t^{\hat{B}}, \delta_t^{\hat{0}}$ with values in $[\underline{\delta}_t, \bar{\delta}_t]$ such that*

$$V_t^B(x_t)(\omega) = -\exp\left(-\gamma x_t(\omega) e^{\int_t^T r(s) ds}\right) \left(E_{\hat{P}}\left[\exp(\hat{B})^{\frac{1}{\delta}} \middle| \mathcal{Y}_t\right](\omega)\right)^\delta \Bigg|_{\delta=\delta_t^{\hat{B}}(\omega)} \quad (4.1)$$

and

$$b_t(\omega) = \frac{e^{-\int_t^T r(s) ds}}{\gamma} \log \frac{(E_{\hat{P}}[\exp(\hat{B})^{1/\delta} | \mathcal{Y}_t](\omega))^\delta}{(E_{\hat{P}}[\exp(\hat{0})^{1/\delta'} | \mathcal{Y}_t](\omega))^{\delta'}} \Bigg|_{\delta'=\delta_t^{\hat{0}}(\omega), \delta=\delta_t^{\hat{B}}(\omega)} \quad (4.2)$$

for almost all $\omega \in \Omega$ and every bounded \mathcal{G}_t -measurable random variable x_t .

Theorem 3. *Consider the setting and the assumptions from Section 2 and recall $\underline{\delta}_t, \bar{\delta}_t$ from (3.2). In case (II), there exists a \mathcal{G}_t -measurable random variable $\delta_t^{\gamma B}$ with values in $[\underline{\delta}_t, \bar{\delta}_t]$ such that*

$$V_t^B(x_t)(\omega) = -\exp\left(-\gamma x_t(\omega) e^{\int_t^T r(s) ds} - \frac{1}{2} E_{\hat{P}}\left[\int_t^T \frac{(\mu_s - r(s))^2}{\sigma_s^2} ds \middle| \mathcal{F}_t\right]\right) \\ \times \left(E_{\hat{P}}\left[\exp(\gamma B)^{\frac{1}{\delta}} \middle| \hat{\mathcal{Y}}_t\right](\omega)\right)^\delta \Bigg|_{\delta=\delta_t^{\gamma B}(\omega)}$$

and

$$b_t(\omega) = \frac{e^{-\int_t^T r(s) ds}}{\gamma} \log \left(E_{\hat{P}}\left[\exp(\gamma B)^{\frac{1}{\delta}} \middle| \hat{\mathcal{Y}}_t\right](\omega)\right)^\delta \Bigg|_{\delta=\delta_t^{\gamma B}(\omega)}$$

for almost all $\omega \in \Omega$ and every bounded \mathcal{G}_t -measurable random variable x_t .

To the best of our knowledge, results like Theorems 2 or 3 have not been available in the literature so far; all previous approaches leading to explicit formulas have only considered situations where the correlation ρ is deterministic and constant in time. One nice feature of all formulas in Theorems 2 and 3 is that the only unknowns are the distortion powers $\delta^{\hat{B}}$, $\delta^{\hat{0}}$ or δ^{γ^B} , and we have precise bounds for these in terms of bounds on the correlation ρ . In general, each such power is random (in a \mathbb{G} -adapted way) and depends on B via \hat{B} . Since we have assumed that \mathcal{G}_0 is trivial, $\delta_0^{\hat{B}}$ is deterministic, but may still depend on \hat{B} . However, if the correlation ρ is deterministic and constant in time, the functions $\underline{\delta}$ and $\bar{\delta}$ in (3.2) coincide and equal $\frac{1}{1-\rho^2}$, and then $\delta^{\hat{B}} = \frac{1}{1-\rho^2}$ becomes constant and independent of B or \hat{B} . This explains why the constant correlation case is easier to handle and understand.

We defer the proofs of Theorems 2 and 3 to Section 4.3, and first compare our results with the existing literature.

4.2 Comparison with the literature

Exponential utility indifference valuation in Brownian settings has been extensively studied, particularly in Markovian models. An overview with a long literature list is provided by Henderson and Hobson [11]. We present here some references and comment first on the different model assumptions and then on the methods and results.

Recall the model in (2.1) and (2.2). Henderson [7, 8], Henderson and Hobson [9, 10], and Musiela and Zariphopoulou [18] all work in a Markovian framework where μ , σ , r and ρ are all constant. [7–10] have a nontraded asset Z satisfying, for some constants $a > 0$ and $b \in \mathbb{R}$,

$$\frac{dZ_s}{Z_s} = b ds + a dY_s, \quad 0 \leq s \leq T, \quad Z_0 > 0, \quad (4.3)$$

and the contingent claim $B = B(Z_T)$ is a function of the terminal value Z_T alone. Like in (2.1), Y is a Brownian motion having correlation ρ with W . [18] contains a slightly more general diffusion setting where $a_s = a(Z_s, s)$ and $b_s = b(Z_s, s)$ may depend on the current level of Z and on time. Monoyios [16] studies a similar model where σ and $\lambda = \frac{\mu}{\sigma}$ are not constant, but σ equals Z and $\lambda_s = \lambda(Z_s)$ is a function of the current level of Z . Grasselli and Hurd [6] and Stoikov and Zariphopoulou [20] consider claims which depend not only on Z_T , but also in a certain way on the trajectory of Z . In contrast to all the above Markovian models, Tehranchi [21] analyzes a more general situation very similar to case (I); but his approach is still restricted to a constant correlation ρ .

To the best of our knowledge, the only article where ρ is not constant is by Benth and Karlsen [2] who study a Markovian setting with $\rho = \rho(Z_s)$ depending on the present level of the nontraded asset Z . They show that the minimal entropy martingale measure can be expressed in terms of the solution of a semilinear PDE for which they prove existence and uniqueness of a classical solution. However, they have no claim B and they also do not derive any general explicit formulas.

Remark. All Markovian models above with *constant* $\mu, \sigma, r, \rho, a, b$ satisfy the measurability conditions for both cases (I) and (II). It is therefore somewhat arbitrary whether one views them as stochastic volatility or rather as executive stock option models. (Indeed, only our general model makes this precise distinction really possible.) The subsequent generalizations in [21], [16], [6] and [20] all head towards our case (I), whereas models from case (II) have not yet been studied for nondeterministic λ or ρ . In that sense, it seems fair to say that our formulation with a clear distinction between cases (I) and (II) represents a significant generalization of previously considered models. \diamond

We now recall and comment on how explicit formulas for the indifference value b are derived in the literature. As in Section 2.1, one usually first derives an expression for the value V^B and then obtains a formula for b via (2.7). In a Markovian model, the usual approach is to condition on the current state of the nontraded asset Z in (4.3), i.e., to write

$$V_t^B(x_t) = v(x_t, z_t, t) := \operatorname{ess\,sup}_{\pi \in \mathcal{A}_t} E[U(X_T^{x_t, \pi} - B(Z_T)) | X_t^{x_t, \pi} = x_t, Z_t = z_t].$$

Henderson [7], Henderson and Hobson [9, 10], and later Musiela and Zariphopoulou [18] first write the Hamilton-Jacobi-Bellman nonlinear PDE for the value function v . Exploiting the scaling properties of the exponential utility function U , they try an ansatz of the form

$$v(x, z, t) = U(x)F(z, t),$$

which results in a *nonlinear* PDE for F . A clever power transformation,

$$F(z, t) = f(z, t)^{\frac{1}{1-\rho^2}}, \tag{4.4}$$

reduces this to a *linear* and solvable PDE for f . This yields an explicit formula for v and thus also for b via (2.7).

The idea to convert a nonlinear to a linear PDE by a power transformation was introduced by Zariphopoulou [22] for optimal portfolio management

problems with nontraded assets when the utility is of the separable CRRA type: the payoff $B(Z_T)$ of the claim is multiplied by a power of the investor's final portfolio value $X_T^{x_t, \pi}$, i.e., $\tilde{U}(X_T^{x_t, \pi}, B(Z_T)) = B(Z_T)|X_T^{x_t, \pi}|^\gamma/\gamma$ with $0 \neq \gamma < 1$. The application of the power transformation (4.4) to exponential utility indifference valuation appeared first in Henderson [7], Henderson and Hobson [9, 10], and later in Musiela and Zariphopoulou [18]. The exponent $\delta := \frac{1}{1-\rho^2}$ from (4.4) is called *distortion power*, a terminology due to Zariphopoulou [22], and the approach is also known as *distortion method*. Henderson [7] and Henderson and Hobson [9, 10] also derive an approximation (for a small number of claims) of the power utility indifference value, which they compare with the exponential indifference value. Henderson [8] examines the latter criterion and incentives for executive stock options in the Markovian model of [7, 9, 10]. Monoyios [16] derives a representation of the optimal measure for the dual problem by combining the distortion method with general duality results. He further considers the optimization problem under power utility, but without random endowment. Grasselli and Hurd [6] and Stoikov and Zariphopoulou [20] present explicit formulas for the exponential utility indifference value of a path-dependent claim on the volatility. But as already mentioned, all these approaches work only in a Markovian model and if the instantaneous correlation ρ between W and Y is constant.

In an alternative approach, Tehranchi [21] obtains an explicit expression for $V_t^B(x_t)$ in (2.6) with $t = 0$. He first proves a Hölder-type inequality which he then applies to determine $V_0^B(x_0)$, and this also yields an explicit formula for the indifference value at time 0. His method has the advantage that it needs no Markovian assumption and can treat general (bounded) \mathcal{Y}_T -measurable claims; but it is still restricted to situations with constant correlation. The distortion power $\delta = \frac{1}{1-\rho^2}$ from (4.4) arises here as an exponent in the Hölder-type inequality.

In all the above approaches, δ plays an important role, and it is crucial that it is deterministic and constant in time. We also use in (3.5) a power transformation with a power δ which must be constant, whereas $\delta = \frac{1}{1-\rho^2}$ in the above methods depends on ρ . This explains why we use two different powers in our proof of Theorem 1: $\bar{\delta}_t$ gives in Proposition 1 an upper bound for $\hat{V}_t^{\hat{B}}$, and $\underline{\delta}_t$ a lower bound in Proposition 2. The deeper reason why we can deal with a random correlation ρ is then a monotonicity property, as will be explained in Section 5.

Remarks. 1) We can in Theorem 2 replace \hat{P} by the restriction Q of \hat{P} to \mathcal{Y}_T , because \hat{B} and $\hat{0}$ are \mathcal{Y}_T -measurable; so for almost all $\omega \in \Omega$, we have

$$b_t(\omega) = \frac{e^{-\int_t^T r(s) ds}}{\gamma} \log \frac{(E_Q[\exp(\hat{B})^{1/\delta} | \mathcal{Y}_t](\omega))^\delta}{(E_Q[\exp(\hat{0})^{1/\delta'} | \mathcal{Y}_t](\omega))^{\delta'}} \Big|_{\delta'=\delta_t^{\hat{0}}(\omega), \delta=\delta_t^{\hat{B}}(\omega)}.$$

Since ρ and λ are \mathbb{Y} -predictable in case (I), (2.4) and (2.1) yield explicitly

$$\frac{dQ}{dP} := E_P \left[\frac{d\hat{P}}{dP} \Big| \mathcal{Y}_T \right] = \mathcal{E} \left(- \int \rho \lambda dY \right)_{0,T}. \quad (4.5)$$

This formula is used by Tehranchi [21] to *define* Q in his setting with constant ρ . Similarly, we could in Theorem 3 replace \hat{P} by the restriction \hat{Q} of \hat{P} to $\hat{\mathcal{Y}}_T$. However, this is less useful because \hat{Q} , unlike Q , has in general no explicit form.

2) Apart from exponential utility, Tehranchi [21] also explicitly determines V_0^B for constant ρ when the investor's utility is of the same separable form as in Zariphopoulou [22], i.e., $\tilde{U}(X_T^{x_t, \pi}, B) = B |X_T^{x_t, \pi}|^\gamma / \gamma$ with $0 \neq \gamma < 1$, or $\tilde{U}(X_T^{x_t, \pi}, B) = B \log X_T^{x_t, \pi}$. Those results could be extended with our techniques as for exponential utility to all times t and to random ρ . But we give no details since this provides no essential new insights and, above all, does not help for finding an indifference value, because the above utilities are not of the form $U(X_T^{x_t, \pi} + B)$ required for a natural formulation.

3) The original motivation for this paper was that we were intrigued by the elegantly simple and yet general approach of Tehranchi [21]. Along the way, we then discovered that not all arguments in [21] seem completely rigorous; the proof there of Lemma 4.2 is not quite clear (measurability of integrands?), and we see no argument why the portfolios constructed in Propositions 3.3–3.5 satisfy the integrability requirements to lie in the respective classes \mathcal{A} of admissible strategies. Moreover, the proofs of these propositions also contain an incorrect statement; in general, a Brownian motion W and a process of the form $W + \int \lambda ds$ do not generate the same filtration or σ -field, even if λ is predictable with respect to the filtration generated by W . A counterexample is given by Dubins et al. [4]. Despite all this, the final results in [21] are essentially correct; one way to circumvent the last problem is contained in the proof of our Theorem 2. \diamond

4.3 Proofs of Theorems 2 and 3

We first need the following general result which says that the class (D) property behaves under a change to an equivalent probability measure in the same

way as martingales. This is very intuitive and probably folklore, but we have not found it anywhere.

Lemma 2. *Denote by Z' the P -density process of a probability measure P' equivalent to P , i.e., $Z'_s := E_P\left[\frac{dP'}{dP}\bigg|\mathcal{G}_s\right]$, $s \in [0, T]$. A \mathbb{G} -adapted RCLL process Λ is of P' -class (D) if and only if $\Lambda Z'$ is of P -class (D) .*

Proof. By symmetry and Bayes' formula it is enough to prove the “only if” part. Take a \mathbb{G} -adapted RCLL process Λ of P' -class (D) and fix $\epsilon > 0$. We want to find $K > 0$ with $\sup_\tau E_P\left[|\Lambda_\tau| Z'_\tau I_{\{|\Lambda_\tau| Z'_\tau > K\}}\right] \leq \epsilon$, where the sup is taken over all \mathbb{G} -stopping times τ . Using that $dP' = Z'_\tau dP$ on \mathcal{G}_τ gives

$$E_P\left[|\Lambda_\tau| Z'_\tau I_{\{|\Lambda_\tau| Z'_\tau > K\}}\right] = E_{P'}\left[|\Lambda_\tau| I_{\{|\Lambda_\tau| Z'_\tau > K\}}\right].$$

Since Λ is of P' -class (D) , $m := 1 \vee \sup_\tau E_{P'}\left[|\Lambda_\tau|\right]$ is finite and there exists $d_1 > 0$, which does not depend on τ , such that

$$A \in \mathcal{G}_T \text{ with } P'[A] \leq d_1, \tau \text{ } \mathbb{G}\text{-stopping time} \implies E_{P'}\left[|\Lambda_\tau| I_A\right] \leq \epsilon. \quad (4.6)$$

Because $P' \ll P$ by assumption, there exists $d_2 > 0$ such that

$$A \in \mathcal{G}_T \text{ with } P[A] \leq d_2 \implies P'[A] \leq d_1. \quad (4.7)$$

Set $K := m/d_2$ and use Markov's inequality to obtain

$$P\left[|\Lambda_\tau| Z'_\tau > K\right] \leq \frac{1}{K} E_P\left[|\Lambda_\tau| Z'_\tau\right] = \frac{1}{K} E_{P'}\left[|\Lambda_\tau|\right] \leq \frac{m}{K} = d_2$$

for any \mathbb{G} -stopping time τ . Now (4.6) and (4.7) yield $E_{P'}\left[|\Lambda_\tau| I_{\{|\Lambda_\tau| Z'_\tau > K\}}\right] \leq \epsilon$, uniformly over τ , which ends the proof. \square

Now we can prove Theorems 2 and 3 by applying Theorem 1.

Proof of Theorem 2. (4.2) follows directly from (2.7) and (4.1). To prove (4.1), we apply Theorem 1 with $\hat{B} := \bar{B} = \gamma B - \frac{1}{2} \int_t^T \lambda_s^2 ds$ and $\hat{\mathbb{H}} := \mathbb{Y}$. Comparing (3.1) with (2.11) shows that it only remains to argue that

- i) \hat{Y} has the representation property in \mathbb{Y} , and
- ii) $\pi \in \mathcal{A}_t \iff \gamma \pi \sigma - \lambda \in \hat{\mathcal{A}}_t$.

The latter follows directly from Lemma 1 which yields that $\exp\left(-\int \gamma \pi \sigma d\hat{W}\right)$ is of P -class (D) if and only if $\exp\left(-\int (\gamma \pi \sigma - \lambda) d\hat{W}\right)$ is of \hat{P} -class (D) , because $\int \lambda^2 ds$ is bounded. Property i) is deduced from Itô's representation theorem in the form of Lemma 1.6.7 of Karatzas and Shreve [13]. In more detail, consider the restriction Q of \hat{P} to \mathcal{Y}_T , given as in (4.5) by

$$\frac{dQ}{dP} = E_P\left[\frac{d\hat{P}}{dP}\bigg|\mathcal{Y}_T\right] = \mathcal{E}\left(-\int \rho \lambda dY\right)_{0,T}$$

because λ and ρ are \mathbb{Y} -predictable. Note that this uses the assumptions of case (I). \hat{Y} is also a (\mathbb{Y}, Q) -Brownian motion, and Lemma 1.6.7 in [13] now yields that any (\mathbb{Y}, Q) -martingale L is of the form $L = L_0 + \int \zeta d\hat{Y}$ for a \mathbb{Y} -predictable ζ with $\int_0^T \zeta_s^2 ds < \infty$ a.s. This crucially needs that $\rho\lambda$ is \mathbb{Y} -predictable, to ensure that $\hat{Y} = Y + \int \rho\lambda ds$ from (2.3) is \mathbb{Y} -adapted. \square

Proof of Theorem 3. As in the proof of Theorem 2, we apply Theorem 1, but now with $\hat{B} := \gamma B$ and $\hat{\mathbb{H}} := \hat{\mathbb{Y}}$. Of course, the $(\hat{\mathbb{H}}, \hat{P})$ -Brownian motion \hat{Y} then has the representation property in $\hat{\mathbb{H}}$. To get rid of the term $\frac{1}{2} \int_t^T \lambda_s^2 ds$ in \bar{B} in (2.11), we use again Itô's representation theorem as in Lemma 1.6.7 of [13] and obtain an \mathbb{F} -predictable process $\eta = (\eta_s)_{t \leq s \leq T}$ with

$$\frac{1}{2} \int_t^T \lambda_s^2 ds = \frac{1}{2} E_{\hat{P}} \left[\int_t^T \lambda_s^2 ds \middle| \mathcal{F}_t \right] + \int_t^T \eta_s d\hat{W}_s \quad \text{and} \quad E_{\hat{P}} \left[\int_t^T \eta_s^2 ds \right] < \infty.$$

Here we use that λ is \mathbb{F} -predictable in case (II), where we recall that $\mathbb{F} = \mathbb{F}^W$. Finally, comparison of (2.11) and (3.1) with $\hat{B} = \gamma B$ shows that it remains to prove that $\pi \in \mathcal{A}_t$ if and only if $\gamma\pi\sigma - \lambda + \eta \in \hat{\mathcal{A}}_t$. But this follows as in the proof of Theorem 2 from Lemma 1, using that $\int \eta d\hat{W}$ is like $\int \lambda^2 ds$ uniformly bounded. \square

5 On the monotonicity in the correlation

In this section, we explain both intuitively and mathematically why we can obtain results even for a random correlation ρ .

For a constant correlation ρ , the abstract optimization problem (3.1) has by Theorem 1 (or from Tehranchi [21] for $t = 0$) an explicit value, namely (3.3) with $\delta = \frac{1}{1-\rho^2}$. This expression is continuous in ρ and increasing in $|\rho|$, for fixed \hat{P} , and the intuition is as follows. The endowment \hat{B} is driven by \hat{Y} , whereas hedging can only be done in \hat{W} which is imperfectly correlated with \hat{Y} . If the correlation between \hat{W} and \hat{Y} is increased, better hedging is possible; so the value of the optimization problem (3.1) decreases. (Note that (3.1) gives us *minus* the maximal expected utility.)

If we can extend the above monotonicity to a general correlation, it is clear why we can get the explicit structure in Theorem 1. Indeed, if ρ is random but lies between two bounds, the corresponding optimization problem must by monotonicity have an explicit expression with the same basic structure — and of course the interpolating distortion power may now be random and depend on \hat{B} .

Let us now introduce more precise notations by writing (3.1) as

$$\hat{V}_t^{\hat{B}}(\rho', \hat{P}) := - \operatorname{ess\,inf}_{\hat{\pi} \in \hat{\mathcal{A}}_t(\rho')} E_{\hat{P}} \left[\exp \left(- \int_t^T \hat{\pi}_s d\hat{W}_s(\rho') + \hat{B} \right) \middle| \mathcal{G}_t \right]$$

for a \mathbb{G} -predictable process ρ' denoting the instantaneous correlation between the (\mathbb{G}, \hat{P}) -Brownian motions $\hat{W}(\rho')$ and \hat{Y} ; the set $\hat{\mathcal{A}}_t(\rho')$ depends on ρ' through the \hat{P} -class (D) condition on $(\exp(-\int_t^s \hat{\pi}_y d\hat{W}_y(\rho')))_{t \leq s \leq T}$. Note that if we change ρ' , only $\hat{W}(\rho')$ and all expressions depending on it will change. This is reasonable; clearly \hat{B} and $\hat{\mathbb{H}}$ should not be affected.

The above intuitive argument now says that if we keep \hat{P} fixed and vary ρ' , we get a monotonicity, which is made precise in the following result.

Proposition 3. *Let \hat{P} be fixed and suppose that ρ' and ρ'' are \mathbb{G} -predictable processes such that $|\rho'| \leq c_1 \leq |\rho''| \leq c_2 < 1$ on $[t, T] \times \Omega$ for some constants c_1 and c_2 . Then $\hat{V}_t^{\hat{B}}(\rho', \hat{P}) \leq \hat{V}_t^{\hat{B}}(\rho'', \hat{P})$ a.s.*

Proof. This follows from applying twice Theorem 1, once for $\hat{V}_t^{\hat{B}}(\rho', \hat{P})$ and once for $\hat{V}_t^{\hat{B}}(\rho'', \hat{P})$, and then using Jensen's inequality. \square

Remark. Proposition 3 says that $\rho' \mapsto \hat{V}_t^{\hat{B}}(\rho', \hat{P})$ is monotonic for correlation processes ρ', ρ'' that can be separated by a constant, uniformly in s and ω . We do not know if the weaker assumption $|\rho'| \leq |\rho''|$ on $[t, T] \times \Omega$ is also sufficient to prove the same conclusion. \diamond

The above intuition and Proposition 3 make it tempting to think that also the value $V_t^B(0)$ in (2.9) is monotonic in $|\rho|$. However, this is not true in general; we give a counterexample in the next paragraph. The crucial point is that \hat{P} itself depends on ρ because W does; this can be seen from (2.1) and (2.4). So the abstract optimization problem (3.1) has the structure $-\hat{V}_t^{\hat{B}}(\rho', \hat{P}(\rho))|_{\rho'=\rho}$, and proving as in Proposition 3 that $\rho' \mapsto \hat{V}_t^{\hat{B}}(\rho', \hat{P}(\rho))$ is monotonic for fixed ρ need not imply that $\rho \mapsto \hat{V}_t^{\hat{B}}(\rho, \hat{P}(\rho))$ is monotonic.

We now show by a counterexample that $\rho \mapsto \hat{V}_t^{\hat{B}}(\rho, \hat{P}(\rho))$ and thus $\rho \mapsto V_t^B(0; \rho) := V_t^B(0)$ from (2.9) are indeed not monotonic in general. In view of Proposition 3, this can only fail in the non-martingale case $\lambda \neq 0$, since otherwise $\hat{P}(\rho) = P$ does not depend on ρ . We take ρ and λ both constant, $t = 0$, and set $\hat{\mathbb{H}} = \mathbb{Y}$ as in case (I). Then Theorem 1 implies

$$\begin{aligned} \hat{V}_0^{\hat{B}}(\rho, \hat{P}(\rho)) &= - \left(E_{\hat{P}(\rho)} \left[\exp(\hat{B})^{1-\rho^2} \right] \right)^{\frac{1}{1-\rho^2}} \\ &= - \left(E_P \left[\exp \left(\hat{B}(1-\rho^2) - \lambda \rho Y_T - \frac{T \lambda^2 \rho^2}{2} \right) \right] \right)^{\frac{1}{1-\rho^2}}, \end{aligned}$$

where we have conditioned on \mathcal{Y}_T under P and used that \hat{B} is \mathcal{Y}_T -measurable. For $\hat{B} = -Y^n := ((-Y_T) \wedge n) \vee (-n)$, $n \in \mathbb{N}$, dominated convergence and an easy calculation yield

$$\begin{aligned} \lim_{n \rightarrow \infty} \hat{V}_0^{-Y^n}(\rho, \hat{P}(\rho)) &= - \left(E_P \left[\exp \left(-Y_T(1 - \rho^2) - \lambda \rho Y_T - \frac{T\lambda^2 \rho^2}{2} \right) \right] \right)^{\frac{1}{1-\rho^2}} \\ &= - \exp \left((-\rho^2 + 2\lambda\rho + 1) \frac{T}{2} \right) =: g(\rho). \end{aligned} \quad (5.1)$$

The mapping $\rho \mapsto g(\rho)$ is clearly not monotonic in $|\rho|$ except in the martingale case $\lambda = 0$. Because of (5.1), the mapping $\rho \mapsto \hat{V}_0^{-Y^n}(\rho, \hat{P}(\rho))$ for n big enough is not monotonic in $|\rho|$ either. If we now consider case (I) with $\gamma = 1$ and $B = -Y^n$, the proof of Theorem 2 implies that

$$V_0^{-Y^n}(0; \rho) := V_0^{-Y^n}(0) = e^{-\lambda^2 T/2} \hat{V}_0^{-Y^n}(\rho, \hat{P}(\rho))$$

so that $\rho \mapsto V_0^{-Y^n}(0; \rho)$ for n big enough is not monotonic in $|\rho|$ either. This completes our example.

Remark. One can directly show that $\hat{V}_0^{-Y_T}(\rho, \hat{P}(\rho)) = g(\rho)$ if one adapts the definition of $\hat{\mathcal{A}}_t(\rho)$. For such an *unbounded* \hat{B} , one stipulates that $(\exp(-\int_t^s \hat{\pi}_y d\hat{W}_y(\rho) + \hat{B}))_{t \leq s \leq T}$ instead of $(\exp(-\int_t^s \hat{\pi}_y d\hat{W}_y(\rho)))_{t \leq s \leq T}$ is of \hat{P} -class (D) . The point is then that one can for this example explicitly determine $L = (\rho^2 - 1)(Y + \lambda \rho s)$ defined in (3.5), and L is obviously a $BMO(\mathbb{G}, \hat{P}(\rho))$ -martingale. \diamond

The above example shows that V^B is in general not monotonic in $|\rho|$. We now explain the intuition for this. In the martingale case $\lambda = 0$, the value $V_0^{-Y_T}(0; \rho) = \hat{V}_0^{-Y_T}(\rho, \hat{P}(\rho)) = -\exp((1 - \rho^2)T/2)$ is clearly monotonic in $|\rho|$, and we have already seen why: Higher correlation permits better hedging, and so the investor runs less risk and has a higher expected utility. For the semimartingale case $\lambda \neq 0$, this effect is still there, but now also interacts with the correlation. Consider for instance the case where $\lambda > 0$ and $\rho > 0$. The optimal strategy $\hat{\pi}^*$ for $\hat{V}_0^0(\rho, \hat{P}(\rho))$ is zero and hence the optimal strategy for $V_0^0(0; \rho)$ is $\pi^* = \frac{\lambda}{\gamma\sigma}$; compare the proof of Theorem 2. This strategy π^* makes a positive investment in the stock S . Adding $-B = Y_T$ leads to a total position with a higher risk, since the correlation ρ between Y and S is positive. To counteract this exposure, the investor will reduce his position in S and smooth out his terminal wealth. Hence he accepts in average a lower return on his portfolio in S to reduce the risk of his total position. So an increase in correlation yields a higher risk exposure for a fixed

strategy; this is compensated by more conservative (smaller) investment in S , leading to a lower return and hence a decrease of the value $V_0^{-Y_T}(0; \rho)$. In total, $\rho \mapsto V_0^{-Y_T}(0; \rho)$ can therefore become decreasing in $|\rho|$ —despite the better hedging possibility. The above argument explains why this *can* happen, and (5.1) shows that it *does* happen for $0 < \rho < \lambda$.

Remark. In a Markovian framework with constant ρ and λ , the result of Proposition 3 has already been established by Henderson [8] who shows that the indifference value b (or, equivalently in that setting, V^B) is increasing in $|\rho|$. Henderson’s analysis at first sight seems to contradict our non-monotonic example, and closer inspection reveals that it crucially depends on fixing some parameter called δ in [8] while varying ρ . But this exactly corresponds to our fixing \hat{P} in Proposition 3 while varying ρ , and it has in both cases a very natural financial interpretation. In fact, the standard viewpoint in financial theory is that the instantaneous Sharpe ratio $\frac{a}{b}$ of the nontraded asset Z in (4.3) is not fixed exogenously, but related with λ via the correlation ρ . This tacit assumption is usually not spelt out explicitly in the finance literature, and the point of our counterexample is to illustrate that monotonicity may fail in its absence. \diamond

6 The multidimensional case

In this section, we extend our main results to the case of more than two Brownian motions. Since most arguments are straightforward generalizations, we only sketch the main differences.

The probabilistic framework consists of an n -dimensional (\mathbb{G}, P) -Brownian motion Y and an m -dimensional (\mathbb{G}, P) -Brownian motion W , each having P -independent components. Instantaneous correlations are now given by a matrix $R = (\rho^{ij})_{\substack{i=1,\dots,n, \\ j=1,\dots,m}}$, with $\rho_s^{ij} := \frac{d\langle Y^i, W^j \rangle_s}{ds}$, and we choose R to be \mathbb{G} -predictable. It can be shown that the symmetric positive semidefinite matrix RR^\top has nonnegative eigenvalues which are all at most 1. We assume that all eigenvalues are bounded away from one uniformly on $\Omega \times [t, T]$, i.e.,

there exists $c < 1$ such that $\text{spectral radius}(RR^\top) \leq c$ a.e. on $\Omega \times [t, T]$.

Recall that $t \in [0, T]$ is fixed. There are m traded risky assets $S = (S_s^j)_{j=1,\dots,m}$ with dynamics

$$dS_s^j = S_s^j \mu_s^j ds + \sum_{k=1}^m S_s^j \sigma_s^{jk} dW_s^k, \quad 0 \leq s \leq T, \quad S_0^j > 0, \quad j = 1, \dots, m;$$

the drift vector $\mu = (\mu_s^j)_{j=1,\dots,m}$ and the volatility matrix $\sigma = (\sigma_s^{jk})_{j,k=1,\dots,m}$ are \mathbb{G} -predictable. We assume that σ is invertible, $\lambda := \sigma^{-1}\mu$ is bounded uniformly (in s and ω) and that there exists a constant C such that

$$C\beta^\top\beta \geq \beta^\top\sigma\sigma^\top\beta \geq \frac{1}{C}\beta^\top\beta \quad \text{on } \Omega \times [0, T] \text{ for all } \beta \in \mathbb{R}^m.$$

(In other words, σ is uniformly both bounded and elliptic.) The processes

$$\hat{W} := W + \int \lambda \, ds \quad \text{and} \quad \hat{Y} := Y + \int R\lambda \, ds$$

are Brownian motions under the minimal martingale measure \hat{P} given by $\frac{d\hat{P}}{dP} := \mathcal{E}(-\int \lambda \, dW)_{0,T}$. All other definitions and model assumptions of Sections 2.1 and 3.1 can be easily translated to this setting and we do not detail this. The multidimensional version of Theorem 1 then reads as follows.

Theorem 4. *Under the above assumptions, denote by $\min\text{EV}(R_s R_s^\top)$ and $\max\text{EV}(R_s R_s^\top) = \text{spectral radius}(R_s R_s^\top)$ the smallest and largest eigenvalues of $R_s R_s^\top$, and define $\underline{\delta}_t$ and $\bar{\delta}_t$ by*

$$\underline{\delta}_t := \inf_{s \in [t, T]} \frac{1}{\|1 - \min\text{EV}(R_s R_s^\top)\|_{L^\infty}}, \quad \bar{\delta}_t := \sup_{s \in [t, T]} \left\| \frac{1}{1 - \max\text{EV}(R_s R_s^\top)} \right\|_{L^\infty}. \quad (6.1)$$

Then there exists a \mathcal{G}_t -measurable random variable $\delta_t^{\hat{B}}$ with values in $[\underline{\delta}_t, \bar{\delta}_t]$ such that

$$-\hat{V}_t^{\hat{B}}(\omega) = \left(E_{\hat{P}} \left[\exp(\hat{B})^{\frac{1}{\delta}} \middle| \mathcal{H}_t \right] (\omega) \right)^\delta \Big|_{\delta = \delta_t^{\hat{B}}(\omega)} \quad (6.2)$$

for almost all $\omega \in \Omega$.

Outline of the proof. This goes similarly to Theorem 1 via analogues of Propositions 1 and 2, and we only point out where significant changes occur. The analogue to (3.8) is, for $t \leq s \leq T$,

$$Z_s^{(\hat{\pi})} := c_t^\delta M_{t,s}^{(\hat{\pi})} \exp \left(\frac{1}{2} \int_t^s \left(|\hat{\pi}_y - \delta R_y^\top \zeta_y|^2 + \delta \zeta_y^\top (\delta(I - R_y R_y^\top) - I) \zeta_y \right) dy \right) \quad (6.3)$$

with $M^{(\hat{\pi})} := \mathcal{E}(\int \delta \zeta \, d\hat{Y} - \int \hat{\pi} \, d\hat{W})$ like in (3.7) and $c_t := E_{\hat{P}}[\exp(\hat{B}/\delta) | \mathcal{H}_t]$ like in (3.5). In (6.3), I denotes the $(n \times n)$ -identity matrix. As in the proof of Proposition 1, the key point is that the integrand in (6.3) with $\delta := \bar{\delta}_t$ is non-negative for every $\hat{\pi} \in \hat{\mathcal{A}}_t$. To see this, one must prove that $\bar{\delta}_t(I - R_y R_y^\top) - I$

is positive semidefinite or, equivalently, that all its eigenvalues are nonnegative. But if α is such an eigenvalue, then $1 - (\alpha + 1)/\bar{\delta}_t$ is an eigenvalue of $R_y R_y^\top$; this implies $1 - (\alpha + 1)/\bar{\delta}_t \leq 1 - 1/\bar{\delta}_t$ by (6.1), and hence $\alpha \geq 0$.

For the analogue of Proposition 2, one defines, for $t \leq s \leq T$,

$$\hat{\pi}_s^* := \underline{\delta}_t R_s^\top \zeta_s + \sqrt{\underline{\delta}_t \zeta_s^\top (I - \underline{\delta}_t (I - R_s R_s^\top)) \zeta_s} (1, 0, \dots, 0)^\top, \quad (6.4)$$

where ζ is now defined in (3.5), (3.6) with $\delta := \underline{\delta}_t$ and $(1, 0, \dots, 0) \in \mathbb{R}^m$. Using (6.1) and a similar reasoning as above, one sees that the expression under the square root in (6.4) is nonnegative, and (6.3) simplifies for $\hat{\pi} = \hat{\pi}^*$ and $\delta = \underline{\delta}_t$ like (3.12) to $Z_s^{(\hat{\pi}^*)} = c_t^{\underline{\delta}_t} M_{t,s}^{(\hat{\pi}^*)}$, $t \leq s \leq T$. As in the proof of Proposition 2, one can show that $M^{(\hat{\pi}^*)}$ is a (\mathbb{G}, \hat{P}) -martingale and that $\hat{\pi}^* \in \hat{\mathcal{A}}_t$.

Finally, (6.2) is proved from the analogues of Propositions 1 and 2 similarly as in the two-dimensional case. This concludes the proof outline. \square

Using Theorem 4, one can of course obtain results like Theorems 2 and 3 also in the multidimensional case. We refrain from giving details because the procedure goes essentially along the same lines as in Section 4. However, we emphasize that it is important to assume that the rank of the volatility matrix σ equals the dimension m of W . (In particular, we typically want at least m risky assets.) This condition, implied by the assumption that σ is invertible, is required to show that the sets $\hat{\mathcal{A}}_t$ and \mathcal{A}_t fit together; compare ii) in the proofs of Theorems 2 and 3.

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