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# Some new BSDE results for an infinite-horizon stochastic control problem

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**Abstract.** We study in a continuous filtration a quadratic BSDE with an unbounded generator and an infinite time horizon. This equation comes from a stochastic control problem in the context of robust utility maximisation. We prove existence and uniqueness, in a suitable class, of a solution to the BSDE, and we show that the BSDE characterises the dynamic value process of the stochastic control problem.

**Keywords:** backward stochastic differential equations, infinite horizon, quadratic BSDE, unbounded solution, stochastic control, robust utility maximisation

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## 0. Introduction

This paper studies a stochastic control problem arising in the context of robust utility maximisation, and proves new results via BSDE techniques. A particular feature is that the problem is formulated and solved for an infinite horizon and that we also obtain new results on a certain infinite-horizon BSDE with quadratic generator.

In loose terms, the basic problem we should like to tackle has the form

$$\text{find } \sup_{\pi} \inf_{Q} \mathbf{U}(\pi, Q),$$

where $\mathbf{U}$ is some utility functional, $\pi$ runs through a set of investment and consumption strategies, and $Q$ through a set of models given by probability measures. In a first step, we focus only on the inner minimisation problem;

thus we think of $\pi$ as being fixed and look for a worst-case model $Q$. The functional $\mathbf{U}(\pi, Q)$ we consider has the form

$$\mathbf{U}(\pi, Q) = E_Q\big[\mathcal{U}^\delta_{0,\infty} + \beta\mathcal{R}^\delta_{0,\infty}(Q)\big],$$

where $\mathcal{U}^\delta_{0,\infty} = \alpha \int_0^\infty S^\delta_s U_s\,ds$ stands for a discounted utility term (whose dependence on the fixed $\pi$ is suppressed) and $\mathcal{R}^\delta_{0,\infty}(Q) = \int_0^\infty S^\delta_s \log Z^Q_s\,ds$ is an entropic penalty term. A precise formulation is given later.

The finite-horizon version of this problem has been studied in Bordigoni/Matoussi/Schweizer [2], who have characterised the dynamic value process $V = V^{(T)}$ of the resulting stochastic control problem as the unique solution of the BSDE

$$dY_t = (\delta_t Y_t - \alpha U_t)\,dt + \frac{1}{2\beta}\,d\langle M\rangle_t + dM_t \tag{0.1a}$$

with a final condition at time $T$. We generalise these results here to an infinite-horizon setting with the terminal condition

$$\lim_{t\to\infty} Y_t = 0. \tag{0.1b}$$

In an unpublished PhD thesis, G. Bordigoni has already shown that $V$ for the infinite-horizon problem is a solution of the BSDE (0.1); but uniqueness and the required integrability (and hence the characterisation of $V$ by the BSDE) remained open. We close this gap here.

In contrast to Bordigoni [1], our approach and main results here are on the side of BSDE theory. Equation (0.1) is a quadratic BSDE in a continuous filtration and has an unbounded generator (due to the presence of $U$) and an infinite horizon. For the finite-horizon case, the classical results of Kobylanski [10] on existence and uniqueness of a bounded solution for quadratic BSDEs in a Brownian filtration have been extended to unbounded solutions in the Brownian setting by Briand/Hu [5, 6], and to bounded solutions in a continuous filtration by Morlais [12]. In infinite-horizon settings, Briand/Hu [4] and later Royer [13] have studied bounded solutions for BSDEs with a Lipschitz generator, and Briand/Confortola [3] have extended these results to bounded solutions for a quadratic generator. The methods in all these papers rely on Girsanov techniques.

Our approach here is quite different. To prove existence of a solution to (0.1), we have adapted the localisation method from Briand/Hu [5], while for uniqueness, we have applied the $\theta$-difference method from Briand/Hu [6]. Both these techniques have so far only been used in finite-horizon settings.

Finally, let us also mention two closely related papers from the finance and economics literature. Schroder/Skiadas [14] study the same BSDE as we do and obtain existence and uniqueness of unbounded solutions, but in a Brownian filtration and with a finite time horizon. Hansen/Sargent/Turmuhambetova/Williams [9] study robustness aspects for infinite-horizon utility maximisation problems; their main ideas and problems are similar to ours, but the approach is rather heuristic, using Hamilton–Jacobi–Bellman equations and formal manipulations in a Markovian setting. For a more detailed discussion of additional references to the literature, we refer to Section 6 of Bordigoni/Matoussi/Schweizer [2].

The paper is structured as follows. After some preliminaries and notations in Section 1, we study in Section 2 the BSDE on a finite horizon. This serves as preparation for the infinite-horizon BSDE studied in Section 3 and gives to that end fairly precise estimates for the solution $Y$. Section 3 establishes existence and uniqueness of a solution $(Y, M)$ for the infinite-horizon BSDE (0.1) and gives a sufficient condition for $\mathcal{E}\big(-\frac{1}{\beta}M\big)$ to be a martingale. In Section 4, we prove by general arguments as in Bordigoni/Matoussi/Schweizer [2] and Bordigoni [1] the existence of a solution to our stochastic control problem and show that its value process $V$ satisfies the boundary condition $\lim\limits_{t\to\infty} V_t = 0$. Finally, Section 5 uses the BSDE results to characterise $V$ as the unique solution, in a suitable space, for the BSDE (0.1), and in particular establishes that $V$ has the required good integrability properties.

## 1. Preliminaries and overview

In this section, we introduce all required notations, the basic BSDEs and the basic optimisation problems. We start with a probability space $(\Omega, \mathcal{F}, P)$ and a time horizon $T \in (0, \infty]$. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions of right-continuity and $P$-completeness, $\mathcal{F}_0$ is $P$-trivial, and we set $\mathcal{F}_\infty := \bigvee\limits_{t\geq 0} \mathcal{F}_t$. The basic ingredients for our optimisation problems are

- parameters $\alpha, \alpha' \in [0, \infty)$ and $\beta \in (0, \infty)$;
- progressively measurable processes $\delta = (\delta_t)_{t\geq 0}$ and $U = (U_t)_{t\geq 0}$;
- an $\mathcal{F}_T$-measurable random variable $U'_T$, with $U'_\infty := 0$ for $T = \infty$.

With these, we can formulate the BSDEs studied here. On the one hand, for a finite horizon $T < \infty$, we introduce the BSDE

$$dY_t = (\delta_t Y_t - \alpha U_t)\,dt + \frac{1}{2\beta}\,d\langle M\rangle_t + dM_t, \qquad Y_T = \alpha' U'_T. \tag{1.1}$$

On the other hand, for an infinite horizon $T = \infty$, the BSDE is

$$dY_t = (\delta_t Y_t - \alpha U_t)\,dt + \frac{1}{2\beta}\,d\langle M\rangle_t + dM_t, \qquad \lim_{t\to\infty} Y_t = 0. \tag{1.2}$$

A solution of (1.1) or (1.2) is a pair $(Y, M)$ satisfying (1.1) or (1.2), respectively, where $Y$ is a $P$-semimartingale and $M$ is a locally $P$-square-integrable local $P$-martingale null at 0.

For the optimisation problems, we first define the discounting process

$$S_t^\delta := \exp\Big(-\int_0^t \delta_s\,ds\Big), \qquad t \geq 0,$$

and for $T < \infty$ the auxiliary quantities, for $0 \leq t \leq T$,

$$\mathcal{U}_{t,T}^\delta := \alpha \int_t^T \frac{S_s^\delta}{S_t^\delta} U_s\,ds + \alpha' \frac{S_T^\delta}{S_t^\delta} U_T' \tag{1.3}$$

$$= \int_t^T \alpha e^{-\int_t^s \delta_r dr} U_s\,ds + \alpha' e^{-\int_t^T \delta_r dr} U_T',$$

$$\mathcal{R}_{t,T}^\delta(Q) := \int_t^T \delta_s \frac{S_s^\delta}{S_t^\delta} \log \frac{Z_s^Q}{Z_t^Q}\,ds + \frac{S_T^\delta}{S_t^\delta} \log \frac{Z_T^Q}{Z_t^Q}, \tag{1.4}$$

for $Q \ll P$ on $\mathcal{F}_T$ with density process $Z^Q$ on $[0,T]$. We consider the cost functional

$$c_T(Q) := \mathcal{U}_{0,T}^\delta + \beta \mathcal{R}_{0,T}^\delta(Q),$$

and the basic stochastic control problem on a finite horizon is to minimise the functional

$$Q \mapsto \Gamma_T(Q) := E_Q[c_T(Q)]$$

over a suitable class of probability measures $Q \ll P$ on $\mathcal{F}_T$. In a classical way, we can choose an adapted RCLL process $V = (V_t)_{0 \leq t \leq T}$ such that

$$V_t = \operatorname*{ess\,inf}_Q E_Q\left[\mathcal{U}_{t,T}^\delta + \beta \mathcal{R}_{t,T}^\delta(Q) \,\middle|\, \mathcal{F}_t\right], \qquad 0 \leq t \leq T.$$

For $T = \infty$, we define similarly, for $t \geq 0$,

$$\mathcal{U}_{t,\infty}^\delta := \alpha \int_t^\infty \frac{S_s^\delta}{S_t^\delta} U_s\,ds = \int_t^\infty \alpha e^{-\int_t^s \delta_r dr} U_s\,ds,$$

$$\mathcal{R}_{t,\infty}^\delta(Q) := \int_t^\infty \delta_s \frac{S_s^\delta}{S_t^\delta} \log \frac{Z_s^Q}{Z_t^Q}\,ds,$$

for $Q \overset{\text{loc}}{\ll} P$ with density process $Z^Q$. We consider the cost functional

$$c_\infty(Q) := \mathcal{U}_{0,\infty}^\delta + \beta \mathcal{R}_{0,\infty}^\delta(Q)$$

and in principle want to minimise the functional

$$Q \mapsto \Gamma_\infty(Q) := E_Q[c_\infty(Q)]$$

over a suitable class of probability measures $Q \overset{\text{loc}}{\ll} P$. (For the precise formulation, we refer to Section 4.) In a similar manner as for $T < \infty$, we can choose an adapted RCLL process $V = (V_t)_{t \geq 0}$ which is again called the dynamic value process of our stochastic control problem. Of course, to be accurate, we should distinguish in notations between $V^{(T)}$ and $V^{(\infty)}$.

Our main results in this paper are

a) existence and uniqueness results for the above BSDEs (both with finite and infinite horizon), and
b) a characterisation of the value process $V = V^{(\infty)}$ for the infinite-horizon setting as the solution of the BSDE (1.2).

## 2. The BSDE on a finite horizon

The main goal of this section is to prove the existence of a solution to the finite-horizon BSDE under weak conditions. This slightly extends previous work and above all serves as preparation for the infinite-horizon case. So we fix $T \in (0, \infty)$ and view $U$ and $\delta$ as processes on $[0, T]$.

**Hypothesis 2.1. Throughout this section, we impose the standing assumptions**

(2.1a) $\mathbb{F}$ is a *continuous filtration*, i.e. all local $(P, \mathbb{F})$-martingales are continuous.

(2.1b) $\delta \geq 0$ is uniformly bounded (by $\bar{\delta}$, say).

Precise assumptions on $U, U'_T$ will be specified below. Now we introduce the quantities

$$
\begin{aligned}
B &:= \alpha \int_0^T S_s^\delta U_s \, ds + \alpha' S_T^\delta U'_T = \mathcal{U}^\delta_{0,T}, \\
B_- &:= \alpha \int_0^T S_s^\delta U_s^- \, ds + \alpha' S_T^\delta (U'_T)^-, \\
B_+ &:= \alpha \int_0^T S_s^\delta U_s^+ \, ds + \alpha' S_T^\delta (U'_T)^+.
\end{aligned}
$$

The BSDE (1.1) under study is

$$
dY_t = (\delta_t Y_t - \alpha U_t) \, dt + \frac{1}{2\beta} \, d\langle M \rangle_t + dM_t, \qquad Y_T = \alpha' U'_T. \tag{2.2}
$$

**Definition 2.2.** A *solution* of (2.2) is a pair of processes $(Y, M)$ satisfying (2.2), where $Y$ is a $P$-semimartingale and $M$ is a locally $P$-square-integrable local $P$-martingale null at 0.

Due to the standing assumption (2.1a), $M$ and then $Y$ are continuous for any solution $(Y, M)$ of (2.2). Our proof of existence applies the localisation method originally developed in a Brownian setting by Briand/Hu [5]. To that end, we need to establish precise a priori estimates in the bounded case. Note that $1/S^\delta = S^{-\delta}$.

**Proposition 2.3 (A priori estimates).** *Suppose that* $\int_0^T |U_s|\,ds$ *and* $U'_T$ *are bounded random variables. Then there exists a unique solution* $(Y, M)$ *to (2.2) such that* $Y$ *is a bounded process. Moreover, we have for* $0 \le t \le T$ *the estimates*

$$S_t^{-\delta}\underline{Y}_t - \int_0^t \alpha e^{\int_s^t \delta_r\,dr} U_s\,ds \le Y_t \le S_t^{-\delta}\overline{Y}_t - \int_0^t \alpha e^{\int_s^t \delta_r\,dr} U_s\,ds, \tag{2.3}$$

*where*

$$\underline{Y}_t := -\beta e^{-\bar{\delta}T} \log E\left[e^{-\frac{1}{\beta}e^{\bar{\delta}T}B}\,\middle|\,\mathcal{F}_t\right] \quad \text{and} \quad \overline{Y}_t := E[B|\mathcal{F}_t]. \tag{2.4}$$

*Proof.* Existence and uniqueness of a solution with $Y$ bounded are immediate from Theorems 2.5 and 2.6 of Morlais [12]. From the definitions of $\underline{Y}$ and $\overline{Y}$ and Itô's formula, it is clear that there exist $\underline{M}$ and $\overline{M}$ such that

$$d\underline{Y}_t = \frac{1}{2\beta}e^{\bar{\delta}T}\,d\langle\underline{M}\rangle_t + d\underline{M}_t, \qquad \underline{Y}_T = B, \tag{2.5}$$

$$d\overline{Y}_t = d\overline{M}_t, \qquad \overline{Y}_T = B. \tag{2.6}$$

If we first set $Y_t^1 := S_t^\delta Y_t$ and $M_t^1 := \int_0^t S_s^\delta\,dM_s$, then the BSDE (2.2) is transformed to

$$dY_t^1 = -\alpha S_t^\delta U_t\,dt + \frac{1}{2\beta}S_t^{-\delta}\,d\langle M^1\rangle_t + dM_t^1, \qquad Y_T^1 = \alpha' S_T^\delta U'_T. \tag{2.7}$$

If we next put $Y_t^2 := Y_t^1 + \int_0^t \alpha S_s^\delta U_s\,ds$ and $M^2 := M^1$, the BSDE (2.7) becomes

$$dY_t^2 = \frac{1}{2\beta}S_t^{-\delta}\,d\langle M^2\rangle_t + dM_t^2, \qquad Y_T^2 = \alpha' S_T^\delta U'_T + \alpha\int_0^T S_s^\delta U_s\,ds = B. \tag{2.8}$$

Because $0 \le S_t^{-\delta} \le e^{\bar{\delta}T}$, we deduce by comparison of (2.5), (2.8) and (2.6) that

$$\underline{Y}_t \le Y_t^2 \le \overline{Y}_t, \qquad 0 \le t \le T.$$

Returning to $Y$ by the formula $Y_t = S_t^{-\delta}Y_t^2 - \int_0^t \alpha e^{\int_s^t \delta_r\,dr} U_s\,ds$, we conclude the proof. $\square$

We now apply the localisation method to get the following existence result.

**Theorem 2.4 (Existence of solution).** *Let us suppose that*

$$E\left[e^{\frac{1}{\beta}e^{\bar{\delta}T}B_-}\right] + E[B_+] < \infty. \tag{2.9}$$

*Then the BSDE (2.2) admits a solution* $(Y, M)$ *which satisfies*

$$S_t^{-\delta}\underline{Y}_t - \int_0^t \alpha e^{\int_s^t \delta_r\,dr} U_s\,ds \le Y_t \le S_t^{-\delta}\overline{Y}_t - \int_0^t \alpha e^{\int_s^t \delta_r\,dr} U_s\,ds \tag{2.10}$$

*for* $0 \le t \le T$*, with* $\underline{Y}, \overline{Y}$ *given in (2.4).*

*Proof.* 1) We first assume that $U'_T$ and $U$ are nonnegative; then $\underline{Y}$ is also nonnegative. For each $n \in I\!N$, we consider $U_t^n := U_t \mathbf{1}_{\left\{\int_0^t U_s\,ds \le n\right\}}$, $0 \le t \le T$, and $U_T^{\prime;n} := U'_T \wedge n$. According to Proposition 2.3, the BSDE

$$dY_t = (\delta_t Y_t - \alpha U_t^n)\,dt + \frac{1}{2\beta}\,d\langle M\rangle_t + dM_t, \qquad Y_T = \alpha' U_T^{\prime;n}$$

admits a unique solution $(Y^n, M^n)$ such that $Y^n$ is a bounded process, and by (2.3), extended from $U_T^{\prime;n}$ and $U^n$ to $U'_T$ and $U$ thanks to nonnegativity,

$$-\int_0^t \alpha e^{\int_s^t \delta_r\,dr} U_s\,ds \le Y_t^n \le S_t^{-\delta} E[B|\mathcal{F}_t].$$

Since $U_T^{\prime;n} \le U_T^{\prime;n+1}$ and $U^n \le U^{n+1}$, the sequence $(Y^n)$ is nondecreasing by a comparison result; this can be obtained similarly as in the proof of Theorem 8 in Mania/Schweizer [11]. For $k \in I\!N$, define the stopping times

$$\tau_k := \inf\left\{t \in [0,T] : \int_0^t \alpha e^{\int_s^t \delta_r\,dr} U_s\,ds + S_t^{-\delta} E[B|\mathcal{F}_t] \ge k\right\} \wedge T$$

and note that $(\tau_k)_{k\in I\!N}$ increases to $T$ stationarily. By construction, the stopped processes $Y^{n;k} := (Y^n)^{\tau_k}$, $n \in I\!N$, are uniformly bounded by $k$. Setting $M^{n;k} := (M^n)^{\tau_k}$, we have

$$Y_t^{n;k} = Y_{\tau_k}^n - \int_t^T \mathbf{1}_{\{s\le\tau_k\}}(\delta_s Y_s^{n;k} - \alpha U_s^n)\,ds - \int_t^T \tfrac{1}{2\beta}\,d\langle M^{n;k}\rangle_s - \int_t^T dM_s^{n;k}.$$

We now take the supremum over $n$ and apply the monotonic stability theorem (see e.g. Lemma 3.3 in Morlais [12]) to obtain, for each $k$, a solution $(Y^k, M^k)$ to the BSDE

$$Y_t^k = \xi_k - \int_t^{\tau_k} (\delta_s Y_s^k - \alpha U_s)\, ds - \int_t^{\tau_k} \tfrac{1}{2\beta}\, d\langle M^k\rangle_s - \int_t^{\tau_k} dM_s^k, \text{ with } \xi_k := \sup_{n\in\mathbb{N}} Y_{\tau_k}^n.$$

More precisely, that result shows that as $n \to \infty$, $M_T^{n;k}$ converges to $M_T^k$ in $L^2$, so that both $M_t^{n;k}$ and $\langle M^{n;k}\rangle_t$ converge uniformly over $t \in [0,T]$ in probability to $M_t^k$ and $\langle M^k\rangle_t$, respectively. Moreover, $\tau_k \le \tau_{k+1}$ by construction; hence $Y_{t\wedge\tau_k}^{n;k+1} = Y_t^{n;k}$ and so we have the localisation property

$$Y_{t\wedge\tau_k}^{k+1} = Y_t^k \quad \text{and} \quad M_{t\wedge\tau_k}^{k+1} = M_t^k.$$

So if we set $\tau_0 := 0$ and define the processes $Y$ and $M$ on $[0,T]$ by

$$Y_t := Y_0^1 + \sum_{k=1}^{\infty} Y_t^k \mathbf{1}_{(\tau_{k-1},\tau_k]}(t) \quad \text{and} \quad M_t := \sum_{k=1}^{\infty} M_t^k \mathbf{1}_{(\tau_{k-1},\tau_k]}(t),$$

the last BSDE can be rewritten as

$$Y_t = \xi_k - \int_t^{\tau_k} (\delta_s Y_s - \alpha U_s)\, ds - \int_t^{\tau_k} \tfrac{1}{2\beta}\, d\langle M\rangle_s - \int_t^{\tau_k} dM_s.$$

Finally we observe that $P$-a.s., $\tau_k = T$ for $k$ large enough. This allows us to send $k \to \infty$ in the previous equation and hence to prove that $(Y, M)$ is a solution to (2.2). The inequality (2.10) is satisfied by the process $Y$ since it holds for each $Y^n$ in view of Proposition 2.3.

2) If $U_T'$ and $U$ are not necessarily nonnegative, we use a double approximation by setting $U_t^{n,p} := U_t^+ \mathbf{1}_{\{\int_0^t |U_s| ds \le n\}} - U_t^- \mathbf{1}_{\{\int_0^t |U_s| ds \le p\}}$ and $U_T'^{,n,p} := (U_T')^+ \wedge n - (U_T')^- \wedge p$. The condition (2.9) is used here to extend (2.3) from the truncated to the general case and to ensure that $\overline{Y}, \underline{Y}$ remain well-defined. In some more detail, we define $\tau_k$ (with $|U|$ and $|B|$) and $Y^{n,p;k}$ and $M^{n,p;k}$ analogously as before. Then $Y^{n,p;k}$ is increasing in $n$ and decreasing in $p$, and it remains bounded by $k$. Arguing as before, we set $Y^k := \inf_p \sup_n Y^{n,p;k}$ to get the existence of $M^k$ such that $\lim_{p\to\infty} \lim_{n\to\infty} M_{s\wedge\tau_k}^{n,p;k} = M_s^k$ and $(Y^k, M^k)$ still solves the BSDE. The rest of the proof is unchanged. □

In connection with the stochastic control problem, it will be important to know when the stochastic exponential $\mathcal{E}\big(-\frac{1}{\beta}M\big)$ is a true martingale, where $M$ comes from the solution of the BSDE (2.2).

**Theorem 2.5.** *Suppose that there exists a constant $\lambda > 1 + \frac{e^{\bar\delta T}-1}{e^{\bar\delta T}}$ such that*

$$E\left[e^{\lambda \frac{1}{\beta} e^{\bar\delta T} B_+}\right] + E\left[e^{\lambda \frac{1}{\beta} e^{\bar\delta T} B_-}\right] < \infty. \tag{2.11}$$

*Then the stochastic exponential $\mathcal{E}\big(-\frac{1}{\beta}M\big)$ is bounded in $L\log L(P)$ and hence a (uniformly integrable) martingale on $[0,T]$.*

*Proof.* Since (2.11) clearly implies (2.9), existence of a solution is ensured by Theorem 2.4. For any stopping time $\tau$ with values in $[0,T]$, the BSDE (2.2) gives

$$\begin{aligned}\mathcal{E}\big(-\tfrac{1}{\beta}M\big)_\tau &= \exp\Big(-\tfrac{1}{\beta}\big(M_\tau + \tfrac{1}{2\beta}\langle M\rangle_\tau\big)\Big)\\ &= \exp\Big(-\tfrac{1}{\beta}\Big(Y_\tau - Y_0 - \int_0^\tau \delta_s Y_s\,ds + \alpha\int_0^\tau U_s\,ds\Big)\Big).\end{aligned}$$

Hence it suffices to prove that there exists a constant $\lambda_1 > 1$ such that

$$E\Big[\exp\Big(-\lambda_1\tfrac{1}{\beta}\Big(Y_\tau - \int_0^\tau \delta_s Y_s\,ds + \alpha\int_0^\tau U_s\,ds\Big)\Big)\Big] \le C,$$

where $C \in (0,\infty)$ is a constant which is independent of $\tau$.

From the estimate (2.10), we have

$$\begin{aligned}Y_\tau + \alpha\int_0^\tau U_s\,ds - \int_0^\tau \delta_t Y_t\,dt \ge S_\tau^{-\delta}\underline{Y}_\tau + \int_0^\tau \alpha\Big(1 - e^{\int_s^\tau \delta_r\,dr}\Big)U_s\,ds\\ - \int_0^\tau \delta_t\Big(S_t^{-\delta}\overline{Y}_t - \int_0^t \alpha e^{\int_s^t \delta_r\,dr}U_s\,ds\Big)\,dt.\end{aligned}$$

But Fubini's theorem gives $\int_0^\tau \delta_t \int_0^t \alpha e^{\int_s^t \delta_r\,dr}U_s\,ds\,dt = \int_0^\tau \alpha\Big(e^{\int_s^\tau \delta_r\,dr} - 1\Big)U_s\,ds$, and so we deduce that

$$Y_\tau - \int_0^\tau \delta_s Y_s\,ds + \alpha\int_0^\tau U_s\,ds \ge S_\tau^{-\delta}\underline{Y}_\tau - \int_0^\tau \delta_t S_t^{-\delta}\overline{Y}_t\,dt. \tag{2.12}$$

Now pick $p > 1$ with $\lambda > p > 1 + \frac{e^{\bar\delta T}-1}{e^{\bar\delta T}}$, and set $\lambda_1 = \frac{\lambda}{p} > 1$. Using (2.12) and (2.4) and setting $L_t^* := \sup_{0\le s\le t} E[B_+|\mathcal{F}_s]$, we obtain with (2.1b) that

$$\begin{aligned}&E\Big[\exp\Big(-\lambda_1\tfrac{1}{\beta}\Big(Y_\tau - \int_0^\tau \delta_s Y_s\,ds + \alpha\int_0^\tau U_s\,ds\Big)\Big)\Big]\\ &\le E\Big[\exp\Big(-\lambda_1\tfrac{1}{\beta}\Big(S_\tau^{-\delta}\underline{Y}_\tau - \int_0^\tau \delta_t S_t^{-\delta}\overline{Y}_t\,dt\Big)\Big)\Big]\\ &\le E\Big[\exp\Big(\lambda_1 \log E\Big[e^{\frac{1}{\beta}e^{\bar\delta T}B_-}\,\Big|\,\mathcal{F}_\tau\Big]\Big)\exp\Big(\lambda_1\tfrac{1}{\beta}L_T^*\int_0^T \delta_t S_t^{-\delta}\,dt\Big)\Big]\\ &\le E\Big[\Big(E\Big[e^{\frac{1}{\beta}e^{\bar\delta T}B_-}\,\Big|\,\mathcal{F}_\tau\Big]\Big)^{\lambda_1} e^{\frac{\lambda_1}{\beta}(e^{\bar\delta T}-1)L_T^*}\Big].\end{aligned}$$

Finally, we set $q := \frac{p}{p-1}$ and $r := (p-1)\frac{e^{\bar\delta T}}{e^{\bar\delta T}-1} > 1$ and use Hölder's inequality, $\lambda_1 p = \lambda$, Jensen's inequality and Doob's inequality in $L^r$ to get

$$E\left[\exp\left(-\lambda_1\tfrac{1}{\beta}\Big(Y_\tau-\int_0^\tau\delta_sY_s\,ds+\alpha\int_0^\tau U_s\,ds\Big)\right)\right]$$
$$\le E\left[\left(E\left[e^{\frac{1}{\beta}e^{\bar\delta T}B_-}\,\Big|\,\mathcal{F}_\tau\right]\right)^{\lambda_1 p}\right]^{1/p}E\left[e^{\frac{\lambda_1 q}{\beta}(e^{\bar\delta T}-1)L^*_T}\right]^{1/q}$$
$$\le C'_rE\left[e^{\lambda\frac{1}{\beta}e^{\bar\delta T}B_-}\right]^{1/p}E\left[e^{\frac{\lambda_1 qr}{\beta}(e^{\bar\delta T}-1)B_+}\right]^{\frac{1}{qr}}=C<\infty,$$

because $\lambda_1 qr\big(e^{\bar\delta T}-1\big)=\lambda e^{\bar\delta T}$. □

## 3. The BSDE on an infinite horizon

In this section, we use BSDE techniques to prove the existence and uniqueness of a solution to the infinite-horizon BSDE under suitable conditions.

**Hypothesis 3.1. Throughout this section, we impose the standing assumptions**

(3.1a) $I\!\!F$ is a *continuous filtration*, i.e. all local $(P,I\!\!F)$-martingales are continuous.

(3.1b) $\delta\ge 0$ is uniformly bounded (by $\bar\delta$, say).

Again, the assumptions on $U=(U_t)_{t\ge0}$ will be specified later; $U'_T$ does not appear here. The BSDE (1.2) under study is now

$$dY_t=(\delta_tY_t-\alpha U_t)\,dt+\frac{1}{2\beta}\,d\langle M\rangle_t+dM_t,\qquad \lim_{t\to\infty}Y_t=0\ P\text{-a.s.},\tag{3.2}$$

and as before, a *solution* of (3.2) is a pair $(Y,M)$ satisfying (3.2), where $Y$ is a $P$-semimartingale and $M$ is a locally $P$-square-integrable local $P$-martingale null at 0.

The first step in tackling (3.2) is to obtain a priori estimates for the finite-horizon version with terminal condition $Y_T=0$. But in contrast to Section 2, we now need the bounds to be uniform in $T$, and so we need stronger assumptions on $U$.

**Definition 3.2.** We say that a random variable $X$ is in $D^{\exp}$ if $E\left[e^{\lambda|X|}\right]<\infty$ for all $\lambda>0$. A progressively measurable process $U=(U_t)_{t\ge0}$ is in $D^{\exp}_{1,T}$, for $T\in(0,\infty]$, if $\int_0^T|U_s|\,ds$ is in $D^{\exp}$, and an RCLL process $Y=(Y_t)_{t\ge0}$ is in $D^{\exp}_{0,T}$, for $T\in(0,\infty]$, if $Y^*_T:=\sup_{0\le t\le T}|Y_t|$ is in $D^{\exp}$. (By convention, $Y^*_\infty:=\sup_{t\ge0}|Y_t|$.)

Let us now consider the BSDE

$$dY_t = (\delta_t Y_t - \alpha U_t)\,dt + \frac{1}{2\beta}\,d\langle M\rangle_t + dM_t, \qquad Y_T = 0. \tag{3.3}$$

**Proposition 3.3 (A priori estimates).** *Suppose that* $\int_0^T |U_s|\,ds$ *is a bounded random variable. Then there exists a unique solution* $(Y, M)$ *to (3.3) such that* $Y$ *is a bounded process. Moreover, we have the estimate*

$$|Y_t| \le \beta \log E\Big[\exp\Big(\tfrac{1}{\beta}\int_t^T \alpha|U_s|\,ds\Big)\,\Big|\,\mathcal{F}_t\Big], \qquad 0 \le t \le T. \tag{3.4}$$

*Proof.* Existence and uniqueness of a solution with $Y$ bounded follow as for Proposition 2.3 from Theorems 2.5 and 2.6 of Morlais [12]. Applying Tanaka's formula then first yields

$$d|Y_t| = \operatorname{sign}(Y_t)\,dY_t + dL_t = \operatorname{sign}(Y_t)\Big((\delta_t Y_t - \alpha U_t)\,dt + \frac{1}{2\beta}\,d\langle M\rangle_t + dM_t\Big) + dL_t,$$

where $L$ is the local time at 0 of the continuous semimartingale $Y$. Next, applying Itô's formula to the bounded process $Z_t := \exp\Big(\frac{1}{\beta}\Big(|Y_t| + \int_0^t \alpha|U_s|\,ds\Big)\Big)$, we obtain

$$\begin{aligned} dZ_t &= \frac{1}{\beta} Z_t \operatorname{sign}(Y_t)\Big((\delta_t Y_t - \alpha U_t)\,dt + \frac{1}{2\beta}\,d\langle M\rangle_t + dM_t\Big) \\ &\quad + \frac{1}{\beta} Z_t\,dL_t + \frac{1}{\beta} Z_t \alpha |U_t|\,dt + \frac{1}{2\beta^2} Z_t\,d\langle M\rangle_t \\ &\ge \frac{1}{\beta} Z_t \operatorname{sign}(Y_t)\,dM_t, \end{aligned}$$

in the sense that the difference of the terms on the two sides of the inequality is an increasing process. So $Z$ is a submartingale, which gives

$$\exp\Big(\tfrac{1}{\beta}\Big(|Y_t| + \int_0^t \alpha|U_s|\,ds\Big)\Big) \le E\Big[\exp\Big(\tfrac{1}{\beta}\int_0^T \alpha|U_s|\,ds\Big)\,\Big|\,\mathcal{F}_t\Big]$$

since $Y_T = 0$, and (3.4) follows. □

From this a priori estimate and the localisation method, we obtain the existence of a solution to the infinite-horizon BSDE (3.2).

**Theorem 3.4 (Existence of solution).** *Let us suppose that*

$$E\Big[\exp\Big(\tfrac{1}{\beta}\int_0^\infty \alpha|U_s|\,ds\Big)\Big] < \infty, \text{ i.e. } \exp\Big(\tfrac{1}{\beta}\int_0^\infty \alpha|U_s|\,ds\Big) \in L^1. \tag{3.5}$$

*Then the BSDE (3.2) admits a solution* $(Y, M)$ *which satisfies*

$$(3.6)\qquad |Y_t| \le \beta \log E\Big[\exp\Big(\tfrac{1}{\beta}\int_t^\infty \alpha |U_s|\,ds\Big)\,\Big|\,\mathcal{F}_t\Big], \qquad t \ge 0.$$

*If* $\exp\Big(\frac{1}{\beta}\int_0^\infty \alpha|U_s|\,ds\Big)$ *is in* $L^r$ *for some* $r > 1$*, then so is* $\exp\big(\frac{1}{\beta}Y^*_\infty\big)$*. If* $U \in D^{\exp}_{1,\infty}$*, then* $Y \in D^{\exp}_{0,\infty}$*.*

*Proof.* 1) We first assume that $U$ is nonnegative and set $U^n_t := U_t \mathbf{1}_{\{\int_0^t U_s ds \le n\}}$ for each $n \in I\!N$. According to Proposition 3.3, the BSDE

$$dY_t = (\delta_t Y_t - \alpha U^n_t)\,dt + \frac{1}{2\beta}\,d\langle M\rangle_t + dM_t, \qquad Y_n = 0$$

on $[0,n]$ admits a unique solution $(Y^n, M^n)$ with $Y^n$ bounded, and by (3.4),

$$|Y^n_t| \le \beta \log E\Big[\exp\Big(\tfrac{1}{\beta}\int_t^\infty \alpha|U_s|\,ds\Big)\,\Big|\,\mathcal{F}_t\Big], \qquad t \in [0,n].$$

If we set $Y^n_t = 0$ and $M^n_t = M^n_n$ for $t > n$, then $(Y^n, M^n)$ also satisfies on $[0, n+1]$ the BSDE

$$dY_t = (\delta_t Y_t - \alpha \mathbf{1}_{\{t\le n\}} U^n_t)\,dt + \frac{1}{2\beta}\,d\langle M\rangle_t + dM_t, \qquad Y_{n+1} = 0.$$

Because $U$ is nonnegative, $\mathbf{1}_{\{t\le n\}}U^n_t \le U^{n+1}_t$ for $t \in [0, n+1]$, and so the sequence $(Y^n)$ is nondecreasing by the comparison theorem. For each $k \in I\!N$, we define the stopping time

$$\tau_k := \inf\Big\{t \ge 0 : \beta \log E\Big[\exp\Big(\tfrac{1}{\beta}\int_t^\infty \alpha|U_s|\,ds\Big)\,\Big|\,\mathcal{F}_t\Big] \ge k\Big\} \wedge k.$$

Introducing the stopped processes $Y^{n;k} := (Y^n)^{\tau_k}$ and $M^{n;k} := (M^n)^{\tau_k}$, we can argue exactly as in the proof of Theorem 2.4 to construct processes $Y$ and $M$, now on $[0,\infty)$, satisfying for each $T$ the BSDE

$$Y_t = Y_T - \int_t^T (\delta_s Y_s - \alpha U_s)\,ds - \int_t^T \tfrac{1}{2\beta}\,d\langle M\rangle_s - \int_t^T dM_s.$$

Since each $Y^n$ satisfies the estimate (3.6) by Proposition 3.3, it follows from the construction that so does $Y$, and this implies due to (3.5) that

$$\lim_{t\to\infty} Y_t = 0 \qquad P\text{-a.s.}$$

2) In the general case where $U$ need not be nonnegative, we use the double approximation $U^{n,p}_t := U^+_t \mathbf{1}_{\{\int_0^t |U_s| ds \le n\}} \mathbf{1}_{\{t\le n\}} - U^-_t \mathbf{1}_{\{\int_0^t |U_s| ds \le p\}} \mathbf{1}_{\{t \le p\}}$, $t \ge 0$, and denote by $(Y^{n,p}, M^{n,p})$ the solution to

$$dY_t = (\delta_t Y_t - \alpha U_t^{n,p})\,dt + \frac{1}{2\beta}\,d\langle M\rangle_t + dM_t, \qquad Y_{n\vee p} = 0.$$

Then the proof goes like for Theorem 2.4, using that $Y^{n,p}$ increases in $n$ and decreases in $p$.

3) The integrability assertions about $Y$ follow from (3.6) and Doob's inequality. □

To get a uniqueness result for the infinite-horizon BSDE (3.2), we need a stronger assumption.

**Theorem 3.5 (Uniqueness of solution).** *Suppose that $U$ is in $D^{\exp}_{1,\infty}$. Then the BSDE (3.2) admits a unique solution $(Y,M)$ with $Y \in D^{\exp}_{0,\infty}$.*

*Proof.* Existence is clear from Theorem 3.4. For uniqueness, let $(Y,M)$ and $(Y',M')$ be two such solutions and note that the martingale part is always unique by the uniqueness of the canonical decomposition of a special semimartingale. Fix $\theta \in (0,1)$ and set $\hat{Y} := Y - \theta Y'$ and $\hat{M} := M - \theta M'$. Then

$$d\hat{Y}_t = \big(\delta_t \hat{Y}_t - \alpha(1-\theta)U_t\big)\,dt + \frac{1}{2\beta}\,d\big(\langle M\rangle_t - \theta\langle M'\rangle_t\big) + d\hat{M}_t. \tag{3.7}$$

Noting that convexity gives

$$d\langle M\rangle_t = d\Big\langle \theta M' + (1-\theta)\frac{\hat{M}}{1-\theta}\Big\rangle_t \le \theta\,d\langle M'\rangle_t + \frac{1}{1-\theta}\,d\langle \hat{M}\rangle_t, \tag{3.8}$$

we rewrite (3.7) as

$$\begin{aligned} d\hat{Y}_t = {} & \big(\delta_t \hat{Y}_t - \alpha(1-\theta)U_t\big)\,dt + d\hat{M}_t \\ & + \frac{1}{2\beta}\,d\Big(\langle M\rangle_t - \theta\langle M'\rangle_t - \frac{1}{1-\theta}\langle\hat{M}\rangle_t\Big) + \frac{1}{2\beta(1-\theta)}\,d\langle\hat{M}\rangle_t. \end{aligned} \tag{3.9}$$

Now Tanaka's formula yields $d\hat{Y}_t^- = -\mathbf{1}_{\{\hat{Y}_t\le 0\}}\,d\hat{Y}_t + \frac{1}{2}\,d\hat{L}_t$, where $\hat{L}$ is the local time at 0 of the process $\hat{Y}$. Applying next Itô's formula to the process $Z_t := \exp\left(\frac{1}{\beta(1-\theta)}\Big(\hat{Y}_t^- + \int_0^t \alpha(1-\theta)|U_s|\,ds\Big)\right)$, $t \ge 0$, we get

$$\begin{aligned} dZ_t &= \frac{1}{\beta(1-\theta)} Z_t\Big(-\mathbf{1}_{\{\hat{Y}_t\le 0\}}\,d\hat{Y}_t + \frac{1}{2}\,d\hat{L}_t + \alpha(1-\theta)|U_t|\,dt\Big) \\ &\quad + \frac{1}{2\beta^2(1-\theta)^2} Z_t \mathbf{1}_{\{\hat{Y}_t\le 0\}}\,d\langle\hat{M}\rangle_t \\ &\ge -\frac{1}{\beta(1-\theta)} Z_t \mathbf{1}_{\{\hat{Y}_t\le 0\}}\,d\hat{M}_t, \end{aligned}$$

where the last inequality uses (3.9) and (3.8). Thus $Z$ is a local submartingale, and so there exists an increasing sequence of stopping times $\tau_n \nearrow \infty$ such that

$$\exp\Big(\tfrac{1}{\beta(1-\theta)}\hat{Y}^-_{t\wedge\tau_n}\Big) \le E\Big[\exp\Big(\tfrac{1}{\beta(1-\theta)}\hat{Y}^-_{T\wedge\tau_n} + \tfrac{1}{\beta}\int\limits_{t\wedge\tau_n}^{T\wedge\tau_n}\alpha|U_s|\,ds\Big)\,\Big|\,\mathcal{F}_t\Big].$$

Because $\hat{Y} \in D^{\exp}_{0,\infty}$, $U \in D^{\exp}_{1,\infty}$ and $\lim\limits_{t\to\infty}\hat{Y}_t = 0$, we obtain for $n \to \infty$ and $T \to \infty$ that

$$\exp\Big(\tfrac{1}{\beta(1-\theta)}\hat{Y}^-_t\Big) \le E\Big[\exp\Big(\tfrac{1}{\beta}\int\limits_t^\infty\alpha|U_s|\,ds\Big)\,\Big|\,\mathcal{F}_t\Big],$$

which is equivalent to

$$(Y_t - \theta Y'_t)^- = \hat{Y}^-_t \le \beta(1-\theta)\log E\Big[\exp\Big(\tfrac{1}{\beta}\int\limits_t^\infty\alpha|U_s|\,ds\Big)\,\Big|\,\mathcal{F}_t\Big].$$

Letting $\theta \to 1$, we deduce that $Y_t \ge Y'_t$, and since a symmetrical argument gives the reverse inequality, the proof is complete. □

As in Section 2, we again want to know when the stochastic exponential $\mathcal{E}\big(-\frac{1}{\beta}M\big)$ is a true martingale, where $M$ now comes from the solution of the BSDE (3.2). However, we only expect to obtain this here on the open interval $[0,\infty)$, and the proof below shows why $T = \infty$ causes a difficulty.

**Theorem 3.6.** *Suppose that $U$ is in $D^{\exp}_{1,\infty}$ and denote by $(Y, M)$ the solution to (3.2) from Theorem 3.4. Then for every finite $T$ and every $r < \infty$, the stochastic exponential $\Big(\mathcal{E}\big(-\frac{1}{\beta}M\big)_t\Big)_{0\le t\le T}$ is bounded in $L^r$, and so $\mathcal{E}\big(-\frac{1}{\beta}M\big)$ is a martingale on $[0,\infty)$.*

*Proof.* Fix $T \in (0,\infty)$ and let $\tau$ be a stopping time with values in $[0, T]$. As in the proof of Theorem 2.5, the BSDE (3.2) gives

$$\begin{aligned}\mathcal{E}\big(-\tfrac{1}{\beta}M\big)_\tau &= \exp\Big(-\tfrac{1}{\beta}\Big(Y_\tau - Y_0 - \int\limits_0^\tau\delta_s Y_s\,ds + \alpha\int\limits_0^\tau U_s\,ds\Big)\Big)\\ &\le \exp\big(\tfrac{1}{\beta}Y_0\big)\exp\Big(\tfrac{1}{\beta}Y^*_T(1+\bar{\delta}T) + \tfrac{1}{\beta}\int\limits_0^\infty\alpha|U_s|\,ds\Big),\end{aligned}$$

and the conclusion follows from Theorem 3.4 because $Y \in D^{\exp}_{0,\infty}$. □

## 4. The stochastic control problem on an infinite horizon

In this section, we prove existence and uniqueness of a solution to the infinite-horizon stochastic control problem.

Let us first give a precise formulation. We recall from Section 1 the underlying filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le\infty}$, the parameters $\alpha \ge 0$, $\beta > 0$ and the processes $\delta = (\delta_t)_{t\ge0}$ and $U = (U_t)_{t\ge0}$. We denote by $\mathcal{Q}$ the set of all probability measures $Q \overset{\text{loc}}{\ll} P$ and by $Z^Q = (Z^Q_t)_{t \geq 0}$ an RCLL version of the density process of $Q$ with respect to $P$. Since $\mathcal{F}_0$ is trivial, we have

$$\begin{aligned}&\{Z^Q \,|\, Q \in \mathcal{Q}\}\\ &\subseteq \{\text{all RCLL } P\text{-martingales } Z = (Z_t)_{t\geq 0} \text{ with } Z \geq 0 \text{ and } Z_0 = 1\} =: \mathcal{Z}.\end{aligned}$$

Now define for any $t \geq 0$ and $Z \in \mathcal{Z}$ in analogy to (1.3) and (1.4)

$$\tilde{\mathcal{U}}^\delta_{t,\infty}(Z) := \alpha \int_t^\infty \frac{S^\delta_s}{S^\delta_t} \frac{Z_s}{Z_t} U_s \, ds,$$

$$\tilde{\mathcal{R}}^\delta_{t,\infty}(Z) := \int_t^\infty \delta_s \frac{S^\delta_s}{S^\delta_t} \frac{Z_s}{Z_t} \log \frac{Z_s}{Z_t} \, ds$$

and the cost functional

$$\tilde{c}_\infty(Z) := \tilde{\mathcal{U}}^\delta_{0,\infty}(Z) + \beta \tilde{\mathcal{R}}^\delta_{0,\infty}(Z).$$

The stochastic control problem studied here is to minimise the functional

$$Z \mapsto \Gamma_\infty(Z) := E_P[\tilde{c}_\infty(Z)]$$

over a subset $\mathcal{Z}_f$ of $\mathcal{Z}$, defined below. Note that by the minimum principle for supermartingales, $Z \in \mathcal{Z}$ remains 0 if it ever hits 0; so both summands of $\tilde{c}_\infty(Z)$ are well-defined.

**Hypothesis 4.1. Throughout this section, we impose the standing assumptions**

(4.1a) There exists some $T_0 \in (0,\infty)$ such that for all $\gamma > 0$,

$$E_P\Big[\exp\Big(\gamma \int_0^{T_0} |U_s| \, ds\Big)\Big] + E_P\Big[\int_{T_0}^\infty \exp(\gamma |U_s|) \mathbf{1}_{\{U_s \neq 0\}} \, ds\Big] < \infty.$$

(4.1b) $0 < \underline{\delta} \leq \delta_t \leq \bar{\delta} < \infty$, uniformly in $(t,\omega)$, for constants $\underline{\delta}, \bar{\delta}$.

The first condition in (4.1a) says that $U$ is in $D^{\exp}_{1,T_0}$; we remark that the indicator function in the second term fixes an obvious oversight in (4.4) of Bordigoni [1]. The condition (4.1b) is natural for an infinite-horizon problem. Note that we do not assume here that $\mathbb{F}$ is a continuous filtration.

**Definition 4.2.** $\mathcal{Z}_f$ denotes the set of all martingales $Z \in \mathcal{Z}$ satisfying $E_P\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(Z)\big] < \infty$.

Our stochastic control problem is slightly more general than the one studied in Chapter 4 of Bordigoni [1] since we do not insist on working on the canonical (Skorohod) path space. The presentation here is linked to Bordigoni [1] and to the slightly different formulation in Section 1 as follows. For any $Q$ with $Z^Q \in \mathcal{Z}_f$, we have under (4.1a) for any $t \geq 0$

$$E_P\big[\tilde{\mathcal{R}}^\delta_{t,\infty}(Z^Q)\,\big|\,\mathcal{F}_t\big] = E_P\left[\int_t^\infty \delta_s \frac{S^\delta_s}{S^\delta_t}\frac{Z^Q_s}{Z^Q_t}\log\frac{Z^Q_s}{Z^Q_t}\,ds\,\middle|\,\mathcal{F}_t\right] = E_Q\left[\mathcal{R}^\delta_{t,\infty}(Q)\,\big|\,\mathcal{F}_t\right],$$

$$E_P\big[\tilde{\mathcal{U}}^\delta_{t,\infty}(Z^Q)\,\big|\,\mathcal{F}_t\big] = E_P\left[\int_t^\infty \frac{S^\delta_s}{S^\delta_t}\frac{Z^Q_s}{Z^Q_t}U_s\,ds\,\middle|\,\mathcal{F}_t\right] = E_Q\left[\mathcal{U}^\delta_{t,\infty}\,\big|\,\mathcal{F}_t\right];$$

this is proved in Lemma 4.6 and Remark 4.10 of Bordigoni [1], essentially by using Bayes' rule. In particular, this shows that

$$\varGamma_\infty(Z^Q) := E_P\big[\tilde{c}_\infty(Z^Q)\big] = E_Q[c_\infty(Q)] =: \varGamma_\infty(Q).$$

Expressing everything under $P$ and working with martingales $Z \in \mathcal{Z}_f$ turns out to be a bit more flexible than working with probability measures $Q \overset{\text{loc}}{\ll} P$. From now on, all expectations without subscript are under $P$.

**Remark 4.3.** Under (4.1b), $\int_t^\infty \delta_s S^\delta_s\,ds = S^\delta_t$ for every $t \geq 0$ and hence also

$$E\left[\int_t^\infty \delta_s \frac{S^\delta_s}{S^\delta_t}\frac{Z_s}{Z_t}\,ds\,\middle|\,\mathcal{F}_t\right] = 1 \qquad \text{for every } t \geq 0 \text{ and any } Z \in \mathcal{Z}.$$

◇

We start with some auxiliary estimates. These are true for any $Z \in \mathcal{Z}$, with the understanding that we set $E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(Z)\big] = +\infty$ for $Z \in \mathcal{Z} \setminus \mathcal{Z}_f$.

**Lemma 4.4.** *For every $Z \in \mathcal{Z}$ and every $T \in (0,\infty)$,*

$$E[Z_T \log Z_T] \leq \frac{1}{\underline{\delta}} e^{\bar{\delta}(T+1)}\big(E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(Z)\big] + e^{-1}\big).$$

*Proof.* Since $t \mapsto E[Z_t \log Z_t]$ is increasing and $z\log z \geq -e^{-1}$, Fubini and (4.1b) give

$$\begin{aligned} E[Z_T \log Z_T] &\leq E\left[\int_T^{T+1} \frac{\delta_s}{\underline{\delta}} S^\delta_s e^{\bar{\delta}s}(Z_s \log Z_s + e^{-1})\,ds\right] \\ &\leq \frac{1}{\underline{\delta}} e^{\bar{\delta}(T+1)} E\Big[\int_0^\infty \delta_s S^\delta_s (Z_s \log Z_s + e^{-1})\,ds\Big] \\ &= \frac{1}{\underline{\delta}} e^{\bar{\delta}(T+1)}\big(E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(Z)\big] + e^{-1}\big) \end{aligned}$$

by Remark 4.3. □

**Proposition 4.5.** *There is a constant $C < \infty$ such that for every $Z \in \mathcal{Z}$,*

$$E\Big[\int_0^\infty S^\delta_s Z_s |U_s|\,ds\Big] \le C\big(1 + E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(Z)\big]\big) \le C\big(1 + \Gamma_\infty(Z)\big). \tag{4.2}$$

*Proof.* This is a simplified version of the proofs for Lemma 4.9 and Proposition 4.11 in Bordigoni [1]. We use several times below the elementary inequality $xy \le e^x + y\log y$ for $x \in I\!R, y \ge 0$, typically applied to $xy = \gamma x \frac{1}{\gamma} y$. Starting with $\tilde{c}_\infty(Z)$, we split $\tilde{\mathcal{U}}^\delta_{0,\infty}(Z)$ into an integral from 0 to $T_0$ and another from $T_0$ to $\infty$ to write first

$$\begin{aligned} E\Big[\int_0^{T_0} S^\delta_s Z_s |U_s|\,ds\Big] &\le E\Big[Z_{T_0}\int_0^{T_0} |U_s|\,ds\Big] \\ &\le E\Big[\exp\Big(\gamma\int_0^{T_0}|U_s|\,ds\Big)\Big] \\ &\quad + \frac{1}{\gamma}E\left[Z_{T_0}(\log Z_{T_0} + |\log\gamma|)\right]. \end{aligned} \tag{4.3}$$

Next, we have due to $S^\delta \le 1$ and Remark 4.3 that

$$\begin{aligned} &E\Big[\int_{T_0}^\infty S^\delta_s Z_s |U_s|\,ds\Big] \\ &\le E\Big[\int_{T_0}^\infty e^{\gamma|U_s|}\mathbf{1}_{\{U_s\neq 0\}}\,ds\Big] + E\Big[\int_{T_0}^\infty \tfrac{1}{\gamma}S^\delta_s Z_s \log(\tfrac{1}{\gamma}S^\delta_s Z_s)\mathbf{1}_{\{U_s\neq 0\}}\,ds\Big] \\ &\le E\Big[\int_{T_0}^\infty e^{\gamma|U_s|}\mathbf{1}_{\{U_s\neq 0\}}\,ds\Big] + \tfrac{1}{\gamma}E\Big[\int_0^\infty \tfrac{\delta_s}{\underline{\delta}}S^\delta_s(Z_s\log Z_s + e^{-1})\,ds\Big] \\ &\quad + \frac{1}{\gamma}|\log\gamma| E\Big[\int_0^\infty \tfrac{\delta_s}{\underline{\delta}}S^\delta_s Z_s\,ds\Big] \\ &= E\Big[\int_{T_0}^\infty e^{\gamma|U_s|}\mathbf{1}_{\{U_s\neq 0\}}\,ds\Big] + \frac{1}{\gamma\underline{\delta}}\big(E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(Z)\big] + e^{-1} + |\log\gamma|\big). \end{aligned} \tag{4.4}$$

Combining (4.3) with Lemma 4.4 and (4.4) gives the left inequality in (4.2) in the form

$$E\Big[\int_0^\infty S^\delta_s Z_s |U_s|\,ds\Big] \le C + E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(Z)\big]\frac{1}{\gamma\underline{\delta}}\big(e^{\bar{\delta}(T_0+1)} + 1\big), \tag{4.5}$$

where the constant $C$ depends on $\gamma$ and also on $U$ via (4.1a). By definition, then,

$$\begin{aligned} \Gamma_\infty(Z) &= E\big[\tilde{\mathcal{U}}^\delta_{0,\infty}(Z)\big] + \beta E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(Z)\big] \\ &\ge -\alpha E\Big[\int_0^\infty S^\delta_s Z_s|U_s|\,ds\Big] + \beta E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(Z)\big] \\ &\ge -\alpha C + E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(Z)\big]\Big(\beta - \frac{\alpha}{\gamma\underline{\delta}}\big(1 + e^{\bar{\delta}(T_0+1)}\big)\Big), \end{aligned}$$

and so the right inequality in (4.2) follows by taking $\gamma$ large enough and choosing a new constant appropriately. □

The above argument also shows that $\Gamma_\infty(Z) \le C\big(1 + E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(Z)\big]\big)$ for all $Z \in \mathcal{Z}$, with a suitable constant. Another direct consequence is the following result that will be used later.

**Corollary 4.6.** *For every $T \ge T_0$, every $Z \in \mathcal{Z}$ and every $\gamma > 0$,*

$$E\Big[\int_T^\infty S^\delta_s Z_s U_s\,ds\Big] \le E\Big[\int_T^\infty e^{\gamma|U_s|}\mathbf{1}_{\{U_s \ne 0\}}\,ds\Big] + \frac{1}{\gamma\underline{\delta}}\big(E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(Z)\big] + e^{-1} + |\log\gamma|\big).$$

With these preparations, we are ready to prove existence and uniqueness of a solution to our stochastic control problem.

**Theorem 4.7.** *Under Hypothesis 4.1, there exists a unique $Z^* \in \mathcal{Z}_f$ that minimises the cost functional $Z \mapsto \Gamma_\infty(Z)$ over all $Z \in \mathcal{Z}_f$.*

*Proof.* This again follows closely the arguments in Bordigoni [1]; see there the proof of Theorem 4.15. Since we optimise over $Z$ instead of $Q$, we need not work on path space and can simplify some arguments.

First of all, uniqueness is clear because $Z \mapsto \Gamma_\infty(Z)$ is strictly convex like $z \mapsto z\log z$. Existence is proved in several steps.

1) Since $Z \equiv 1$ is in $\mathcal{Z}_f$ and $E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(1)\big] = 0$, (4.1a) and Proposition 4.5 imply that $-\infty < \inf_{Z\in\mathcal{Z}_f} \Gamma_\infty(Z) < \infty$. So we can take a sequence $(Z^n)_{n\in\mathbb{N}}$ in $\mathcal{Z}_f$ such that $\Gamma_\infty(Z^n)$ decreases to $\inf_{Z\in\mathcal{Z}_f}\Gamma_\infty(Z)$ as $n \to \infty$. Combining the well-known Komlós-type result in Lemma A1.1 in Delbaen/Schachermayer [7] with a diagonalisation argument produces a sequence $(\bar{Z}^n)_{n\in\mathbb{N}}$ with $\bar{Z}^n \in \mathrm{conv}(Z^n, Z^{n+1}, \dots)$ for all $n$ and such that with probability 1,

$$\lim_{n\to\infty} \bar{Z}^n_r =: \bar{Z}^\infty_r \qquad \text{exists in } [0,\infty] \text{ for all } r \in \mathbb{Q}^+.$$

Since each $\bar{Z}^n$ is like the $Z^n$ a martingale $\ge 0$ with expectation 1, Fatou's lemma yields that each $\bar{Z}^\infty_r$ is integrable and $(\bar{Z}^\infty_r)_{r\in\mathbb{Q}^+}$ is a supermartingale. By a standard argument (see Dellacherie/Meyer [8], Theorem VI.2), we can therefore extend $(\bar{Z}^\infty_r)_{r\in\mathbb{Q}^+}$ to a process $Z^* = (Z^*_t)_{t\ge0}$ with RCLL trajectories and such that $Z^*$ is a supermartingale $\ge 0$ (now over $[0,\infty)$ instead of $\mathbb{Q}^+$). In fact, we can take $Z^*_t := \lim_{r\searrow t,\, r\in\mathbb{Q}^+} \bar{Z}^\infty_r$.

2) In order to show that $Z^*$ is even a martingale and in $\mathcal{Z}_f$, we first use Lemma 4.4 to obtain for each $r \in \mathbb{Q}^+$ that

$$\sup_{n\in\mathbb{N}} E\left[\bar{Z}^n_r \log \bar{Z}^n_r\right] \le \frac{1}{\underline{\delta}} e^{\bar{\delta}(r+1)}\Big(\sup_{n\in\mathbb{N}} E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(\bar{Z}^n)\big] + e^{-1}\Big) < \infty,$$

because by Proposition 4.5 and convexity of $Z \mapsto \Gamma_\infty(Z)$,

$$E\big[\tilde{\mathcal{R}}^{\delta}_{0,\infty}(\bar{Z}^n)\big] \le C\big(1 + \Gamma_\infty(\bar{Z}^n)\big) \le C\Big(1 + \sup_{m\ge n} \Gamma_\infty(Z^m)\Big) \le C\big(1 + \Gamma_\infty(Z^1)\big) < \infty.$$

So $(\bar{Z}^n_r)_{n\in\mathbb{N}}$ is uniformly integrable for each $r \in \mathbb{Q}^+$, and this implies

$$E\left[\bar{Z}^\infty_r\right] = \lim_{n\to\infty} E\left[\bar{Z}^n_r\right] = 1 \qquad \text{for all } r \in \mathbb{Q}^+,$$

which means that the supermartingale $\bar{Z}^\infty$ is a martingale (over $\mathbb{Q}^+$). Using Doob's maximal inequality and the fact that $(\bar{Z}^n_m)_{n\in\mathbb{N}}$ converges to $Z^*_m$ in $L^1$ for every $m \in \mathbb{N}$ next shows that with probability 1, $(\bar{Z}^n)_{n\in\mathbb{N}}$ converges to $Z^*$ uniformly on compact subsets of $[0,\infty)$, and the same uniform integrability argument as above then yields that also $Z^*$ is a martingale $\big($over $[0,\infty)\big)$, hence in $\mathcal{Z}$. Finally, Fatou's lemma, Remark 4.3 and Proposition 4.5 give

$$E\big[\tilde{\mathcal{R}}^{\delta}_{0,\infty}(Z^*)\big] = E\Big[\int_0^\infty \delta_s S^\delta_s Z^*_s \log Z^*_s \, ds\Big] \le \liminf_{n\to\infty} E\big[\tilde{\mathcal{R}}^{\delta}_{0,\infty}(\bar{Z}^n)\big] \le \sup_{n\in\mathbb{N}} C\big(1 + \Gamma_\infty(\bar{Z}^n)\big) < \infty$$

so that $Z^*$ is in $\mathcal{Z}_f$.

3) To show that $Z^*$ is optimal, we want to prove that $Z \mapsto \Gamma_\infty(Z)$ is lower semicontinuous along the sequence $(\bar{Z}^n)_{n\in\mathbb{N}}$, because we then get by convexity

$$\Gamma_\infty(Z^*) \le \liminf_{n\to\infty} \Gamma_\infty(\bar{Z}^n) \le \liminf_{n\to\infty} \Gamma_\infty(Z^n) = \inf_{Z\in\mathcal{Z}_f} \Gamma_\infty(Z).$$

We have just seen in step 2) that $E\big[\tilde{\mathcal{R}}^{\delta}_{0,\infty}(Z^*)\big] \le \liminf_{n\to\infty} E\big[\tilde{\mathcal{R}}^{\delta}_{0,\infty}(\bar{Z}^n)\big]$ so that it only remains to prove the analogous inequality for the part with $\tilde{\mathcal{U}}^{\delta}_{0,\infty}(Z)$. Now $U \in D^{\exp}_{1,T_0}$ by (4.1a), and so we can use the finite-horizon results in Bordigoni/Matoussi/Schweizer [2] to obtain

$$E\Big[\int_0^{T_0} S^\delta_s Z^*_s U_s \, ds\Big] = E\Big[Z^*_{T_0} \int_0^{T_0} S^\delta_s U_s \, ds\Big] \le \liminf_{n\to\infty} E\Big[\bar{Z}^n_{T_0} \int_0^{T_0} S^\delta_s U_s \, ds\Big] = \liminf_{n\to\infty} E\Big[\int_0^{T_0} S^\delta_s \bar{Z}^n_s U_s \, ds\Big];$$

see step 4) in the proof of Theorem 9 in Bordigoni/Matoussi/Schweizer [2]. We then split the remaining integral from $T_0$ to $\infty$ into one integral from $T_0$ to $T \ge T_0$ and another from $T$ to $\infty$. The integral over the finite interval $(T_0, T]$ is again treated as above, using that (4.1a) also gives $U \in D^{\exp}_{1,T}$ by Jensen's inequality. Finally, Corollary 4.6 gives

$$E\Big[\int_T^\infty S^\delta_s \bar{Z}^n_s U_s \, ds\Big] \le E\Big[\int_T^\infty e^{\gamma|U_s|} \mathbf{1}_{\{U_s \ne 0\}} \, ds\Big] + \frac{1}{\gamma\underline{\delta}}\Big(\sup_{n\in\mathbb{N}} E\big[\tilde{\mathcal{R}}^{\delta}_{0,\infty}(\bar{Z}^n)\big] + e^{-1} + |\log\gamma|\Big)$$

for all $\gamma > 0$ and all $n \in I\!N$, and the same estimate holds for $Z^*$ instead of $\bar{Z}^n$ as well. Choosing first $\gamma$ large to make the second summand above small and then, using (4.1a), $T$ large to get the first summand small as well, we deduce that

$$\lim_{T\to\infty} \sup \left\{ E\Big[\int_T^\infty S_s^\delta Z_s U_s\,ds\Big] : Z = Z^* \text{ or } Z = \bar{Z}^n \text{ for some } n \in I\!N \right\} = 0,$$

and so we obtain after putting everything together that

$$E\Big[\int_0^\infty S_s^\delta Z_s^* U_s\,ds\Big] \le \liminf_{n\to\infty} E\Big[\int_0^\infty S_s^\delta \bar{Z}_s^n U_s\,ds\Big].$$

This completes the proof. □

As in the finite-horizon case treated in Bordigoni/Matoussi/Schweizer [2], one can show that the optimal $Z^*$ from Theorem 4.7 is strictly positive. If there exists $Q^*$ with density process $Z^*$ (e.g. as in Bordigoni [1] if one works on path space), this translates into saying that $Q^* \overset{\text{loc}}{\approx} P$. The proof of positivity can be found in Bordigoni [1], Theorem 4.18, and largely parallels that of Theorem 12 in Bordigoni/Matoussi/Schweizer [2].

Also as in Bordigoni/Matoussi/Schweizer [2], one can show that the *martingale optimality principle* holds in our setting; see Proposition 4.19 and Corollary 4.20 in Bordigoni [1] for details of this standard argument. As a consequence, the optimal $Z^*$ from Theorem 4.7 is also conditionally optimal at any time $t$ or even stopping time $\tau$. To properly formulate this, we denote by $V = (V_t)_{t\ge 0}$ an RCLL version of the process

$$V_t := \operatorname*{ess\,inf}_{Z\in\mathcal{Z}_f} E\big[\tilde{\mathcal{U}}_{t,\infty}^\delta(Z) + \beta\tilde{\mathcal{R}}_{t,\infty}^\delta(Z)\,\big|\,\mathcal{F}_t\big] =: \operatorname*{ess\,inf}_{Z\in\mathcal{Z}_f} J_t(Z), \qquad t \ge 0. \tag{4.6}$$

Then conditional optimality says that

$$V_t = J_t(Z^*) \qquad P\text{-a.s. for all } t \ge 0,$$

and we now use this to describe the behaviour of $V_t$ as $t \to \infty$.

**Proposition 4.8.** *Under Hypothesis 4.1,*

$$\lim_{t\to\infty} V_t = 0 \qquad P\text{-a.s.}$$

*Proof.* This is analogous to the proof of Lemma 4.22 in Bordigoni [1]. Since $Z \equiv 1$ is in $\mathcal{Z}_f$, $S^\delta$ is decreasing and $E\big[\tilde{\mathcal{R}}_{t,\infty}^\delta(Z)\,\big|\,\mathcal{F}_t\big] \ge 0$, we have

$$V_t = J_t(Z^*) \le J_t(1) \le \alpha E\Big[\int_t^\infty |U_s|\,ds\,\Big|\,\mathcal{F}_t\Big]$$

which yields $\limsup_{t\to\infty} V_t \le 0$ due to (4.1a). To get a lower bound for $V_t$, we use analogous arguments as in the proof of Proposition 4.5 to first obtain

$$\left| E\left[ \int_t^\infty \frac{S_s^\delta}{S_t^\delta} \frac{Z_s^*}{Z_t^*} U_s \, ds \,\middle|\, \mathcal{F}_t \right] \right| \le E\Big[ \int_t^\infty e^{\gamma |U_s|} \mathbf{1}_{\{U_s \neq 0\}} \, ds \,\Big|\, \mathcal{F}_t \Big] + \frac{1}{\gamma \underline{\delta}} \Big( E\big[ \tilde{\mathcal{R}}^\delta_{t,\infty}(Z^*) \,\big|\, \mathcal{F}_t \big] + e^{-1} + |\log \gamma| \Big).$$

Hence

$$\begin{aligned} V_t = J_t(Z^*) &\ge -\alpha E\Big[ \int_t^\infty e^{\gamma |U_s|} \mathbf{1}_{\{U_s \neq 0\}} \, ds \,\Big|\, \mathcal{F}_t \Big] \\ &\quad - \frac{\alpha}{\gamma \underline{\delta}} (e^{-1} + |\log \gamma|) + \Big( \beta - \frac{\alpha}{\gamma \underline{\delta}} \Big) E\big[ \tilde{\mathcal{R}}^\delta_{t,\infty}(Z^*) \,\big|\, \mathcal{F}_t \big], \end{aligned}$$

and taking $\gamma$ so large that $\beta - \frac{\alpha}{\gamma \underline{\delta}} \ge 0$, we get from $E\big[ \tilde{\mathcal{R}}^\delta_{t,\infty}(Z^*) \,\big|\, \mathcal{F}_t \big] \ge 0$ and (4.1a) that

$$\liminf_{t\to\infty} V_t \ge -\frac{\alpha}{\gamma \underline{\delta}} (e^{-1} + |\log \gamma|) \qquad P\text{-a.s.}$$

Since $\gamma$ is arbitrary, we conclude that $\liminf_{t\to\infty} V_t \ge 0$ $P$-a.s., which completes the proof. □

All results in this section so far hold for a general filtration. If $I\!\!F$ is continuous, one can in addition show as in Bordigoni/Matoussi/Schweizer [2] that $V$ obeys the dynamics

$$dV_t = (\delta_t V_t - \alpha U_t) \, dt + \frac{1}{2\beta} \, d\langle M \rangle_t + dM_t$$

for some (continuous) local martingale $M$; see Theorem 4.27 in Bordigoni [1] for a detailed proof. Together with Proposition 4.8, this explains where the infinite-horizon BSDE (1.2) comes from. Since the above derivation uses no essential new ideas in comparison with Bordigoni/Matoussi/Schweizer [2], we refrain from giving more details.

## 5. Solving the stochastic control problem via the BSDE

Our goal in this section is to use the results on the infinite-horizon BSDE (3.2) for a characterisation of the dynamic value process $V$ for the stochastic control problem from Section 4. As just mentioned, we could have shown that $V$ solves (3.2), but this is not enough: The uniqueness result in Theorem 3.5 only holds for solutions $(Y, M)$ with $Y \in D^{\exp}_{0,\infty}$, and we do not know at this point how to argue directly that $V$ from the control problem is in $D^{\exp}_{0,\infty}$. The BSDE techniques developed so far will enable us to prove this. To that end, we now show how one can construct from a (particular) solution to the BSDE (3.2) a (and actually the, by uniqueness) solution for the infinite-horizon stochastic control problem.

**Hypothesis 5.1. Throughout this section, we impose the standing assumptions**

(5.1a) $I\!\!F$ is a *continuous filtration*, i.e. all local $(P, I\!\!F)$-martingales are continuous.

(5.1b) $0 < \underline{\delta} \le \delta_t \le \bar{\delta} < \infty$, uniformly in $(t, \omega)$, for constants $\underline{\delta}, \bar{\delta}$.

Conditions on $U$ will be specified below, when we successively treat three cases.

Our arguments rely substantially on the finite-horizon results proved in Bordigoni/Matoussi/Schweizer [2] so that we very briefly recall these here. Fix $T < \infty$ and consider on $[0, T]$ the BSDE

$$dY_t = (\delta_t Y_t - \alpha U_t)\, dt + \frac{1}{2\beta}\, d\langle M \rangle_t + dM_t, \qquad Y_T = \alpha' U'_T. \tag{2.2}$$

Recall from (1.3) and (1.4) the definitions of $\mathcal{U}^\delta_{t,T}$ and $\mathcal{R}^\delta_{t,T}$, and assume that $U$ (as a process on $[0, T]$) is in $D^{\exp}_{1,T}$ and $U'_T$ is in $D^{\exp}$. Then Theorem 17 of Bordigoni/Matoussi/Schweizer [2] states that (2.2) has a unique solution $(Y, M)$ in $D^{\exp}_{0,T} \times \mathcal{M}_{0,\mathrm{loc}}(P)$, that $\bar{Z} := \mathcal{E}\big( -\frac{1}{\beta} M \big)$ is a martingale on $[0, T]$ with $E\big[\bar{Z}_T \log \bar{Z}_T\big] < \infty$, and that for any martingale $Z \ge 0$ on $[0, T]$ with $Z_0 = 1$ and $E[Z_T \log Z_T] < \infty$, we have for any stopping time $\tau \le T$ that

$$Y_\tau = E\left[ \frac{\bar{Z}_T}{\bar{Z}_\tau} \mathcal{U}^\delta_{\tau,T} + \beta \tilde{\mathcal{R}}^\delta_{\tau,T}(\bar{Z}) \,\middle|\, \mathcal{F}_\tau \right] \le E\left[ \frac{Z_T}{Z_\tau} \mathcal{U}^\delta_{\tau,T} + \beta \tilde{\mathcal{R}}^\delta_{\tau,T}(Z) \,\middle|\, \mathcal{F}_\tau \right].$$

$\big($This reformulates the statement that the dynamic value process of the finite-horizon stochastic control problem is the unique solution of (2.2).$\big)$ For $\tau \equiv 0$, this reduces to

$$\begin{aligned} Y_0 \le {} & E\Big[ Z_T\, \alpha \int_0^T S^\delta_s U_s\, ds + Z_T\, \alpha' S^\delta_T U'_T \Big] \\ & + \beta E\Big[ \int_0^T \delta_s S^\delta_s Z_s \log Z_s\, ds + S^\delta_T Z_T \log Z_T \Big], \end{aligned} \tag{5.2}$$

with equality for $Z = \bar{Z}$.

### 5.1. The bounded case

Let us now first study the case where $\int_0^\infty |U_s|\, ds$ is bounded. This is of course a restrictive assumption, but it allows fairly simple arguments and provides a basic building block. We shall see that the proofs in more general cases follow the same scheme.

**Proposition 5.2.** *Suppose that* $\int_0^\infty |U_s|\,ds$ *is a bounded random variable. For any solution* $(Y, M)$ *to the infinite-horizon BSDE (3.2) with* $Y$ *bounded, we then have for any* $t \geq 0$

$$Y_t = \operatorname*{ess\,inf}_{Z\in\mathcal{Z}_f} E\big[\tilde{\mathcal{U}}^\delta_{t,\infty}(Z) + \beta\tilde{\mathcal{R}}^\delta_{t,\infty}(Z)\,\big|\,\mathcal{F}_t\big].$$

*Proof.* Without loss of generality, we prove the result for $t = 0$.

1) Let us start by arguing that for any $Z \in \mathcal{Z}_f$, we have

$$Y_0 \leq E\big[\tilde{\mathcal{U}}^\delta_{0,\infty}(Z) + \beta\tilde{\mathcal{R}}^\delta_{0,\infty}(Z)\big]. \tag{5.3}$$

We first note that the nonnegative function $g(s) := E[\delta_s S^\delta_s(Z_s \log Z_s + e^{-1})]$ satisfies, by Remark 4.3 and Fubini,

$$\int_0^\infty g(s)\,ds = E\Big[\int_0^\infty \delta_s S^\delta_s(Z_s \log Z_s + e^{-1})\,ds\Big] = E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(Z)\big] + e^{-1} < \infty.$$

This implies that there exists a sequence of deterministic times $T_n \nearrow \infty$ such that

$$\lim_{n\to\infty} E\big[S^\delta_{T_n}(Z_{T_n}\log Z_{T_n} + e^{-1})\big] \leq \lim_{n\to\infty} \tfrac{1}{\underline{\delta}}\, g(T_n) = 0$$

and therefore also

$$\lim_{n\to\infty} E[S^\delta_{T_n} Z_{T_n}\log Z_{T_n}] = 0, \tag{5.4}$$

since $S^\delta_{T_n} \leq e^{-\underline{\delta}T_n} \to 0$. Moreover, because $Y$ is bounded and $Z \geq 0$ is a martingale, we have

$$\lim_{n\to\infty} \big|E[S^\delta_{T_n} Z_{T_n} Y_{T_n}]\big| \leq \lim_{n\to\infty} e^{-\underline{\delta}T_n}\|Y\|_\infty E[Z_{T_n}] = 0. \tag{5.5}$$

Now $Y$ is bounded, hence in $D^{\exp}_{0,T_n}$, and satisfies the finite-horizon BSDE (2.2) with final value $\alpha' U'_{T_n} := Y_{T_n}$. Moreover, $Z \in \mathcal{Z}_f$ verifies $E[Z_{T_n}\log Z_{T_n}] < \infty$ due to Lemma 4.4, and so the finite-horizon results tell us that

$$\begin{aligned}Y_0 \leq{}& E\Big[\alpha\int_0^{T_n} S^\delta_s Z_s U_s\,ds + S^\delta_{T_n} Z_{T_n} Y_{T_n}\Big]\\ &+ \beta E\Big[\int_0^{T_n} \delta_s S^\delta_s Z_s \log Z_s\,ds + S^\delta_{T_n} Z_{T_n}\log Z_{T_n}\Big].\end{aligned} \tag{5.6}$$

On the right-hand side, the second and the fourth summand tend to 0 as $n \to \infty$ by (5.5) and (5.4), respectively. Next, Fatou's lemma yields

$$E\Big[\int_0^\infty S^\delta_s Z_s|U_s|\,ds\Big] \le \liminf_{n\to\infty} E\Big[Z_{T_n}\int_0^{T_n} S^\delta_s|U_s|\,ds\Big]$$
$$\le \liminf_{n\to\infty} E\Big[Z_{T_n}\int_0^\infty |U_s|\,ds\Big] < \infty$$

since $\int_0^\infty |U_s|\,ds$ is bounded and $Z \ge 0$ is a martingale. From dominated convergence, we thus deduce that the first summand in (5.6) converges to $E\Big[\alpha\int_0^\infty S^\delta_s Z_s U_s\,ds\Big] = E\big[\tilde{\mathcal{U}}^\delta_{0,\infty}(Z)\big]$ as $n\to\infty$. Finally,

$$\int_0^{T_n} \delta_s S^\delta_s Z_s \log Z_s\,ds = \int_0^{T_n} \delta_s S^\delta_s (Z_s\log Z_s + e^{-1})\,ds - e^{-1}(1 - S^\delta_{T_n})$$

implies via monotone integration and by using $\int_0^\infty \delta_s S^\delta_s\,ds = 1$ that the third summand in (5.6) converges to $\beta E\Big[\int_0^\infty \delta_s S^\delta_s Z_s\log Z_s\,ds\Big] = \beta E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(Z)\big]$ as $n\to\infty$. Putting everything together gives (5.3).

2) Now define $\bar Z := \mathcal{E}\big(-\frac{1}{\beta}M\big)$. Since $\int_0^\infty |U_s|\,ds$ is bounded, $U$ is in $D^{\exp}_{1,\infty}$; and since $(Y,M)$ solves (3.2), Theorem 3.6 tells us that $\bar Z$ is a martingale on $[0,\infty)$ so that $\bar Z\in\mathcal{Z}$. We want to prove that $\bar Z$ is even in $\mathcal{Z}_f$. To that end, we apply the finite-horizon results to write

$$Y_0 = E\Big[\bar Z_{T_n}\alpha\int_0^{T_n} S^\delta_s U_s\,ds + S^\delta_{T_n}\bar Z_{T_n} Y_{T_n}\Big]$$
$$+ \beta E\Big[\int_0^{T_n}\delta_s S^\delta_s \bar Z_s\log\bar Z_s\,ds + S^\delta_{T_n}\bar Z_{T_n}\log\bar Z_{T_n}\Big].$$

Using Fatou's lemma and $z\log z \ge -e^{-1}$ therefore gives

$$\begin{aligned}
&\beta E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(\bar Z)\big]\\
&= \beta E\Big[\int_0^\infty \delta_s S^\delta_s \bar Z_s\log\bar Z_s\,ds\Big]\\
&\le \liminf_{n\to\infty} \beta E\Big[\int_0^{T_n} \delta_s S^\delta_s \bar Z_s\log\bar Z_s\,ds\Big]\\
&= \liminf_{n\to\infty}\Big(Y_0 - \beta E\big[S^\delta_{T_n}\bar Z_{T_n}\log\bar Z_{T_n}\big] - E\Big[\bar Z_{T_n}\alpha\int_0^{T_n} S^\delta_s U_s\,ds + S^\delta_{T_n}\bar Z_{T_n}Y_{T_n}\Big]\Big)\\
&\le \liminf_{n\to\infty}\Big(Y_0 + \beta e^{-1} + \alpha\Big\|\int_0^\infty |U_s|\,ds\Big\|_{L^\infty} E\big[\bar Z_{T_n}\big] + \|Y^*_\infty\|_{L^\infty} E\big[\bar Z_{T_n}\big]\Big) < \infty
\end{aligned}$$

because $Y$ and $\int_0^\infty |U_s|\,ds$ are bounded and $\bar Z \ge 0$ is a martingale. Hence $\bar Z$ is indeed in $\mathcal{Z}_f$.

3) Since $\bar{Z}$ is in $\mathcal{Z}_f$ by step 2) and satisfies (5.2) with equality, the same argument as in step 1) shows that the inequality in (5.3) becomes an equality for $Z = \bar{Z}$. Hence $\bar{Z}$ attains the infimum, and the proof is complete. □

### 5.2. The positive case

We now turn to the case where $U$ is nonnegative and satisfies some integrability condition. Note that $U \geq 0$ is a fairly natural assumption. Indeed, if we think of a full-fledged robust control problem for utility maximisation, then $U_t$ typically represents the utility $\mathbf{U}(c_t)$ from consumption at time $t$, where we still optimise over $c$ in a second step. As a consumption rate, $c_t \geq 0$; so $U_t = \mathbf{U}(c_t) \geq 0$ for any nonnegative utility function $\mathbf{U}$ on $[0,\infty)$, like e.g. power utility $\mathbf{U}(x) = \frac{1}{\gamma}x^\gamma$ for $\gamma \in (0,1)$.

**Theorem 5.3.** *Suppose that $U \geq 0$ and $U$ is in $D^{\exp}_{1,\infty}$. For the solution $(Y, M)$ to (3.2) from Theorem 3.4, we then have for any $t \geq 0$*

$$Y_t = \operatorname*{ess\,inf}_{Z \in \mathcal{Z}_f} E\big[\tilde{\mathcal{U}}^\delta_{t,\infty}(Z) + \beta\tilde{\mathcal{R}}^\delta_{t,\infty}(Z) \,\big|\, \mathcal{F}_t\big].$$

*Proof.* Without loss of generality, we again argue for $t = 0$. The overall structure of the proof is like for Proposition 5.2, but we first need to recall the construction of $(Y, M)$. For each $n \in I\!N$, set $U^n_t := U_t \mathbf{1}_{\{\int_0^t U_s ds \leq n\}}$ and denote by $(Y^n, M^n)$ with $Y^n$ bounded the solution to the BSDE

$$dY_t = (\delta_t Y_t - \alpha U^n_t)\,dt + \frac{1}{2\beta}\,d\langle M\rangle_t + dM_t, \qquad Y_n = 0$$

on $[0,n]$. Extending $(Y^n, M^n)$ to $[0,\infty)$ by setting $Y^n_t = 0$, $M^n_t = M^n_n$ for $t > n$, we then get on $[0,\infty)$ a solution $(Y^n, M^n)$ to the BSDE

$$(5.7)\qquad dY_t = (\delta_t Y_t - \alpha \mathbf{1}_{\{t \leq n\}} U^n_t)\,dt + \frac{1}{2\beta}\,d\langle M\rangle_t + dM_t, \qquad \lim_{t\to\infty} Y_t = 0,$$

and $Y_t = \nearrow\text{-}\lim_{n\to\infty} Y^n_t$ for all $t \geq 0$. We first prove that for any $Z \in \mathcal{Z}_f$,

$$(5.8)\qquad Y_0 \leq E\big[\tilde{\mathcal{U}}^\delta_{0,\infty}(Z) + \beta\tilde{\mathcal{R}}^\delta_{0,\infty}(Z)\big].$$

Indeed, applying Proposition 5.2 to the process $\big(\mathbf{1}_{\{t\leq n\}}U^n_t\big)_{t\geq 0}$ and the solution to (5.7) gives

$$Y^n_0 \leq E\Big[\alpha \int_0^\infty S^\delta_s Z_s \mathbf{1}_{\{s\leq n\}} U^n_s\,ds + \beta \int_0^\infty \delta_s S^\delta_s Z_s \log Z_s\,ds\Big],$$

and (5.8) follows by monotone integration since $U \geq 0$.

Now set $\bar{Z} := \mathcal{E}\big(-\frac{1}{\beta}M\big)$ so that $\bar{Z} \in \mathcal{Z}$ by Theorem 3.6; this uses the integrability assumption on $U$. To prove that $\bar{Z}$ is even in $\mathcal{Z}_f$, we first note that by Proposition 5.2 and its proof, we have equality in (5.8) for the choice $Z = \bar{Z}^n := \mathcal{E}\big(-\frac{1}{\beta}M^n\big)$ so that

$$Y_0^n = E\Big[\alpha \int_0^\infty S_s^\delta \bar{Z}_s^n \mathbf{1}_{\{s\le n\}} U_s^n\,ds + \beta \int_0^\infty \delta_s S_s^\delta \bar{Z}_s^n \log \bar{Z}_s^n\,ds\Big].$$

But by construction, $(Y_0^n)_{n\in I\!N}$ increases to $Y_0$, and $(M^n)_{n\in I\!N}$ and $(\langle M^n\rangle)_{n\in I\!N}$ converge to $M$ and $\langle M\rangle$ locally uniformly in probability so that also $\bar{Z}^n \to \bar{Z}$ locally uniformly in probability as $n\to\infty$. Hence Fatou's lemma yields

$$Y_0 \ge E\Big[\alpha \int_0^\infty S_s^\delta \bar{Z}_s U_s\,ds + \beta \int_0^\infty \delta_s S_s^\delta \bar{Z}_s \log \bar{Z}_s\,ds\Big] = E\big[\tilde{\mathcal{U}}_{0,\infty}^\delta(\bar{Z}) + \beta \tilde{\mathcal{R}}_{0,\infty}^\delta(\bar{Z})\big],$$

and so $\bar{Z}\in\mathcal{Z}_f$ because $U\ge 0$. Since (5.8) gives the converse inequality, we actually have equality in (5.8) for $Z=\bar{Z}$, and this completes the proof. □

### 5.3. The general case

Finally, we study a situation where $U$ can be real-valued. Then we need slightly stronger integrability assumptions.

**Theorem 5.4.** *Suppose that $U$ is in $D_{1,\infty}^{\exp}$ and that $U$ also satisfies (4.1a), i.e. there exists some $T_0\in(0,\infty)$ such that for all $\gamma>0$,*

$$E_P\Big[\exp\Big(\gamma \int_0^{T_0} |U_s|\,ds\Big)\Big] + E_P\Big[\int_{T_0}^\infty \exp(\gamma|U_s|)\mathbf{1}_{\{U_s\ne 0\}}\,ds\Big] < \infty.$$

*For the solution $(Y,M)$ to (3.2) from Theorem 3.4, we then have for any $t\ge 0$*

$$Y_t = \operatorname*{ess\,inf}_{Z\in\mathcal{Z}_f} E\big[\tilde{\mathcal{U}}_{t,\infty}^\delta(Z) + \beta\tilde{\mathcal{R}}_{t,\infty}^\delta(Z)\,\big|\,\mathcal{F}_t\big].$$

*Proof.* As already in the last proof, we argue for $t=0$ without loss of generality and again first recall from the proof of Theorem 3.4 the construction of $(Y,M)$. For $n,p\in I\!N$, set

$$U_t^{n,p} := U_t^+ \mathbf{1}_{\{\int_0^t |U_s|ds\le n\}}\mathbf{1}_{\{t\le n\}} - U_t^- \mathbf{1}_{\{\int_0^t |U_s|ds\le p\}}\mathbf{1}_{\{t\le p\}}$$

and denote by $(Y^{n,p},M^{n,p})$ with $Y^{n,p}$ bounded the solution to the BSDE

$$dY_t = (\delta_t Y_t - \alpha U_t)\,dt + \frac{1}{2\beta}\,d\langle M\rangle_t + dM_t, \qquad Y_{n\vee p}=0$$

on $[0,n\vee p]$. We extend $(Y^{n,p},M^{n,p})$ to $[0,\infty)$ by setting $Y_t^{n,p}=0$ and $M_t^{n,p}=M_{n\vee p}^{n,p}$ for $t>n\vee p$ to get on $[0,\infty)$ a solution to the BSDE

$$dY_t = (\delta_t Y_t - \alpha\mathbf{1}_{\{t\le n\vee p\}}U_t^{n,p})\,dt + \frac{1}{2\beta}\,d\langle M\rangle_t + dM_t, \qquad \lim_{t\to\infty} Y_t = 0. \tag{5.9}$$

Then $Y_t = \nearrow\text{-}\lim_{n\to\infty} \searrow\text{-}\lim_{p\to\infty} Y_t^{n,p}$ and $M_t = \lim_{n\to\infty}\lim_{p\to\infty} M_t^{n,p}$. We proceed on a familiar path.

1) First we prove that for any $Z \in \mathcal{Z}_f$,

$$Y_0 \le E\big[\tilde{\mathcal{U}}^\delta_{0,\infty}(Z) + \beta\tilde{\mathcal{R}}^\delta_{0,\infty}(Z)\big]. \tag{5.10}$$

Like in Section 5.2, using Proposition 5.2 gives

$$Y_0^{n,p} \le E\Big[\alpha\int_0^\infty S_s^\delta Z_s \mathbf{1}_{\{s\le n\vee p\}} U_s^{n,p}\,ds + \beta\int_0^\infty \delta_s S_s^\delta Z_s \log Z_s\,ds\Big],$$

and (5.10) follows by letting $p\to\infty$ and then $n\to\infty$, provided we can use dominated convergence. But this is ensured by the first estimate in Proposition 4.5; indeed, (4.1a) yields $E\Big[\int_0^\infty S_s^\delta Z_s|U_s|\,ds\Big] < \infty$ for $Z \in \mathcal{Z}_f$.

2) Thanks to the first integrability assumption on $U$, Theorem 3.6 implies that the process $\bar{Z} := \mathcal{E}\big(-\frac{1}{\beta}M\big)$ is in $\mathcal{Z}$. To show it is even in $\mathcal{Z}_f$, we use Proposition 5.2 for $\bar{Z}^{n,p} := \mathcal{E}\big(-\frac{1}{\beta}M^{n,p}\big)$ to get

$$Y_0^{n,p} = E\Big[\alpha\int_0^\infty S_s^\delta \bar{Z}_s^{n,p}\mathbf{1}_{\{s\le n\vee p\}} U_s^{n,p}\,ds + \beta\int_0^\infty \delta_s S_s^\delta \bar{Z}_s^{n,p}\log\bar{Z}_s^{n,p}\,ds\Big]. \tag{5.11}$$

But (4.5) in the proof of Proposition 4.5 gives

$$\begin{aligned}\Big|E\Big[\alpha\int_0^\infty S_s^\delta \bar{Z}_s^{n,p}\mathbf{1}_{\{s\le n\vee p\}}U_s^{n,p}\,ds\Big]\Big| &\le E\Big[\alpha\int_0^\infty S_s^\delta\bar{Z}_s^{n,p}|U_s|\,ds\Big]\\ &\le C_{\gamma,U} + E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}\big(\bar{Z}^{n,p}\big)\big]\frac{1}{\gamma\underline{\delta}}\big(e^{\bar{\delta}(T_0+1)}+1\big),\end{aligned}$$

where the constant $C_{\gamma,U}$ depends on $\gamma$ and $U$ via (4.1a), but not on $n$ and $p$. Plugging this estimate with a minus sign into (5.11) and taking $\gamma$ big enough yields

$$\sup_{n,p\in\mathbb{N}} E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}\big(\bar{Z}^{n,p}\big)\big] \le C\Big(1+\sup_{n,p\in\mathbb{N}} Y_0^{n,p}\Big) < \infty, \tag{5.12}$$

because applying the a priori estimate (3.6) from Theorem 3.4 to (5.9) tells us that

$$|Y_0^{n,p}| \le \beta\log E\Big[\exp\Big(\tfrac{1}{\beta}\int_0^\infty \alpha|U_s|\,ds\Big)\Big] < \infty$$

for all $n$ and $p$, by using the definition of $U^{n,p}$. As $n\to\infty$ and $p\to\infty$, we have locally uniformly in probability $M^{n,p}\to M$ and $\langle M^{n,p}\rangle \to \langle M\rangle$, hence also $\bar{Z}^{n,p}\to\bar{Z}$, and so $\bar{Z}\in\mathcal{Z}_f$ because $E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(\bar{Z})\big]<\infty$ by Fatou's lemma and (5.12).

3) To prove that we have equality in (5.10) for $Z=\bar{Z}$, we start from the equality in (5.11). As $n\to\infty$ and $p\to\infty$, $Y_0^{n,p}$ tends to $Y_0$ and

$\liminf_{n\to\infty}\liminf_{p\to\infty} E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}\big(\bar{Z}^{n,p}\big)\big] \ \geq \ E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(\bar{Z})\big]$ by Fatou's lemma. Because $Z \mapsto E\big[\tilde{\mathcal{R}}^\delta_{0,\infty}(Z)\big]$ is bounded along the sequence $\big(\bar{Z}^{n,p}\big)_{n,p\in\mathbb{N}}$ by (5.12), almost the same argument as in step 3) of the proof of Theorem 4.7 gives

$$\begin{aligned}\liminf_{n\to\infty}\liminf_{p\to\infty} E\Big[\alpha\int_0^\infty S^\delta_s \bar{Z}^{n,p}_s \mathbf{1}_{\{s\leq n\vee p\}} U^{n,p}_s\,ds\Big] &\geq E\Big[\alpha\int_0^\infty S^\delta_s \bar{Z}_s U_s\,ds\Big]\\ &= E\big[\tilde{\mathcal{U}}^\delta_{0,\infty}(\bar{Z})\big].\end{aligned}$$

Note that this exploits the integrability assumption (4.1a). Therefore (5.11) implies that $Y_0 \geq E\big[\tilde{\mathcal{U}}^\delta_{0,\infty}(\bar{Z}) + \beta\tilde{\mathcal{R}}^\delta_{0,\infty}(\bar{Z})\big]$, and so we must have equality due to (5.10) since $\bar{Z}\in\mathcal{Z}_f$. This completes the proof. □

### 5.4. Consequences for the stochastic control problem

As in (4.6), denote by $V=(V_t)_{t\geq 0}$ the dynamic value process of the infinite-horizon stochastic control problem. We already mentioned at the end of Section 4 that if $\mathbb{F}$ is continuous, then $V$ satisfies the infinite-horizon BSDE (3.2). For the proof, we referred to Bordigoni [1]; let us just note here that the required assumptions are Hypothesis 4.1 plus continuity of $\mathbb{F}$, i.e. Hypothesis 5.1 plus (4.1a). Under a slightly stronger condition, we can now even prove a BSDE characterisation for $V$.

**Theorem 5.5.** *Assume Hypothesis 5.1 and that $U$ is in $D^{\exp}_{1,\infty}$. If in addition either $U\geq 0$ or $U$ satisfies* (4.1a)*, then $V$ is the first component of the unique solution in $D^{\exp}_{0,\infty}\times\mathcal{M}_{0,\mathrm{loc}}(P)$ to the infinite-horizon BSDE (3.2). In particular, $V\in D^{\exp}_{0,\infty}$.*

*Proof.* By Theorem 3.5, (3.2) has a unique solution $(Y,M)$ with $Y\in D^{\exp}_{0,\infty}$; and by the definition of $V$ in (4.6) and either Theorem 5.3 or Theorem 5.4, $Y$ coincides with $V$. □

**Remark 5.6. 1)** The second case of Theorem 5.5 is the infinite-horizon analogue to the finite-horizon Theorem 17 in Bordigoni/Matoussi/Schweizer [2], with assumptions and conclusions almost exactly parallel. The only difference lies in the conditions on $U$: In (4.1a), we need $U\in D^{\exp}_{1,T_0}$, but also an exponential moment control over $U$ on the infinite time interval $[T_0,\infty)$. See Remark 4.28 in Bordigoni [1] for a more detailed comment on this point. The result for $U\geq 0$ has no precedent.

**2)** Our approach for $T=\infty$ here is different from Bordigoni [1] in that we show for the solution of the BSDE (3.2) that it satisfies the defining property (4.6) of the value process $V$. As a bonus, we are able to deduce that $V$ is indeed in $D^{\exp}_{0,\infty}$; this was conjectured, but not proved in Bordigoni [1].

**3)** Again in remarkable analogy to the finite-horizon results in Bordigoni/Matoussi/Schweizer [2], we obtain existence of a solution to the stochastic control problem for a general filtration $I\!\!F$. But the integrability property $V \in D^{\exp}_{0,\infty}$ is only known for continuous $I\!\!F$, since its proof exploits the BSDE results. Like in Bordigoni/Matoussi/Schweizer [2], we do not know if $V \in D^{\exp}_{0,\infty}$ also holds for general $I\!\!F$. ◇

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