Comparison of Some Key Approaches to Hedging in Incomplete Markets

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Abstract.: The paper provides a numerical comparison of local risk minimisation and mean-variance hedging for some key variations of stochastic volatility models. A hedging and pricing framework is established for both approaches. Important quantitative differences become apparent that have implications for the implementation of hedging strategies under stochastic volatility.

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1 Introduction

At present their is much uncertainty in the choice of the right pricing measure for the hedging of derivatives in incomplete markets. Incompleteness can arise for instance in the presence of stochastic volatility as will be studied in the following. This paper provides comparative numerical results for probably the two most important hedging methodologies, namely local risk minimisation and global mean-variance hedging.

We first describe the theoretical framework that underpins these two approaches. We then present some comparative studies on expected squared total costs and the asymptotics of these costs, differences in prices and optimal hedge ratios. The numerical results are obtained for variations of the Heston and the Stein-Stein stochastic volatility models.

To produce accurate and reliable estimates combinations of partial differential equations and simulation techniques have been developed that are of independent interest. The numerical work also includes explicit solutions for certain key quantities that are required for mean-variance hedging. It turns out that mean-variance hedging is far more difficult to implement than what has been currently attempted for most stochastic volatility models. In particular the mean-variance pricing measure is in some cases difficult to identify and to characterise. Furthermore, the corresponding optimal hedge, due to its global characterisation, no longer appears as a simple combination of partial derivatives with respect to state variables. It has more the character of an optimal control strategy.

The importance of the study is that it documents for some classes of stochastic volatility models the typical quantitative differences that arise for two major hedging approaches. We conclude by drawing attention to certain observations that have implications for the practical implementation of stochastic volatility models.

2 A Markovian Stochastic Volatility Framework

We consider a frictionless market in continuous time with a single primary asset available for trade. We denote by $S = \{S_t,\, 0 \le t \le T\}$ the price process for this asset defined on the filtered probability space $(\Omega, \mathcal{F}, P)$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ satisfying the usual conditions for some fixed but arbitrary time horizon $T \in (0, \infty)$.

We introduce the discounted price process $X = \{X_t = \frac{S_t}{B_t},\, 0 \le t \le T\}$, where $B = \{B_t,\, 0 \le t \le T\}$ represents the savings account that accumulates interest at the continuously compounding interest rate.

We consider a general two-factor stochastic volatility model of the form

$$\begin{aligned} dX_t &= X_t \left(\mu(t, Y_t)\, dt + Y_t\, dW_t^1 \right) \\ dY_t &= a(t, Y_t)\, dt + b(t, Y_t)(\varrho\, dW_t^1 + \sqrt{1-\varrho^2}\, dW_t^2) \end{aligned} \tag{2.1}$$

for $0 \leq t \leq T$ with given deterministic initial values $X_0 \in (0, \infty)$ and $Y_0 \in (0, \infty)$. Here the function μ is a given appreciation rate. The volatility component Y evolves according to a separate stochastic differential equation with drift function a, diffusion function b and constant correlation $\varrho \in [-1, 1]$. W^1 and W^2 denote independent standard Wiener processes under P. The component Y allows for an additional source of randomness but is not available as a traded asset.

To ensure that this Markovian framework provides a viable asset price model we assume appropriate conditions hold for the functions μ, a, b and ϱ so that the system of stochastic differential equations (2.1) admits a unique strong continuous solution for the vector process (X, Y) with a strictly positive discounted price process X and strictly positive volatility process Y. We take the filtration $\mathbb{F}$ to be the natural filtration generated by W^1 and W^2, where $\mathcal{F}_0$ is the trivial σ-algebra $\{\emptyset, \Omega\}$.

In order to price and hedge derivatives in an arbitrage free manner we assume that there exists an equivalent local martingale measure (ELMM) Q such that X is a local Q-martingale. That is the probability measures Q and P have the same null sets and

$$E_Q(X_t \,|\, \mathcal{F}_s) = X_s \tag{2.2}$$

for $0 \leq s \leq t \leq T$.

We denote by $\mathbb{P}$ the set of all ELMMs Q. Our financial market is characterised by the system (2.1) together with the filtration $\mathbb{F}$ and is called incomplete if $\mathbb{P}$ contains more than one element.

In this paper we are in principle interested in the hedging of European style contingent claims with an $\mathcal{F}_T$-measurable square integrable random payoff H based on the dynamics given by (2.1). A specific choice for H which we will use later on for all numerical examples is the European put payoff given by

$$H = h(X_T) = (K - X_T)^+. \tag{2.3}$$

The requirement of $\mathcal{F}_T$-measurability and square integrability for the payoff H allows for many types of path dependent contingent claims and possibly even dependence on the evolution of the volatility process Y.

Subject to certain restrictions on the functions μ, a, b and parameter ϱ we can ensure, via an application of the Girsanov transformation, that there is an ELMM Q such that X is a local Q-martingale.

The condition that X should be a local Q-martingale fixes the effect of the Girsanov transformation on W^1 but allows for many different transformation effects on the independent W^2. Consequently if $|\varrho| < 1$ the set $\mathbb{P}$ contains more than one element and our financial market is therefore incomplete.

In order to price and hedge derivatives in this incomplete market setting we need to somehow fix the ELMM Q. Currently there is no general agreement on how to choose a specific ELMM Q with a number of alternatives being considered in the literature.

In this paper we will consider two quadratic approaches to hedging in incomplete markets; these are local risk-minimisation and mean-variance hedging. For either of these two approaches we require hedging strategies of the form $\varphi = (\vartheta, \eta)$, where ϑ is a predictable X-integrable process and η is an adapted process such that the value process $V(\varphi) = \{V_t(\varphi),\, 0 \leq t \leq T\}$ with

$$V_t(\varphi) = \vartheta_t\, X_t + \eta_t \tag{2.4}$$

is right-continuous for $0 \leq t \leq T$. The hedging strategy $\varphi = (\vartheta, \eta)$ means that we form at time t a portfolio with ϑ_t units of the traded risky asset X_t and η_t units of the savings account B_t.

The cost process $C(\varphi) = \{C_t(\varphi),\, 0 \leq t \leq T\}$ is then given by

$$C_t(\varphi) = V_t(\varphi) - \int_0^t \vartheta_s\, dX_s \tag{2.5}$$

for $0 \leq t \leq T$ and $\varphi = (\vartheta, \eta)$. A hedging strategy φ is considered to be self-financing if $C(\varphi)$ is P-a.s. constant over the time interval $[0, T]$ and φ is called mean self-financing if $C(\varphi)$ is a P-martingale.

3 Local Risk-Minimisation

Intuitively the goal of local risk-minimisation is to minimise the local risk which can be interpreted as the conditional second moment of cost increments under the measure P at each time instant.

With local risk-minimisation we only consider hedging strategies which replicate the contingent claim H at time T; that is we only allow hedging strategies φ such that

$$V_T(\varphi) = H \qquad\qquad P-\text{a.s.} \tag{3.1}$$

Subject to certain technical conditions it can be shown that finding a local risk-minimising strategy is equivalent to finding a decomposition of H in the form

$$H = H_0^{\text{lr}} + \int_0^T \xi_s^{\text{lr}}\, dX_s + L_T^{\text{lr}}, \tag{3.2}$$

where H_0^{lr} is constant, ξ^{lr} is a predictable square integrable process and $L^{\mathrm{lr}} = \{L_t^{\mathrm{lr}},\, 0 \leq t \leq T\}$ is a square integrable P-martingale with $L_0^{\mathrm{lr}} = 0$ and the product process $L^{\mathrm{lr}} M$ is in addition a P-martingale. The representation (3.2) is usually referred to as the Föllmer-Schweizer decomposition of H (see Föllmer & Schweizer (1991)).

The local risk-minimising hedging strategy is then given by

$$\vartheta_t^{\mathrm{lr}} = \xi_t^{\mathrm{lr}} \tag{3.3}$$

and

$$\eta_t^{\mathrm{lr}} = V_t(\varphi^{\mathrm{lr}}) - \vartheta_t^{\mathrm{lr}}\, X_t, \tag{3.4}$$

where

$$V_t(\varphi^{\mathrm{lr}}) = C_t(\varphi^{\mathrm{lr}}) + \int_0^t \vartheta_s^{\mathrm{lr}}\, dX_s \tag{3.5}$$

with

$$C_t(\varphi^{\mathrm{lr}}) = H_0^{\mathrm{lr}} + L_t^{\mathrm{lr}} \tag{3.6}$$

for $0 \leq t \leq T$.

As is shown in Föllmer & Schweizer (1991) and Schweizer (1995) there exists a measure $\hat{P}$, the so called minimal ELMM, such that

$$V_t(\varphi^{\mathrm{lr}}) = E_{\hat{P}}(H \mid \mathcal{F}_t) \tag{3.7}$$

for $0 \leq t \leq T$, where the conditional expectation in (3.7) is taken under $\hat{P}$. The measure $\hat{P}$ is identified, subject to certain integrability condition, by the Radon-Nikodym derivative

$$\frac{d\hat{P}}{dP} = Z_T^{\mathrm{lr}}, \tag{3.8}$$

where

$$Z_t^{\mathrm{lr}} = \exp\left(-\frac{1}{2}\int_0^t \left(\frac{\mu(s, Y_s)}{Y_s}\right)^2 ds - \int_0^t \frac{\mu(s, Y_s)}{Y_s}\, dW_s^1\right) \tag{3.9}$$

for $0 \leq t \leq T$.

Assuming Z^{lr} is a P-martingale, the Girsanov transformation can be used to show that the processes $\hat{W}^1$ and $\hat{W}^2$ defined by

$$\hat{W}_t^1 = W_t^1 + \int_0^t \frac{\mu(s, Y_S)}{Y_s}\, ds \tag{3.10}$$

and

$$\hat{W}_t^2 = W_t^2 \tag{3.11}$$

for $0 \le t \le T$ are independent Wiener processes under $\hat{P}$. Consequently using $\hat{W}^1$ and $\hat{W}^2$ the system of stochastic differential equations (2.1) becomes

$$\begin{aligned} dX_t &= X_t\, Y_t\, d\hat{W}_t^1 \\ dY_t &= \left(a(t, Y_t) - \frac{\varrho}{Y_t} (b\, \mu)(t, Y_t) \right) dt \\ &\quad + b(t, Y_t) \left(\varrho\, d\hat{W}_t^1 + \sqrt{1 - \varrho^2}\, d\hat{W}_t^2 \right) \end{aligned} \tag{3.12}$$

for $0 \le t \le T$.

Taking contingent claims of the form $H = h(X_T)$ for some given function $h : [0, \infty) \to I\!R$ and using the Markov property we can rewrite (3.7) in the form

$$\begin{aligned} V_t(\varphi^{\mathrm{lr}}) &= E_{\hat{P}}(h(X_T) \mid \mathcal{F}_t) \\ &= v_{\hat{P}}(t, X_t, Y_t) \end{aligned} \tag{3.13}$$

for some function $v_{\hat{P}}(t, x, y)$ defined on $[0, T] \times (0, \infty) \times (0, \infty)$. Subject to certain regularity conditions we can show that $v_{\hat{P}}$ is the solution to the partial differential equation

$$\frac{\partial v_{\hat{P}}}{\partial t} + \left(a - \frac{\varrho\, b\, \mu}{y} \right) \frac{\partial v_{\hat{P}}}{\partial y} + \frac{1}{2} \left(x^2\, y^2\, \frac{\partial^2 v_{\hat{P}}}{\partial x^2} + b^2\, \frac{\partial^2 v_{\hat{P}}}{\partial y^2} + 2\, \varrho\, x\, y\, b\, \frac{\partial^2 v_{\hat{P}}}{\partial x\, \partial y} \right) = 0 \tag{3.14}$$

on $(0, T) \times (0, \infty) \times (0, \infty)$ with boundary condition

$$v_{\hat{P}}(T, x, y) = h(x) \tag{3.15}$$

for $x, y \in (0, \infty)$.

Now it follows by application of the Ito formula together with (3.14) that

$$V_t(\varphi^{\mathrm{lr}}) = V_0(\varphi^{\mathrm{lr}}) + \int_0^t \vartheta_s^{\mathrm{lr}}\, dX_s + L_t^{\mathrm{lr}}, \tag{3.16}$$

where

$$\vartheta_t^{\mathrm{lr}} = \frac{\partial v_{\hat{P}}}{\partial x}(t, X_t, Y_t) + \frac{\varrho}{X_t\, Y_t} \left(b\, \frac{\partial v_{\hat{P}}}{\partial y} \right) (t, X_t, Y_t) \tag{3.17}$$

and

$$L_t^{\mathrm{lr}} = \int_0^t \left(b\, \sqrt{1 - \varrho^2}\, \frac{\partial v_{\hat{P}}}{\partial y} \right) (s, X_s, Y_s)\, dW_s^2 \tag{3.18}$$

for $0 \leq t \leq T$.

Using (3.6) and (3.18) we see that the conditional expected squared cost on the interval $[t, T]$ for the locally risk-minimising strategy φ^{lr}, denoted by R_t^{lr}, is given by

$$\begin{aligned} R_t^{\mathrm{lr}} &= E\left[\left(C_T(\varphi^{\mathrm{lr}}) - C_t(\varphi^{\mathrm{lr}})\right)^2 \,\middle|\, \mathcal{F}_t\right] \\ &= E\left[\int_t^T (1-\varrho^2)\left(b\left(\frac{\partial v_{\hat{P}}}{\partial y}\right)\right)^2 (s, X_s, Y_s)\, ds \,\middle|\, \mathcal{F}_t\right]. \end{aligned} \tag{3.19}$$

4 Mean-Variance Hedging

In this section we consider an alternative approach to hedging in incomplete markets based on what is called mean-variance hedging. Intuitively the goal here is to minimise the global risk over the entire time interval $[0, T]$. This contrasts with local risk-minimisation which focuses on minimisation of the second moments of cost increments.

With mean-variance hedging we allow strategies which do not fully replicate the contingent claim H at time T. However, we minimise the expected squared net loss at time T given in the form

$$R_0^{\mathrm{mvo}} = E\left((H - V_T(\varphi))^2\right) \tag{4.1}$$

over an appropriate choice of initiated value V_0 and hedge ratio ϑ. The initial value and hedge ratio which minimise this quantity is called the mean-variance optimal strategy and is denoted by $(V_0^{\mathrm{mvo}}, \vartheta_0^{\mathrm{mvo}})$. For a more precise specification of mean-variance hedging see Heath, Platen & Schweizer (ange).

Under suitable conditions it can be shown that there exists an ELMM $\tilde{P}$, the so-called variance-optimal ELMM, such that the contingent claim H admits a decomposition of the form

$$H = H_0^{\mathrm{mvo}} + \int_0^T \xi_s^{\mathrm{mvo}}\, dX_s + L_T^{\mathrm{mvo}}, \tag{4.2}$$

where

$$H_0^{\mathrm{mvo}} = E_{\tilde{P}}(H) \tag{4.3}$$

and ξ^{mvo} is a predictable square integrable process and L^{mvo} is a square integrable P-martingale with $L_0^{\mathrm{mvo}} = 0$.

If we choose

$$\eta_t^{\mathrm{mvo}} = H_0^{\mathrm{mvo}} + \int_0^t \vartheta_s^{\mathrm{mvo}}\, dX_s - \vartheta_t^{\mathrm{mvo}} X_t \tag{4.4}$$

and

$$C_t(\varphi^{\mathrm{mvo}}) = H_0^{\mathrm{mvo}} \tag{4.5}$$

for $0 \le t \le T$, then with this interpretation we see that the cost process is constant. In this sense our mean variance optimal strategy is self-financing. However, the net loss at time T is given by

$$\begin{aligned} H - V_T(\varphi^{\mathrm{mvo}}) &= H - H_0^{\mathrm{mvo}} - \int_0^T \vartheta_s^{\mathrm{mvo}} \, dX_s \\ &= L_T^{\mathrm{mvo}}. \end{aligned} \tag{4.6}$$

Under suitable conditions and with $\varrho = 0$ it can be shown that the variance optimal measure $\tilde{P}$ can be identified from its Radon-Nikodym derivative in the form

$$\frac{d\tilde{P}}{dP} = Z_T^{\mathrm{mvo}}, \tag{4.7}$$

where

$$\begin{aligned} Z_t^{\mathrm{mvo}} &= \exp\left(- \int_0^t \frac{\mu(s, Y_s)}{Y_s} \, dW_s^1 - \int_0^t \nu_s^{\mathrm{mvo}} \, dW_s^2 \right. \\ &\qquad \left. - \frac{1}{2} \int_0^t \left[\left(\frac{\mu(s, Y_s)}{Y_s} \right)^2 + (\nu_s^{\mathrm{mvo}})^2 \right] ds \right) \end{aligned} \tag{4.8}$$

with

$$\nu_t^{\mathrm{mvo}} = b(t, Y_t) \, \frac{\partial J}{\partial y}(t, Y_t) \tag{4.9}$$

and

$$J(t, y) = -\log E \left[\exp \left(- \int_t^T \left(\frac{\mu(s, Y_s^{t,y})}{Y_s^{t,y}} \right)^2 ds \right) \right] \tag{4.10}$$

for $t \le t \le T$. Here we denote by $Y^{t,y}$ the volatility process that starts at time t with value y and evolves according to the stochastic differential equation (2.1).

Applying the Feynman-Kac formula to the function $\exp(-J)$ and using a transformation of variables back to the function J it can be shown that, under appropriate conditions for a, b and μ, then J satisfies the partial differential equation

$$\frac{\partial J}{\partial t} + a \, \frac{\partial J}{\partial y} + \frac{1}{2} \, b^2 \, \frac{\partial^2 J}{\partial y^2} - \frac{1}{2} \, b^2 \left(\frac{\partial J}{\partial y} \right)^2 + \left(\frac{\mu}{y} \right)^2 = 0 \tag{4.11}$$

on $(0, T) \times (0, \infty)$ with boundary conditions

$$J(T, y) = 0.$$

Assuming Z^{mvo} is a P-martingale an application of the Girsanov transformation shows that the processes $\tilde{W}^1$ and $\tilde{W}^2$ defined by

$$\tilde{W}_t^1 = W_t^1 + \int_0^t \frac{\mu(s, Y_s)}{Y_s}\, ds, \tag{4.12}$$

and

$$\tilde{W}_t^2 = W_t^2 + \int_0^t \vartheta_s^{\text{mvo}}\, ds \tag{4.13}$$

for $0 \leq t \leq T$, are independent Wiener processes under $\tilde{P}$. Hence with respect to $\tilde{W}^1$ and $\tilde{W}^2$ the system of stochastic differential equations (2.1) becomes

$$\begin{aligned} dX_t &= X_t\, Y_t\, d\tilde{W}_t^1 \\ dY_t &= \left[a(t, Y_t) - b^2(t, Y_t)\, \frac{\partial J}{\partial y}(t, Y_t)\right] dt \\ &\quad + b(t, Y_t)\, d\tilde{W}_t^2 \end{aligned} \tag{4.14}$$

for $t \leq t \leq T$. Note that we have assumed $\varrho = 0$.

As in the case for local risk minimisation we consider European contingent claims of the form $H = h(X_T)$. For this type of payoff and again using the Markov property we can express by (4.3) and (4.5) the initial value $V_0(\varphi^{\text{mvo}})$ in the form

$$V_0^{\text{mvo}} = H_0^{\text{mvo}} = E_{\tilde{P}}[H] = v_{\tilde{P}}(0, X_0, Y_0) \tag{4.15}$$

for some function $v_{\tilde{P}}(t, x, y)$ defined on $[0, T] \times (0, \infty) \times (0, \infty)$ such that

$$v_{\tilde{P}}(t, X_t, Y_t) = E_{\tilde{P}}[H \mid \mathcal{F}_t]. \tag{4.16}$$

Subject to certain regularity conditions, it can be shown that $v_{\tilde{P}}$ is the solution of the partial differential equation

$$\frac{\partial v_{\tilde{P}}}{\partial t} + \left[a - b^2\, \frac{\partial J}{\partial y}\right] \frac{\partial v_{\tilde{P}}}{\partial y} + \frac{1}{2}\, x^2\, y^2\, \frac{\partial^2 v_{\tilde{P}}}{\partial x^2} + \frac{1}{2}\, b^2\, \frac{\partial^2 v_{\tilde{P}}}{\partial y^2} = 0 \tag{4.17}$$

on $(0, T) \times (0, \infty) \times (0, \infty)$ with boundary condition

$$v_{\tilde{P}}(T, x, y) = h(x) \tag{4.18}$$

for $x, y \in (0, \infty)$.

Similar to the case for local risk minimisation we can apply the Ito formula combined with (4.14) and (4.17) to obtain

$$v_{\tilde{P}}(t, X_t, Y_t) = v_{\tilde{P}}(0, X_0, Y_0) + \int_0^t \xi_s^{\text{mvo}}\, dX_s + L_t^{\text{mvo}}, \tag{4.19}$$

where

$$\xi_t^{\text{mvo}} = \frac{\partial v_{\tilde{P}}}{\partial x}(s, X_s, Y_s) \tag{4.20}$$

and

$$L_t^{\text{mvo}} = \int_0^t \left(b \, \frac{\partial v_{\tilde{P}}}{\partial y} \right) (s, X_s, Y_s) \, d\tilde{W}_s^2 \tag{4.21}$$

for $t \leq t \leq T$.

Also, under suitable conditions, it can be shown that the conditional expected squared net loss over the interval $[t, T]$, see (4.1), is given by

$$R_t^{\text{mvo}} = E\left[\int_t^T e^{-J(s,Y_s)} \, b^2(s, Y_s) \left(\frac{\partial v_{\tilde{P}}}{\partial y}(s, X_s, Y_s) \right)^2 ds \,\middle|\, \mathcal{F}_t \right]. \tag{4.22}$$

Furthermore, the mean variance optimal strategy $(V_0^{\text{mvo}}, \vartheta^{\text{mvo}})$, is given by

$$V_0^{\text{mvo}} = v_{\tilde{P}}(0, X_0, Y_0) \tag{4.23}$$

and

$$\vartheta_t^{\text{mvo}} = \xi_t^{\text{mvo}} + \frac{\mu(t, Y_t)}{X_t \, Y_t^2} \left(v_{\tilde{P}}(t, X_t, Y_t) - H_0^{\text{mvo}} - \int_0^t \vartheta_s^{\text{mvo}} \, dX_s \right). \tag{4.24}$$

Thus in the case of mean variance hedging the optimal hedge ratio ϑ^{mvo} is in general not equal to ξ^{mvo} which is the integrand appearing in the decomposition (4.2). This might not have been expected based on the results obtained for local risk minimisation and is due to the fact that ϑ_t^{mvo} has more the character of a control variable.

Finally, in the case where $\tilde{P} = \hat{P}$, so that $v_{\tilde{P}} = v_{\hat{P}}$, and, again subject to certain conditions, see Heath, Platen & Schweizer (ange), it can be shown that

$$R_t^{\text{mvo}} = E\left[\int_t^T e^{-J(s,Y_s)} \, (1 - \varrho^2) \, b^2(s, Y_s) \left(\frac{\partial v_{\hat{P}}}{\partial y}(s, X_s, Y_s) \right)^2 ds \,\middle|\, \mathcal{F}_t \right] \tag{4.25}$$

which is similar to (4.22) but includes the case $\varrho \neq 0$.

5 Some Specific Models

In this section we will consider the application of both local risk minimisation and mean variance hedging to four stochastic volatility models. The purpose of this study is to compare and also visualise results detained for the two hedging approaches for the given models. This will provide insight into qualitative and quantitative differences for the two quadratic hedging approaches.

Model	Type	Volatility Dynamics Y	Appreciation Rate μ
S1	Stein/Stein	$dY_t = \delta\,(\beta - Y_t)\,dt + k\,dW_t^2$	$\mu(t, Y_t) = \Delta\,Y_t$
S2	as above	as above	$\mu(t, Y_t) = \gamma\,Y_t^2$
H1	Heston	$dY_t^2 = \kappa\,(\theta - Y_t^2)\,dt + \Sigma\,Y_t\,(\varrho\,dW_t^1 + \sqrt{1-\varrho^2}\,dW_t^2)$	$\mu(t, Y_t) = \Delta\,Y_t$
H2	as above	$dY_t^2 = \kappa\,(\theta - Y_t^2)\,dt + \Sigma\,Y_t\,dW_t^2$	$\mu(t, Y_t) = \gamma\,Y_t^2$

Table 1: Model specifications.

The models which we examine are based on the Stein & Stein (1991) and Heston (1993) type stochastic volatility models with two different specifications for the appreciation rate function μ.

The four models with their specifications are summarised in Table 1. We assume that the constants δ, β, k, κ, θ, Σ are non-negative, with Δ and γ real valued and $\varrho \in [-1, 1]$. Note that non-zero correlation is allowed only for model H1. For the Heston type models H1 and H2 a stochastic differential equation for the volatility component Y can be obtained via Ito's formula as follows:

$$dY_t = \left(\frac{4\,\kappa\,(\theta - Y_t^2) - \Sigma^2}{8\,Y_t}\right) dt + \frac{\Sigma}{2}\left(\varrho\,dW_t^1 + \sqrt{1-\varrho^2}\,dW_t^2\right). \tag{5.1}$$

For models S1 and H1 it can be shown, see Heath, Platen & Schweizer (ange), that $\tilde{P} = \hat{P}$ and that

$$J(t, y) = \Delta^2 (T - t) \tag{5.2}$$

for $(t, y) \in [0, T] \times (0, \infty)$. Comparing (3.19) and (4.25) this means that

$$\begin{aligned} R_0^{\mathrm{mvo}} &= E\left[\int_0^T e^{-\Delta^2 (T-s)}\,(1-\varrho^2)\,b^2(s, Y_s)\left(\frac{\partial v_{\tilde{P}}}{\partial y}(s, X_s, Y_s)\right)^2 ds\right] \\ &\geq e^{-\Delta^2 T}\,R_0^{\mathrm{lr}}. \end{aligned} \tag{5.3}$$

In addition it can be shown that the local risk minimising strategy is given by (3.17).

In the next section we compute the local risk minimising strategies for both models S1 and H1 based on the formulae (3.12), (3.14), (3.17), (3.19). We note that the derivations and technical details provided in the papers Heath, Platen & Schweizer (ange) and Schweizer (1991) do not fully cover the case of $\varrho \neq 0$ for the model H1 that have also been included for comparative purposes in our study. However, the numerical results obtained do not indicate any particular problems with this case.

For the models S2 and H2 it can be shown, see again Heath, Platen & Schweizer (ange), that both the local risk minimising and mean-variance hedging strategies exist for the case of a European put option. Note that for mean-variance hedging existence of the optimal strategy is established only for a sufficiently small time horizon T. However in this case as well the numerical experiments have been successfully performed for long time scales without apparent difficulties as will be seen in the next section.

For the models S2 and H2 we have from (4.10) and Table 1 the function

$$J(t,y) = -\log E\left[\exp\left(-\gamma^2 \int_t^T (Y_s^{t,y})^2\, ds\right)\right]. \tag{5.4}$$

Fortunately for both models this function can be computed explicitly, see again Heath, Platen & Schweizer (ange). In the case of model S2 the function (5.4) here denoted by the symbol J_{S2} has the form

$$J_{\mathrm{S2}}(t,y) = f_0(T-t) + f_1(T-t)\,\frac{y}{k} + f_2(T-t)\,\frac{y^2}{k^2}, \tag{5.5}$$

where

$$\begin{aligned}
f_2(\tau) &= \frac{\lambda\,\gamma_1\, e^{-2\gamma_1\tau}}{\lambda+\gamma_1-\lambda\, e^{-2\gamma_1\tau}} - \lambda,\\
f_1(\tau) &= \frac{1}{1+2\,\lambda\,\psi(\tau)}\left((2\,D-D')\,e^{-2\gamma_1\tau} - 2\,D\,e^{-2\gamma_1\tau}\right) + D',\\
f_0(\tau) &= \frac{1}{2}\log(1+2\,\lambda\,\psi(\tau)) - \left(\lambda + \frac{\delta^2\,\beta^2}{2\,k^2}\left(\frac{\delta^2}{\gamma_1^2}-1\right)\right)\tau - \frac{2\,D^2\,\psi(\tau)}{1+2\,\lambda\,\psi(\tau)}\\
&\quad + \frac{\delta^2\,\beta}{k\,\gamma_1^2}\left(\frac{1}{1+2\,\lambda\,\psi(\tau)}\left(2\,D\,e^{-\gamma_1\tau} - \left(D-\frac{1}{2}D'\right)e^{-2\gamma_1\tau}\right) - \left(D+\frac{1}{2}D'\right)\right)
\end{aligned}$$

with constants

$$\gamma_1 = \sqrt{2\,k^2\,\gamma^2+\delta^2},\ \lambda = \frac{\delta-\gamma_1}{2},\ D = \frac{\delta\,\beta}{2\,k}\left(1-\frac{\delta^2}{\gamma_1^2}\right),\ D' = \frac{\delta\,\beta}{k}\left(1-\frac{\delta}{\gamma_1}\right)$$

and function

$$\psi(\tau) = \frac{1 - e^{-2\gamma_1 \tau}}{2\,\gamma_1}.$$

The $\tilde{P}$ dynamics for the volatility component Y can be obtained from (4.14) with the formula

$$\frac{\partial J_{\mathrm{S2}}}{\partial y}(t,y) = \frac{f_1(T-t)}{k} + \frac{2\,f_2(T-t)\,y}{k^2}. \tag{5.6}$$

In model H2 the function (5.4) is denoted by J_{H2} and is given by the expression

$$J_{\mathrm{H2}}(t,y) = -\log g(T-t) + \alpha(T-t)\,\gamma^2\,y^2, \tag{5.7}$$

where

$$\begin{aligned}
\alpha(\tau) &= \frac{2\,(e^{\Gamma_\tau} - 1)}{(\Gamma+\kappa)(e^{\Gamma_\tau}-1) + 2\,\Gamma}, \\
g(\tau) &= \left(\frac{2\,\Gamma\,e^{\frac{\Gamma+\kappa}{2}\tau}}{(\Gamma+\kappa)(e^{\Gamma_\tau}-1) + 2\,\Gamma}\right)^{\frac{2\kappa\vartheta}{\Sigma^2}}, \\
\Gamma &= \sqrt{2\,\gamma^2\,\Sigma^2 + \kappa^2}
\end{aligned}$$

Similarly the $\tilde{P}$ dynamics for the volatility component Y can be obtained from

$$\frac{\partial J_{\mathrm{H2}}}{\partial y}(t,y) = \alpha(T-t)\,2\,\gamma^2\,y. \tag{5.8}$$

For a justification of the approach using partial differential equations which is applied in the next section to all four combination of models, see (Heath & Schweizer 1998).

6 Computational Issues and Results

The purpose of this section is to compare actual numerical results for both hedging approaches for the models previously introduced. Emphasis will be placed on experiments which highlight differences in key quantities including prices, expected squared total costs and hedge ratios. For the four models and two hedging frameworks extensive experimentation has been performed with different parameter sets. Only a small subset of these results can be covered in this paper. Nevertheless these results indicate some crucial differences between the two approaches that might be of more general interest. In total eight different hedging problems had to be solved with corresponding numerical tools developed. For all numerical experiments considered here the contingent claim was taken to be a European put, see (2.3). This ensures the payoff function h is bounded and avoids certain integrability problems.

To solve numerically the partial differential equations (3.14)–(3.15) and (4.17)–(4.18) we employed finite difference approximations based on the Crank-Nicholson scheme. Some experimentation was also performed using the fully implicit scheme. To handle the two-dimensional structures appearing in (3.14) and (4.17) we used the method of fractional steps or operator splitting. For a discussion on these and related techniques, see Fletcher (1988) and Hoffman (1993).

Fractional step methods are usually easier to implement in the case where there is no correlation in the diffusion terms, that is $\varrho = 0$, and thus the term in (3.14) corresponding to the cross-term partial derivative $\frac{\partial^2 v_{\hat{P}}}{\partial x\, \partial y}$ is zero. In model H1 which allows for non-zero correlation we obtained an orthogonalised system of equations by introducing the transformation

$$U_t = \ln(X_t) - \frac{\varrho}{\Sigma} Y_t^2 \tag{6.1}$$

for $0 \le t \le T$ and $\Sigma > 0$.

By Ito's formula, together with (3.12) and (5.1) the evolution of U is governed by the stochastic differential equation

$$dU_t = \left[\left(\frac{\varrho\,\kappa}{\Sigma} - \frac{1}{2}\right) Y_t^2 - \frac{\varrho\,\kappa\,\beta}{\Sigma}\right] dt + Y_t \left[(1-\varrho^2)\, d\hat{W}_t^1 - \varrho\,\sqrt{1-\varrho^2}\, d\hat{W}_t^2\right] \tag{6.2}$$

for $0 \le t \le T$. Using this transformation for a European put option with strike price K, and subject to certain regularity conditions, we can apply the Kolmogorov backward equation to obtain a transformed function $u_{\hat{P}}$ defined on $[0,T] \times (0,\infty) \times (0,\infty)$ which is the solution of the partial differential equation

$$\begin{aligned} \frac{\partial u_{\hat{P}}}{\partial t} \quad &+ \quad \left[\left(\frac{\varrho\,\kappa}{\Sigma} - \frac{1}{2}\right) Y_t^2 - \frac{\varrho\,\kappa\,\beta}{\Sigma}\right] \frac{\partial u_{\hat{P}}}{\partial h} + \left(\frac{4\,\kappa\,\beta - \Sigma^2}{8\,y} - \frac{\kappa\,y}{2} - \frac{\varrho\,\Sigma\,\Delta}{2}\right) \frac{\partial u_{\hat{P}}}{\partial y} \\ &+ \quad -\frac{1}{2}\, y^2\,(1-\varrho^2)\,\frac{\partial u_{\hat{P}}}{\partial h^2} + \frac{\Sigma^2}{8}\,\frac{\partial u_{\hat{P}}}{\partial y^2} \end{aligned} \tag{6.3}$$

on $(0,T) \times (0,\infty) \times (0,\infty)$ with boundary condition

$$u_{\hat{P}}(T,h,y) = \left(K - \exp\left(h + \frac{\varrho\,y^2}{\Sigma}\right)\right)^+. \tag{6.4}$$

In terms of the original pricing function $v_{\hat{P}}$ we have the relation

$$v_{\hat{P}}(t,x,y) = u_{\hat{P}}(t, \ln(x) - \frac{\varrho\,y^2}{\Sigma}, y). \tag{6.5}$$

As noted previously for the model H1 we have $\tilde{P} = \hat{P}$ and the corresponding local risk minimising and mean-variance prices are the same.

For the numerical experiments described in this paper the following default parameter values were used: $X_0 = 100.0$, $Y_0 = 0.2$, $K = 100.0$, $\Delta = 0.5$, $\gamma = 2.5$, $\delta = 5.0$, $\beta = 0.2$, $k = 0.3$, $\kappa = 5.0$, $\beta = 0.04$, $\Sigma = 0.6$ and $\varrho = 0.0$.

To compute the expected squared costs on the interval $[t, T]$ given by (3.19) and (4.22) respectively, we introduce the functions ζ^{lr} and ζ^{mvo} defined on $[0, T] \times (0, \infty) \times (0, \infty)$ given by

$$\zeta^{\mathrm{lr}}(t, x, y) = (1 - \varrho^2)\, b^2(t, y) \left(\frac{\partial v_{\hat{P}}}{\partial y}(t, x, y) \right)^2 \tag{6.6}$$

and

$$\zeta^{\mathrm{mvo}}(t, x, y) = (1 - \varrho^2)\, e^{-J(t,y)}\, b^2(t, y) \left(\frac{\partial v_{\tilde{P}}}{\partial y}(t, x, y) \right)^2 \tag{6.7}$$

for $(t, x, y) \in [0, T] \times (0, \infty) \times (0, \infty)$.

Since $R_t^{\mathrm{lr}} = E\left(\int_t^T \zeta^{\mathrm{lr}}(s, X_s, Y_s)\, ds \,|\, \mathcal{F}_t \right)$ we can, assuming appropriate regularity conditions, apply the Kolmogorov backward equation together with (2.1) to show that there is a function r^{lr} defined on $[0, T] \times (0, \infty) \times (0, \infty)$ such that

$$r^{\mathrm{lr}}(t, X_t, Y_t) = R_t^{\mathrm{lr}}$$

is the solution to the partial differential equation

$$\frac{\partial r^{\mathrm{lr}}}{\partial t} + x\, \mu\, \frac{\partial r^{\mathrm{lr}}}{\partial x} + a\, \frac{\partial r^{\mathrm{lr}}}{\partial y} + \frac{1}{2} \left(x^2\, y^2\, \frac{\partial^2 r^{\mathrm{lr}}}{\partial x^2} + b^2\, \frac{\partial^2 r^{\mathrm{lr}}}{\partial y^2} + 2\, x\, y\, b\, \varrho\, \frac{\partial^2 r^{\mathrm{lr}}}{\partial x\, \partial y} \right) + \zeta^{\mathrm{lr}} = 0 \tag{6.8}$$

on $(0, T) \times (0, \infty) \times (0, \infty)$ with boundary condition

$$r^{\mathrm{lr}}(T, x, y) = 0 \tag{6.9}$$

for $(x, y) \in (0, \infty) \times (0, \infty)$. Since $R_t^{\mathrm{mvo}} = E\left(\int_t^T \zeta^{\mathrm{mvo}}(s, X_s, Y_s)\, ds \,|\, \mathcal{F}_t \right)$ a completely analogous result holds for the function r^{mvo} with ζ^{mvo} replacing ζ^{lr} in (6.8).

Here we have used the system of equations (2.1) because for both hedging approaches the expected squared costs are computed under the real-world measure P. Note that for numerical solvers applied to (6.8) together with (6.9) the solutions to the pricing functions $v_{\hat{P}}$ and $v_{\tilde{P}}$ need to be pre-computed or at least made available at the current time step. For the model H1 with $\varrho \neq 0$ the transformed variable U_t, see (6.1), can be introduced to obtain orthogonalised equations for both hedging approaches, as has been explained for the pricing function $v_{\hat{P}}$.

To illustrate the difference in expected squared costs $(R_0^{\mathrm{lr}} - R_0^{\mathrm{mvo}})$ over the time interval $[0, T]$ we show in Figure 1 for the model H1 these differences using different

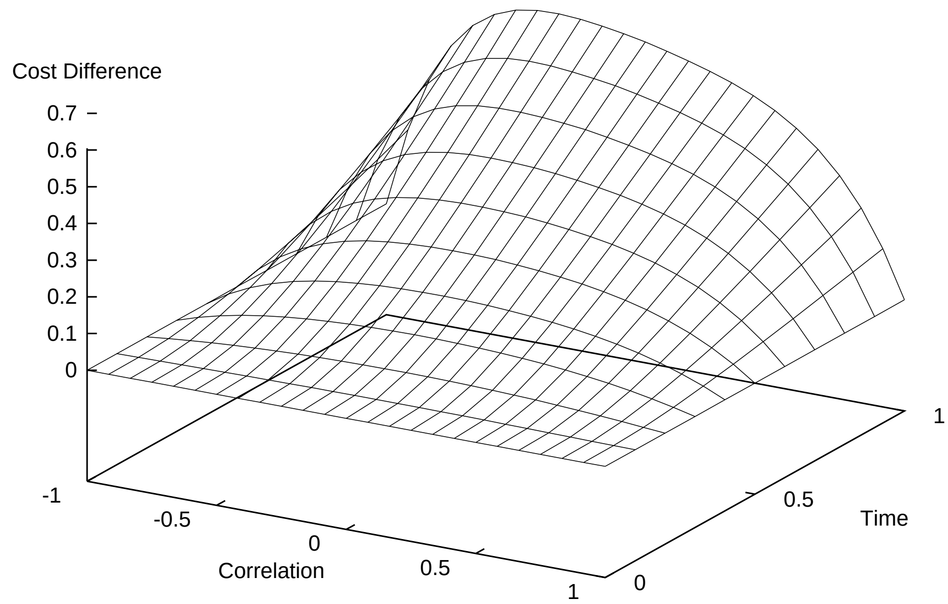

Figure 1: Expected squared costs differences ($R_0^{\mathrm{lr}} - R_0^{\mathrm{mvo}}$) for the model H1.

values for the correlation parameter ϱ and time to maturity T. The absolute values of expected squared costs increase as T increases. For $T = 1.0$ and $\varrho = 0.0$ the computed values for prices and expected squared costs were $V_0(\varphi^{\mathrm{lr}}) = V_0(\varphi^{\mathrm{mvo}}) = 7.691$, $R_0^{\mathrm{lr}} = 4.257$ and $R_0^{\mathrm{mvo}} = 3.685$. For $T = 1.0$ and $\varrho = -0.5$ the computed values were $V_0(\varphi^{\mathrm{lr}}) = V_0(\varphi^{\mathrm{mvo}}) = 10.662$, $R_0^{\mathrm{lr}} = 4.429$ and $R_0^{\mathrm{mvo}} = 3.836$. Both R_0^{lr} and R_0^{mvo} tend to zero as $|\varrho|$ tends to 1 as can be expected from equations (3.19) and (4.24).

For increasing time to maturity T our numerical results indicate that R_0^{mvo} tends to zero. This observation is highlighted in Figure 2 which displays both R_0^{lr} and R_0^{mvo} over the time interval $[0, 100]$. In this sense the market can be considered as being "asymptotically complete" with respect to the mean-variance criterion. Similar results which raise important questions concerning asymptotic completeness are obtained for the other models H1, S2 and H2.

For the models S2 and H2 the quadratic drift specifications, see Table 1, mean that $\hat{P} \neq \tilde{P}$ and consequently different prices are usually obtained for the two distinct measures and hedging strategies. Figure 3 illustrates these price differences for the model H2 using different values for time to maturity T and moneyness $\ln(\frac{X_0}{K})$.

For at the money options typical price differences of the order of 2–3% were obtained. For example, with input values $T = 1.0$ and $X_0 = K = 100.0$ the computed prices were $V_0(\varphi^{\mathrm{lr}}) = 7.6945$ and $V_0(\varphi^{\mathrm{mvo}}) = 7.892$. However for $T = 1.0$ and $\ln(\frac{X_0}{K}) = 0.3$ the estimated prices were $V_0(\varphi^{\mathrm{lr}}) = 0.764$ and $V_0(\varphi^{\mathrm{mvo}}) = 0.848$. For

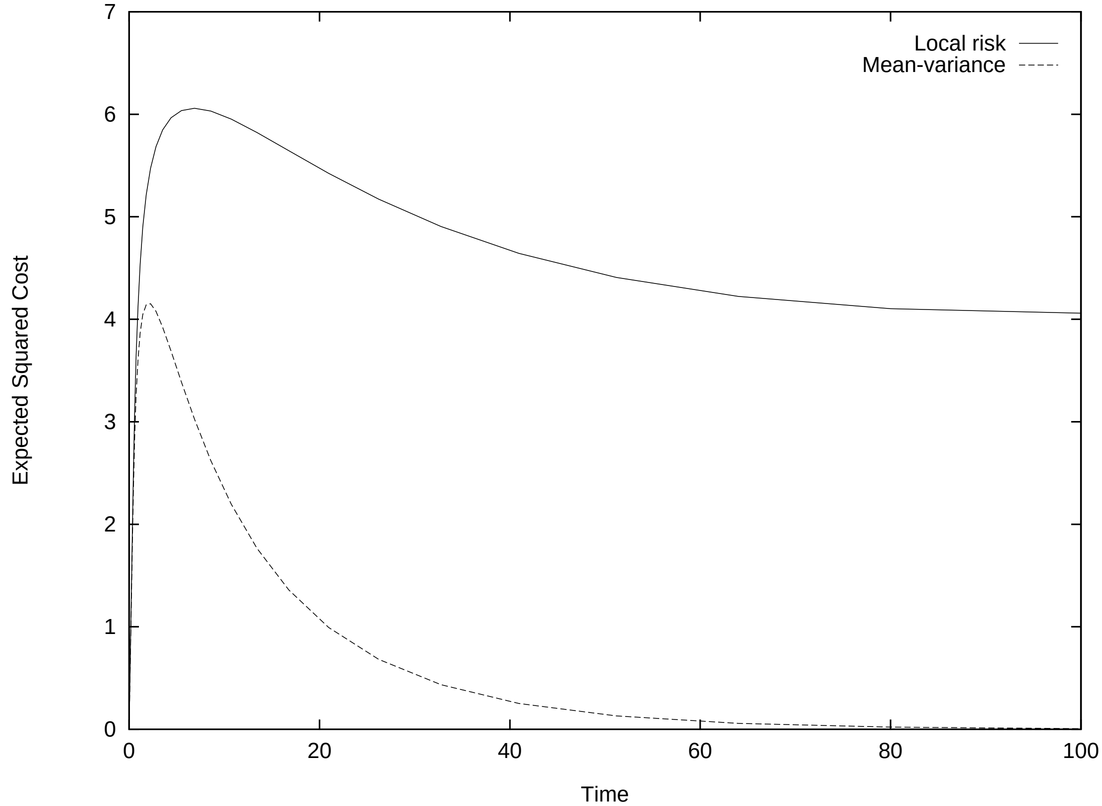

Figure 2: Expected squared costs $R_0^{\rm lr}$ and $R_0^{\rm mvo}$ over long time periods for the model S1.

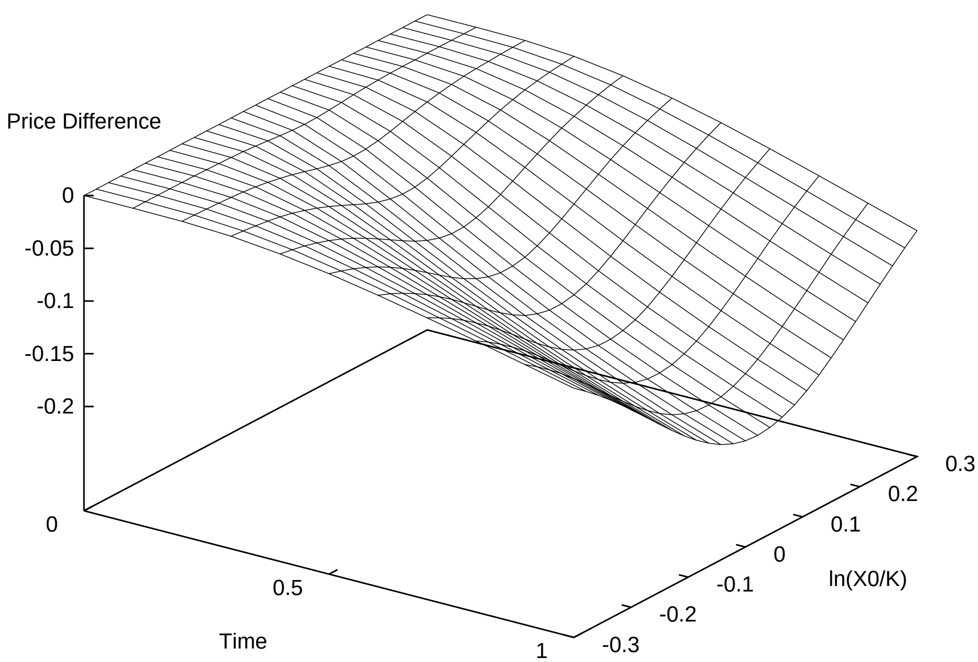

Figure 3: Price differences $(V_0(\varphi^{\rm lr}) - V_0(\varphi^{\rm mvo}))$ for the model H2.

all data points computed local risk minimisation prices were less than corresponding mean-variance prices, hence the differences shown in Figure 3 are negative. This means that for the parameter set and model considered there is no obvious best candidate when choosing between the two hedging approaches. Mean-variance hedging delivers lower expected squared costs but it also results in higher prices.

We will now consider the computation of hedge ratios ϑ^{lr} and ϑ^{mvo} for local risk minimising and mean-variance hedging given by (3.17) and (4.23), respectively. Our aim will be to obtain approximate hedge ratios at equi-spaced discrete times $0 = t_0 < t_1 < \ldots < t_N = T$ with step size $t_i - t_{i-1} = \frac{T}{N}$ for $i \in \{1, \ldots, N\}$ using simulation techniques. Noting the form of (3.17) and (4.23) it is apparent that the price functions $v_{\hat{P}}$ and $v_{\tilde{P}}$ need to be pre-computed in order to calculate hedge ratios.

Once $v_{\hat{P}}$ and $v_{\tilde{P}}$ are determined, say on a discrete grid by a numerical solver, the partial derivatives appearing in (3.17) and (4.23) can be approximated using finite differences.

To simulate via a Monte-Carlo method a given sample path for the vector (X, Y) under the measure P, an order 1.0 weak predictor-corrector numerical scheme, see Kloeden & Platen (1999), was applied to the system of equations (2.1) to obtain a set of estimates $(\bar{X}_{t_i}, \bar{Y}_{t_i})$ for (X_{t_i}, Y_{t_i}) for $i \in \{0, \ldots, N\}$ with $\bar{X}_0 = X_0$ and $\bar{Y}_0 = Y_0$. From these a set of approximate values for the hedge ratio $\bar{\vartheta}^{\mathrm{lr}}_{t_i}$ and integrand $\bar{\xi}^{\mathrm{mvo}}_{t_i}$, $i \in \{0, \ldots, N\}$ which correspond to $\vartheta^{\mathrm{lr}}_{t_i}$ and $\xi^{\mathrm{mvo}}_{t_i}$ respectively, can be obtained. One problem with this procedure is that the set of points $(t_i, \bar{X}_{t_i}, \bar{Y}_{t_i})$ for $i \in \{0, \ldots, N\}$ may not lie on the grid used to compute $v_{\hat{P}}$ and $v_{\tilde{P}}$. This difficulty can be overcome by the application of multi-dimensional interpolation methods.

The estimates $\bar{\vartheta}^{\mathrm{mvo}}_{t_i}$, $i \in \{0, \ldots, N\}$ for the mean variance hedge ratio can now be obtained from the Euler type approximation scheme

$$\bar{\vartheta}^{\mathrm{mvo}}_{t_i} = \bar{\xi}^{\mathrm{mvo}}_{t_i} + \frac{\mu(t_i, \bar{Y}_{t_i})}{\bar{X}_{t_i} \bar{Y}^2_{t_i}} \left(v_{\tilde{P}}(t_i, \bar{X}_{t_i}, \bar{Y}_{t_i}) - v_{\tilde{P}}(0, X_0, Y_0) - \sum_{j=0}^{i-1} \bar{\xi}^{\mathrm{mvo}}_{t_j} (\bar{X}_{t_{j+1}} - \bar{X}_{t_j}) \right) \tag{6.10}$$

for $i \in \{1, \ldots, N\}$. In the case of the models S2 and H2 we have $\hat{P} \neq \tilde{P}$. In general this means that $v_{\hat{P}} \neq v_{\tilde{P}}$ and $\frac{\partial v_{\hat{P}}}{\partial x} \neq \frac{\partial v_{\tilde{P}}}{\partial x}$ and consequently it follows from (3.17), (4.20) and (4.23) with $\varrho = 0$ that for the initial hedge ratios $\bar{\vartheta}^{\mathrm{lr}}_0 \neq \bar{\vartheta}^{\mathrm{mvo}}_0$. For models S1 and H1, since $v_{\hat{P}} = v_{\tilde{P}}$, we then get equal initial hedge ratios $\bar{\vartheta}^{\mathrm{lr}}_0 = \bar{\vartheta}^{\mathrm{mvo}}_0$. This equality does not in general hold for $t \in (0, T)$.

Figure 4 and 5 plot the linearly interpolated hedge ratios $\bar{\vartheta}^{\mathrm{lr}}_{t_i}$ and $\bar{\vartheta}^{\mathrm{mvo}}_{t_i}$, $i \in \{0, \ldots, N\}$ for a European put option for model S2. Figure 4 displays hedge ratios for a sample path ending in-the-money whereas Figure 5 shows hedge ratios for a different sample path ending out-of-the-money. The trajectories for $X/100$ and Y for both sample paths are illustrated in Figure 6. Note that the mean variance

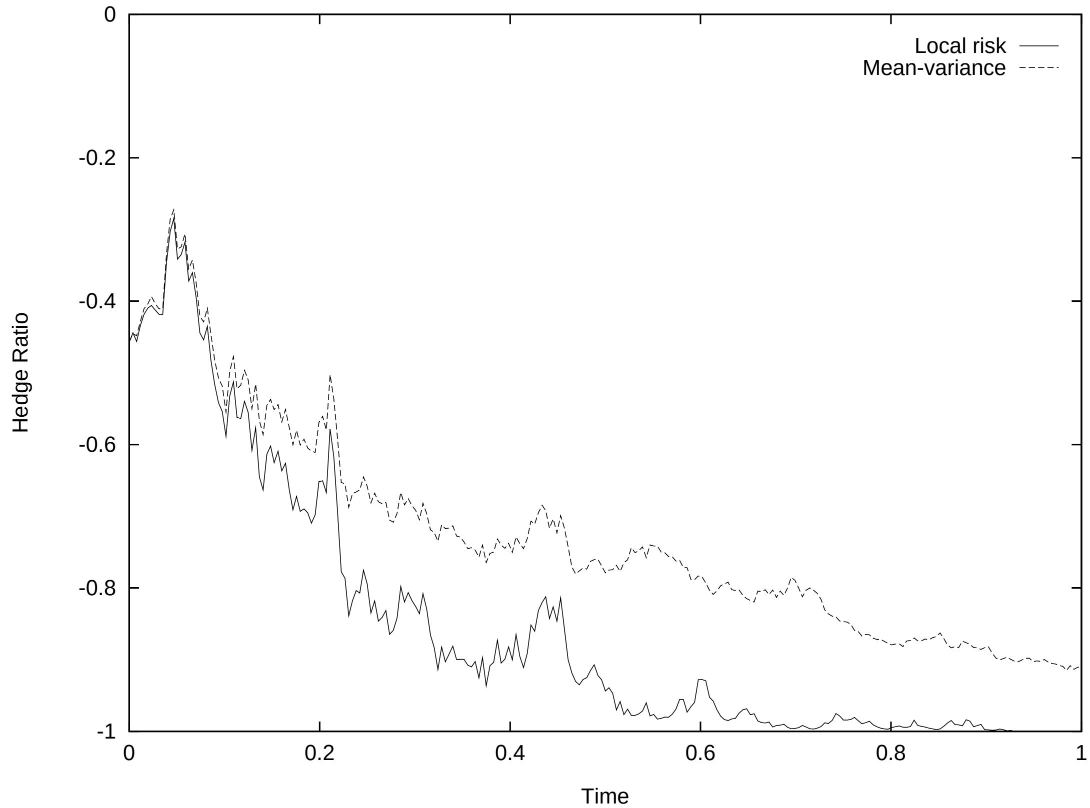

Figure 4: Hedge ratios: Sample path ending in-the-money.

hedge ratio takes values in the interval $(0, -1)$ at maturity. This indicates that there is no full replication of the contingent claim.

In the case of the linear drift models S1 and H1 the factor $\frac{\mu(t_i, \bar{Y}_{t_i})}{\bar{X}_{t_i} \bar{Y}_{t_i}^2}$ appearing in (6.10) becomes $\frac{\Delta}{\bar{X}_{t_i} \bar{Y}_{t_i}}$. This factor becomes $\frac{\gamma}{\bar{X}_{t_i}}$ for the quadratic drift models S2 and H2. For the given default parameter set the approximate volatility values $\bar{Y}_{t_i}$, $i \in \{0, \ldots, N\}$ can become quite small. Consequently for the linear drift models large fluctuations in the mean variance (compared to local risk) hedge ratios can occur. Simulation experiments have shown that these differences are not so apparent for the quadratic drift models.

7 Conclusion

This paper documents some of the differences between local risk minimisation and mean variance hedging for some specific stochastic volatility models. Over long time periods it seems that the mean variance criterion leads to a form of asymptotic completeness which is not the case for local risk minimisation. For the models S2 and H2 mean variance hedging delivered lower expected squared costs but high prices. The results described in this paper raise a number of interesting theoretical and practical questions for future research.

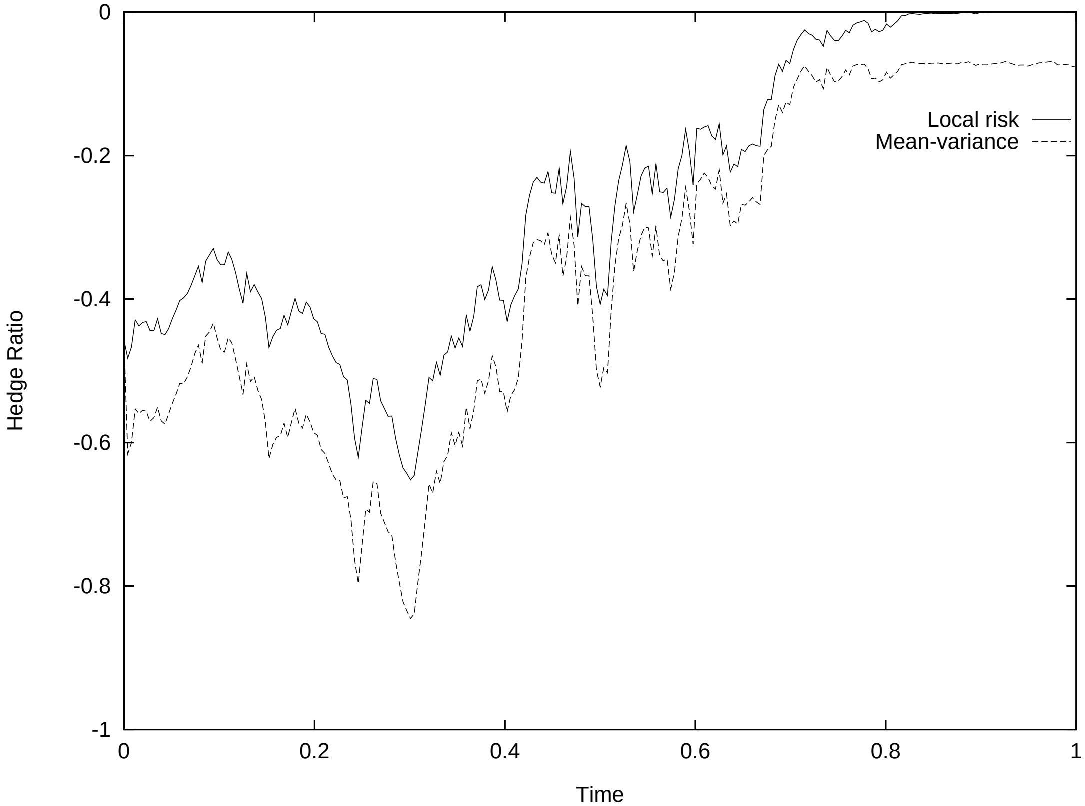

Figure 5: Hedge ratios: Sample path ending out-of-the-money.

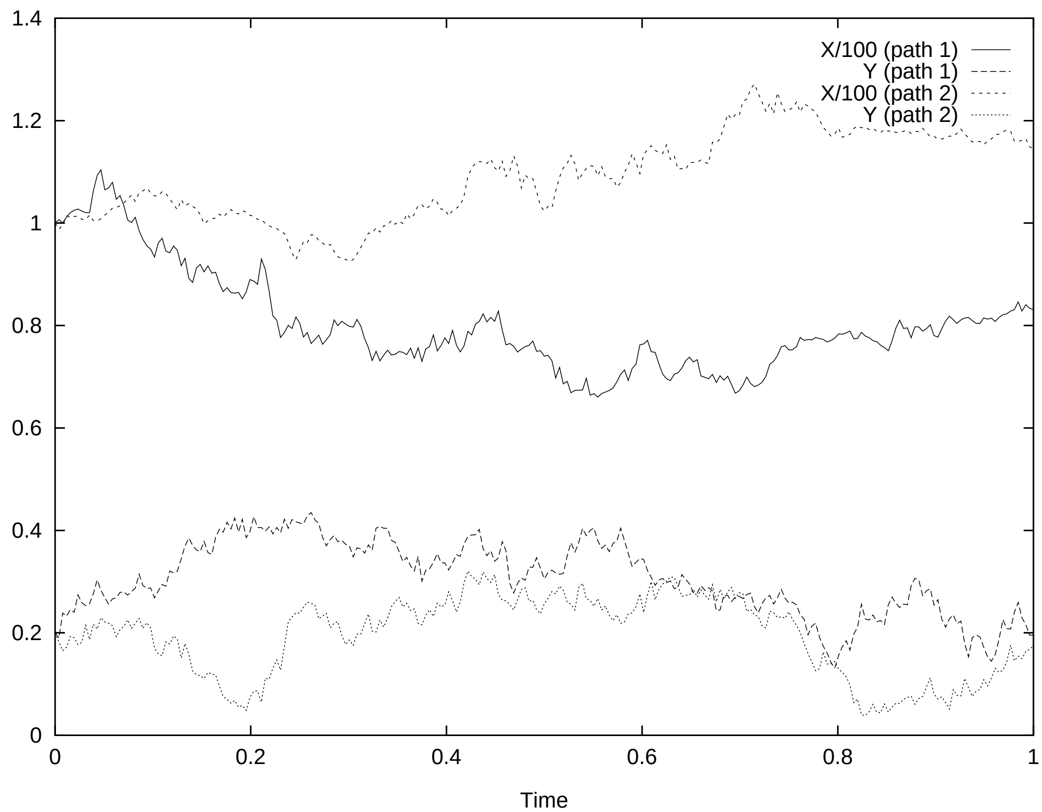

Figure 6: Two pairs of sample paths

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