A Microeconomic Approach to Diffusion Models for Stock Prices*

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Abstract:

This paper studies a class of diffusion models for stock prices derived by a microeconomic approach. We consider discrete-time processes resulting from a market equilibrium and then apply an invariance principle to obtain a continuous-time model. The resulting process is an Ornstein-Uhlenbeck process in a random environment, and we analyze its qualitative behavior. In particular, we provide simple criteria for the stability or instability of the corresponding stock price model, and we give explicit formulae for the invariant distributions in the recurrent case.

Key words:

stock price models, invariance principle, Ornstein-Uhlenbeck process, random environment, invariant distribution, noise traders, information traders

1. Introduction

The price evolution of a risky asset in a financial market is usually described as a stochastic process $S = (S_t)_{t\geq 0}$ on some probability space $(\Omega, \mathcal{F}, P)$. A famous example is the introduction of *Brownian motion* by Bachelier (1900) as a model for price fluctuation on the Paris stock market. A rigorous construction of Brownian motion as a stochastic process was given by Wiener (1923), and a standard Brownian motion with constant variance 1 is often called a *Wiener process*. Samuelson (1964) suggested replacing Brownian motion by *geometric Brownian motion*, i.e., by the pathwise solution

$$S_t = S_0 \exp\big(\sigma W_t + mt\big) \tag{1.1}$$

of the linear stochastic differential equation

$$dS_t = \sigma S_t \, dW_t + \mu S_t \, dt, \tag{1.2}$$

with respect to a Wiener process $W = (W_t)$, where $m = \mu - \frac{1}{2}\sigma^2$. This model is now widely used as a standard reference model, in particular in the context of option pricing and hedging; see Black/Scholes (1973). There have been several attempts to derive (1.1) from economic considerations. Bachelier (1900) and later Samuelson (1965) both concluded from a heuristic equilibrium argument that the price process S, properly discounted, should be a martingale. If one additionally assumes that price increments are stationary and that the paths of S are continuous, then Lévy's theorem implies that S must be a Brownian motion. If we assume instead that the increments of the process $\log S$ are stationary then we are led to the canonical model (1.1).

In recent years, there has been a renewed interest in the derivation of diffusion models for stock prices. From a microeconomic point of view, Kreps (1982) showed that the model (1.1) can be sustained in a suitably chosen rational expectations equilibrium where all agents believe in that model; see also Bick (1987) and Brockett/Witt (1991). In this approach the beliefs and the preferences of the agents have to be specified in a delicate way, and so it does not explain why (1.1) should play the role of a robust reference model. Another motivation is provided by the theory of option hedging. The standard hedging arguments assume that one considers a small investor whose actions do not influence prices. But if hedging is carried out on a large scale, this assumption is no longer realistic since the implementation of trading strategies is likely to affect the underlying stock price process. This calls for a closer look at the microeconomic picture behind the random fluctuation of stock prices, and for a distinction of different types of agents' behavior.

Our goal in this paper is to carry out a case study for the derivation of diffusion models for stock prices which combines the microeconomic point of view with an invariance principle. The robustness of diffusion models such as (1.1) hence will be explained by a functional central limit theorem. In section 2 we consider a model for the fluctuation of stock prices in discrete time. The stock price S_k in period k is an equilibrium price determined by the demand of the agents who are active on the market in that period. Individual demand may involve liquidity demand, a subjective assessment of what an adequate price should be, as well as a technical demand arising from dynamic strategies of portfolio insurance. In this paper we shall model these relations in the setting of a simple log-linear structure of individual excess demand.

If liquidity demand is the only source of randomness in our model, and if this randomness has a classical i.i.d. structure, then logarithmic stock prices $X_k = \log S_k$ perform a random walk with a drift. By the invariance principle, this random walk converges under suitable rescaling to a Brownian motion

$$X_t = \sigma W_t + mt \tag{1.3}$$

with volatility σ and drift parameter m. This means that stock prices themselves will converge to a geometric Brownian motion (1.1); see, e.g., Duffie/Protter (1992). But if excess demand involves other components then we should expect a different equilibrium price structure and, via an invariance principle, a different limiting diffusion model. Following Black (1986), we consider in particular a class of *information traders* and a class of *noise traders*. In our model the excess demand of information traders depends in a log-linear way on their perceptions of an underlying fundamental level, while noise traders take the proposed equilibrium price as a signal for that level. In a log-linear simplification, the demand of technical traders is analogous to the demand of noise traders. The process of logarithmic stock prices, centered around a sequence of aggregates of perceived fundamental levels, then takes the form

$$X_k - X_{k-1} = \beta_k X_{k-1} + \varepsilon_k. \tag{1.4}$$

Here we have two sources of randomness. The quantities (ε_k) are averages of individual liquidity demand. The behavioral quantities (β_k) aggregate the individual agents' ways of reacting to a proposed equilibrium price based on their perceptions of what an adequate price should be. In this second source, randomness may for instance appear as a fluctuation in the proportion between different types of agents. Information traders contribute negative values to β_k. If only information traders are active on the market, the logarithmic price process therefore performs a recurrent fluctuation of Ornstein-Uhlenbeck type around the aggregate fundamental levels. But if the effect of noise trading and of technical trading becomes too dominant then β_k may assume positive values. In such periods the Ornstein-Uhlenbeck process changes from its usual recurrent to a highly transient behavior. With a different interpretation, a model of the form (1.4) is also considered by Orléan/Robin (1991) who discuss its stability, using a criterion of Brandt (1986).

To make the qualitative behavior of the process (1.4) more transparent and explicit, we apply in section 3 an invariance principle to its two sources of randomness to pass to a diffusion model in continuous time. This leads to a stochastic differential equation of the form

$$dX_t = X_t(\tilde{\sigma}_t d\tilde{W}_t + \tilde{m}_t dt) + \sigma_t dW_t + m_t dt \tag{1.5}$$

where $\tilde{W}$ and W are Wiener processes with covariance process $d\langle \tilde{W}, W\rangle_t = \varrho_t dt$. In the simple special case

$$dX_t = \tilde{m} X_t dt + \sigma dW_t, \tag{1.6}$$

the price process is a *geometric Ornstein-Uhlenbeck process.* For $\tilde{m} < 0$, this process is ergodic with a log-normal invariant distribution and provides a natural reference model in the class of *stationary* price processes. The solution of the general case (1.5) may be viewed as an *Ornstein-Uhlenbeck process in a random environment.* In section 4 we investigate its qualitative behavior under the assumption that $e_t = (\tilde{\sigma}_t, \tilde{m}_t, \sigma_t, m_t, \varrho_t)$ is an ergodic process. Extending a result of Brandt (1986) from discrete time to the setting of (1.5), we derive a bound for the aggregate effect of noise trading which assures that the induced price fluctuation remains a stationary ergodic process. Beyond that bound the price process becomes highly transient. In fact we shall show that the paths converge either to 0 or to ∞, and that their growth or decay exceeds an exponential rate.

In section 5 we consider the special case where the process e is a deterministic constant. The random environment of the Ornstein-Uhlenbeck process is then described by the Wiener process $\tilde{W}$. For $\tilde{m} < \frac{1}{2}\tilde{\sigma}^2$ the price fluctuation is ergodic, and we can give explicit formulae for the density of the invariant distribution. But the situation is more volatile than in the classical case (1.6). Typically, the invariant distribution is a mixture of normal distributions, and the mixing measure on the variances has unbounded support. In particular we obtain fatter tails than in the classical model (1.1), and S does not have moments of any order. Explicit examples also show that a continuous but singular distribution may appear as the mixing measure.

The present paper works out some ideas which were stated in an informal manner in Föllmer (1991). Thanks are due to Alan Kirman whose suggestion to consider models with a random fluctuation between different groups of agents provided the initial motivation for this work; see also Kirman (1993).

2. The micro-economic model

Let us describe a simple model for the evolution of the price of a speculative asset. We consider a finite set A of agents who are active on the market. Given a proposed stock price p, each agent $a \in A$ forms an *excess demand* $e_a(p)$. The *equilibrium stock price* S is then determined by the market clearing condition of zero total excess demand. Since we think of a temporal sequence of markets at discrete times t_k $(k = 0, 1, ...)$, we add a time subscript k to A, e_a and S. We also add a parameter ω which summarizes all variables other than p which may influence agents' decisions. We view ω as a sample point in some underlying probability space $(\Omega, \mathcal{F}, P)$, and so the temporal sequence of equilibrium prices becomes a stochastic process

$$S_k(\omega) \qquad (k = 0, 1, ...) \tag{2.1}$$

defined by the implicit equations

$$\sum_{a \in A_k(\omega)} e_{a,k}\big(S_k(\omega), \omega\big) = 0. \tag{2.2}$$

In order to derive explicit results with a minimum of technicalities, we assume that the individual excess demand in period k is of the log-linear form

$$(2.3) \qquad e_{a,k}(p,\omega) = \alpha_{a,k}(\omega) \log \frac{\hat{S}_{a,k}(\omega)}{p} + \delta_{a,k}(\omega),$$

typically with $\alpha_{a,k} \geq 0$. Here $\delta_{a,k}$ may be viewed as a liquidity demand, and $\hat{S}_{a,k}$ denotes an individual reference level of agent a for period k. For instance, $\hat{S}_{a,k}$ could be interpreted as a price expectation for the following period, but we will also consider other interpretations in the examples below. Clearly, the particular form (2.3) is too simple to be a realistic model of excess demand and can only be viewed as a first approximation to a more general demand behavior. In particular, we have not imposed any explicit budget constraint and we leave aside interest rates. The log-linear excess demand function is often used in monetary models; see for instance Cagan (1956), Gourieroux/Laffont/Monfort (1982) or Laidler (1985).

The explicit form (2.3) of individual excess demand permits us to solve (2.2) for the equilibrium stock price to obtain

$$(2.4) \qquad \log S_k(\omega) = \sum_{a \in A_k(\omega)} \bar{\alpha}_{a,k}(\omega) \log \hat{S}_{a,k}(\omega) + \delta_k(\omega)$$

with

$$(2.5) \qquad \bar{\alpha}_{a,k} = \Big(\sum_{a \in A_k} \alpha_{a,k}\Big)^{-1} \alpha_{a,k}, \qquad \delta_k = \Big(\sum_{a \in A_k} \alpha_{a,k}\Big)^{-1} \sum_{a \in A_k} \delta_{a,k}.$$

Thus, the actual logarithmic equilibrium stock price is a weighted average of individual logarithmic price assessments and of liquidity demands.

(2.6) **Remark.** In a model of a *rational expectations equilibrium*, one would assume that all agents have beliefs consistent with the underlying probability measure P. Specifically, let us suppose that the individual price assessments are given by

$$(2.7) \qquad \hat{S}_{a,k} = E\left[S_{k+1} | \mathcal{F}_{t_k}\right],$$

where $\mathcal{F}_{t_k}$ is the σ-algebra of events observable to all agents up to time t_k. In the absence of liquidity demand, (2.4) and (2.7) would imply

$$(2.8) \qquad S_k = E\left[S_{k+1} | \mathcal{F}_{t_k}\right]$$

so that stock prices form a *martingale*; see Tirole (1982). Note that there is no discounting here since our excess demand itself does not contain a discount factor. In contrast to this approach, we do *not* assume in our subsequent discussion that the objective probability measure P on Ω is known to the agents.

Consider the simple special case $\hat{S}_{a,k} = S_{k-1}$ so that

$$\log S_k(\omega) = \log S_{k-1}(\omega) + \delta_k(\omega). \tag{2.9}$$

Under standard i.i.d. assumptions on the sequence (δ_k) of aggregate liquidity demands, the logarithmic price process $X = \log S$ becomes a random walk. Under a passage from discrete to continuous time with suitable rescaling, the process X would converge to a Brownian motion with drift. Thus, the resulting price process $S = \exp(X)$ would be a geometric Brownian motion as in (1.1). But as soon as the individual assessments $\hat{S}_{a,k}$ depend in a more complex manner on past experience, on individual perceptions of fundamental values or of the proposed price taken as a signal, we must expect that other diffusion models will appear in the limit. In the following examples we try to capture, in a very simplified manner, different types of agents' behavior by introducing different specifications of the individual reference level $\hat{S}_{a,k}$.

(2.10) **Examples.** **1)** For an *information trader*, or *fundamentalist*, the individual reference level is determined by his current perception $F_{a,k}(\omega)$ of the fundamental value of the asset and by his belief how the actual stock price will be attracted to that value. Specifically, let us assume that an information trader chooses a reference level of the form

$$\log \hat{S}_{a,k} = \log S_{k-1} + \beta_{a,k}(\log S_{k-1} - \log F_{a,k}) \tag{2.11}$$

with a random coefficient $\beta_{a,k}(\omega) \leq 0$. If only such information traders were active on the market, the resulting price process would be of the form

$$\log S_k = \log S_{k-1} + \beta_k(\log S_{k-1} - \log F_k) + \delta_k \tag{2.12}$$

with

$$\beta_k = \sum_{a \in A_k} \bar{\alpha}_{a,k}\beta_{a,k}, \qquad \log F_k = \frac{1}{\beta_k} \sum_{a \in A_k} \bar{\alpha}_{a,k}\beta_{a,k} \log F_{a,k}. \tag{2.13}$$

As will be explained more carefully below, this suggests that the logarithmic price process induced by information traders behaves like an Ornstein-Uhlenbeck process around a time-dependent level. It will be an *Ornstein-Uhlenbeck process in a random environment* since both the levels and the coefficients (β_k) are random processes.
2) In a simplified model of *noise trading*, we assume that the reference level of a noise trader is of the form

$$\log \hat{S}_{a,k} = \log S_{k-1} + \gamma_{a,k}(\log S_{k-1} - \log p) \tag{2.14}$$

with some random coefficient $\gamma_{a,k}(\omega) \leq 0$. Thus, the proposed price is taken seriously as a signal and replaces the fundamental quantity $F_{a,k}$ in (2.11). Suppose that only such noise traders are active on the market. Then we have

$$\log S_k = \log S_{k-1} + \gamma_k(\log S_{k-1} - \log S_k) + \delta_k, \tag{2.15}$$

hence

$$\log S_k = \log S_{k-1} + \varepsilon_k, \tag{2.16}$$

where

$$\varepsilon_k = (1+\gamma_k)^{-1}\delta_k, \qquad \gamma_k = \sum_{a\in A_k} \bar{\alpha}_{a,k}\gamma_{a,k}. \tag{2.17}$$

This suggests that the logarithmic price process should have the structure of a random walk. We will see below that the effect of noise trading becomes more drastic if noise traders interact with information traders.
3) In a simplistic log-linear approximation, the technical excess demand arising from dynamical strategies of portfolio insurance (see for instance Black/Jones (1987)) would take the form

$$\alpha_{a,k}(\log S_{k-1} - \log p) \tag{2.18}$$

with a random coefficient $\alpha_{a,k}(\omega) \le 0$. Taken together with the agent's liquidity demand, the resulting excess demand is of the form (2.3) with $\hat{S}_{a,k} = S_{k-1}$, but now with a coefficient of negative sign. If only such traders are active on the market, it follows as in 2) that the resulting logarithmic price process will have the structure of a random walk. Here again, the effect of such traders will become more drastic if they start to interact with information traders.

Let us now study the interactive effect of the different types of behavior described in the preceding examples. To this end we assume that the individual reference level is of the form

$$\log \hat{S}_{a,k} = \log S_{k-1} + \beta_{a,k}(\log S_{k-1} - \log F_{a,k}) + \gamma_{a,k}(\log S_{k-1} - \log p) \tag{2.19}$$

with random coefficients $\beta_{a,k}(\omega) \le 0$ and $\gamma_{a,k}(\omega) \le 0$. As in (2.12) and (2.16), the resulting price process takes the form

$$\log S_k = \log S_{k-1} + \beta_k(\log S_{k-1} - \log F_k) + \varepsilon_k \tag{2.20}$$

where

$$\gamma_k = \sum_{a\in A_k} \bar{\alpha}_{a,k}\gamma_{a,k}, \qquad \beta_k = (1+\gamma_k)^{-1}\sum_{a\in A_k} \bar{\alpha}_{a,k}\beta_{a,k}, \qquad \varepsilon_k = (1+\gamma_k)^{-1}\delta_k, \tag{2.21}$$

and where F_k is a logarithmic mixture of the individual assessments $F_{a,k}$. Note that the random coefficients β_k may become positive if the effect of either the noise traders (large absolute values of γ_k) or the portfolio insurers (negative values of $\alpha_{a,k}$) becomes too strong.

Equation (2.20) will be the basis for our subsequent analysis. Note that it contains three, in general correlated, sources of randomness: the behavioral quantities (β_k), the aggregate liquidity demand (ε_k), and the uncertainty about the fundamentals contained in the aggregate quantities (F_k). If we define the level L_k recursively by

$$L_k = (1+\beta_k)L_{k-1} - \beta_k \log F_k, \qquad L_0 = \log F_0,$$

then the process

$$X_k = \log S_k - L_k \qquad (k = 0, 1, ...)$$

satisfies

$$X_k - X_{k-1} = \beta_k X_{k-1} + \varepsilon_k. \tag{2.22}$$

Thus, the logarithmic price process may be viewed as an *Ornstein-Uhlenbeck process in a random environment* which fluctuates around the time-dependent process (L_k). More precisely, the process (X_k) is an Ornstein-Uhlenbeck process centered around 0 where both the additive quantities (ε_k) and the multiplicative quantities (β_k) are random processes. Recall that β_k may become positive if either noise trading or portfolio insurance becomes too dominant. In such periods, the Ornstein-Uhlenbeck process will change from its usual recurrent behavior to a highly unstable transient behavior. These features will be analyzed in more detail for the diffusion approximation obtained in the next section.

In this paper, we are not interested in that part of the randomness which is induced by fluctuations in the fundamentals. In fact, experiments clearly show that one should expect a random fluctuation of asset prices even if the uncertainty about the fundamentals is completely eliminated; cf., e.g., Smith/Suchanek/Williams (1988). In the context of our model, we will concentrate on the effect of the two sources (ε_k) and (β_k) of randomness. Therefore we may as well assume that the sequence (F_k), and hence the sequence (L_k), is equal to some deterministic constant. Thus, our price process is of the form

$$S_k = \exp(X_k + L) \qquad (k = 0, 1, ...) \tag{2.23}$$

where the process (X_k) is given by equation (2.22).

(2.24) **Remark.** Price processes of the type (2.22) have previously appeared in the literature; see for instance Froot/Obstfeld (1991), Shiller (1981), Summers (1986) or West (1988). In our present approach, we motivate these processes from a microeconomic point of view, in terms of assumptions at the level of individual agents. This will allow us to study the effect that different types of behavior and a random fluctuation of the proportion between different groups of agents have on the resulting equilibrium stock price process. Similar models have been suggested or studied by Black (1986), Day/Huang (1989), De Long/Shleifer/Summers/Waldmann (1990a,b), Föllmer (1974), Frankel/Froot (1986), Goodhart (1988), Grossman (1988), Hart/Kreps (1986) and Kirman (1993), among others.

3. Convergence to a diffusion model

In this section, we shall obtain a continuous-time stock price process S by a passage to the limit from the discrete-time equilibrium price processes derived in the previous section. The convergence concept we use is weak convergence on the Skorohod space D^d of all $\mathbb{R}^d$-valued right-continuous functions with left limits on $[0, \infty)$, endowed with the Skorohod topology; see for instance Billingsley (1968) or Jacod/Shiryayev (1987). A similar approach to obtain a continuous-time model from a sequence of suitably specified discrete-time processes was taken by Nelson (1990).

For each n, we consider a process $(S_k^n)_{k=0,1,\dots}$ given by

$$S_k^n = \exp(X_k^n + L) = S_0^n \exp(X_k^n - X_0^n) \tag{3.1}$$

with

$$X_k^n - X_{k-1}^n = \beta_k^n X_{k-1}^n + \varepsilon_k^n \tag{3.2}$$

as in (2.22). We assume that the initial value $S_0^n = S_0$ is fixed.

If $(Z_k^n)_{k=0,1,\dots}$ is any discrete-time process, we identify Z^n with the continuous-time process $Z_t^n := Z_{[nt]}^n$ $(0 \le t < \infty)$ whose paths are right-continuous. Our goal is to obtain a convergence result for the processes $(S^n)_{n \in \mathbb{N}}$, and this will be achieved by applying an invariance principle to the processes

$$\tilde{Z}_t^n = \sum_{k=1}^{[nt]} \beta_k^n, \qquad Z_t^n = \sum_{k=1}^{[nt]} \varepsilon_k^n \tag{3.3}$$

induced by the two sources of randomness in equation (3.2). Note first that equation (3.1) is equivalent to the stochastic differential equation

$$dX_t^n = X_{t-}^n d\tilde{Z}_t^n + dZ_t^n \tag{3.4}$$

in terms of $\tilde{Z}^n$ and Z^n. Under standard assumptions on the processes (β_k^n) and (ε_k^n), the sequence $(\tilde{Z}^n, Z^n)$ is "good" in the sense of the following definition, and converges in distribution to a continuous semimartingale $(\tilde{Z}, Z)$ whose pathwise covariance process will be denoted by $\langle \tilde{Z}, Z \rangle$; see for instance Duffie/Protter (1992) and Kurtz/Protter (1991a,b).

Definition. Suppose that for each n, we have a filtration $\mathbb{F}^n = (\mathcal{F}_t^n)_{t \ge 0}$ on $(\Omega^n, \mathcal{F}^n, P^n)$, and that there is also a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \ge 0}$ on $(\Omega, \mathcal{F}, P)$, all filtrations satisfying the usual conditions of right-continuity and completeness. For each n, let Z^n be a semimartingale and H^n an $\mathbb{F}^n$-adapted process, both on $(\Omega^n, \mathcal{F}^n, P^n)$, and let Z and H be $\mathbb{F}$-adapted processes on $(\Omega, \mathcal{F}, P)$. All these processes are assumed to have paths in D^1. The sequence $(Z^n)_{n \in \mathbb{N}}$ is called *good* if for any such sequence $(H^n)_{n \in \mathbb{N}}$, the weak convergence of (H^n, Z^n) to (H, Z) in D^2 implies that Z is a semimartingale and that $(H^n, Z^n, \int H_-^n \, dZ^n)$ converges weakly to $(H, Z, \int H_- \, dZ)$ in D^3.

(3.5) **Theorem.** *Suppose that* $(\tilde{Z}^n, Z^n)$ *is good and converges in distribution to the continuous semimartingale* $(\tilde{Z}, Z)$*. Then* $(\tilde{Z}^n, Z^n, X^n)$ *converges in distribution to* $(\tilde{Z}, Z, X)$ *where* X *is the strong solution*

$$X_t = \exp(\tilde{Z}_t - \frac{1}{2}\langle\tilde{Z}\rangle_t)(X_0 + \int_0^t \exp\big(-(\tilde{Z}_s - \frac{1}{2}\langle\tilde{Z}\rangle_s)\big)d(Z - \langle\tilde{Z}, Z\rangle)_s) \tag{3.6}$$

of the stochastic differential equation

$$dX_t = X_t d\tilde{Z}_t + dZ_t. \tag{3.7}$$

Moreover, the price processes S^n *(n = 1,2,...) converge in distribution to the process*

$$S_t = \exp(X_t + L). \tag{3.8}$$

Proof. The weak convergence of $(\tilde{Z}^n, Z^n, X^n)$ in D^3 to $(\tilde{Z}, Z, X)$ follows from the results of Słomiński (1989); see also Kurtz/Protter (1991a,b). Itô's formula implies that (3.6) solves (3.7); cf. Protter (1990), Theorem V.52. Finally, the weak convergence of S^n to S follows by the continuous mapping theorem.

(3.9) **Examples. 1)** Suppose that the first source of randomness in equation (3.2) can be neglected as we pass to the continuous-time limit, i.e., assume that $\tilde{Z} = 0$. Then (3.6) reduces to

$$X_t = X_0 + Z_t. \tag{3.10}$$

Under homogeneity assumptions on the second source of randomness, Z will be a Brownian motion with constant variance and constant drift:

$$dZ_t = \sigma dW_t + m dt. \tag{3.11}$$

In that case, the price process S is given by the usual reference model

$$S_t = S_0 \exp(\sigma W_t + mt) \tag{3.12}$$

of *geometric Brownian motion.*
2) Suppose that, in the limit, the first source of randomness in equation (3.2) produces an absolutely continuous drift, but no additional noise. In other words, assume that

$$\tilde{Z}_t = \int_0^t \tilde{m}_s ds \qquad (t \geq 0). \tag{3.13}$$

If Z is given by (3.11) with $m = 0$, the limiting equation (3.7) reduces to

$$dX_t = \tilde{m}_t X_t dt + \sigma dW_t. \tag{3.14}$$

In the special homogeneous case where $\tilde{m}_t(\cdot) = \tilde{m}$, we obtain

$$dX_t = \tilde{m}X_t dt + \sigma dW_t, \tag{3.15}$$

i.e., X is an *Ornstein-Uhlenbeck process*, transient for $\tilde{m} > 0$ and recurrent for $\tilde{m} < 0$. In this model, the logarithmic price process $\log S$ performs an Ornstein-Uhlenbeck fluctuation around the level L, and the price process S will be called a *geometric Ornstein-Uhlenbeck process.* In the *recurrent* case $\tilde{m} < 0$, this process may be viewed as a canonical reference model in the class of *stationary* price processes. In the general case (3.14) where the coefficient fluctuates randomly, the process X is an *Ornstein-Uhlenbeck process in a random environment.*

In the next section we consider a general class of such processes in a random environment and study their qualitative behavior in situations where the environment is ergodic.

4. Ornstein-Uhlenbeck processes in an ergodic random medium

In this section we want to analyze the behavior of the limiting stock price process S corresponding to equation (3.7) in a situation where the two sources of randomness form a stationary ergodic environment. More precisely, we assume that $(\tilde{Z}, Z)$ is of the form

$$d\tilde{Z}_t = \tilde{m}_t dt + \tilde{\sigma}_t d\tilde{W}_t, \qquad dZ_t = m_t dt + \sigma_t dW_t \tag{4.1}$$

with

$$d\langle \tilde{Z}, Z\rangle_t = \tilde{\sigma}_t \sigma_t d\langle \tilde{W}, W\rangle_t = \gamma_t dt, \tag{4.2}$$

where $\tilde{W}, W$ are standard Brownian motions with random correlation (ϱ_t) so that $\gamma_t = \tilde{\sigma}_t \sigma_t \varrho_t$. We assume that the process

$$e_t = (\tilde{m}_t, \tilde{\sigma}_t, m_t, \sigma_t, \gamma_t) \text{ is ergodic,} \tag{4.3}$$

and that (e_t) and the white noise processes corresponding to $\tilde{W}, W$ are defined for all times $t \in I\!\!R$ on some probability space $(\Omega, \mathcal{F}, P)$. We also introduce the integrability assumptions

$$\tilde{m}_t, \tilde{\sigma}_t^2, m_t, \sigma_t^2, \gamma_t \in \mathcal{L}^1(\Omega, \mathcal{F}, P). \tag{4.4}$$

Let us first prove some general facts concerning the pathwise asymptotic behavior of the stock price process $S = (S_t)$.

(4.5) **Theorem.** *If*

$$\tilde{c} = E[\tilde{m}_0 - \frac{1}{2}\tilde{\sigma}_0^2] > 0, \tag{4.6}$$

then the stock price S exhibits superexponential growth or decay, i.e.,

$$\lim_{t\uparrow\infty} \frac{1}{t} \log|\log S_t| = \tilde{c} \qquad P-a.s. \tag{4.7}$$

Proof. 1) The continuous martingale

$$\tilde{M}_t = \int_0^t \tilde{\sigma}_s d\tilde{W}_s \qquad (t \geq 0) \tag{4.8}$$

satisfies

$$\lim_{t\uparrow\infty} \frac{\tilde{M}_t}{t} = 0 \qquad P-a.s. \tag{4.9}$$

This is clear if $E[\tilde{\sigma}_0^2] = 0$. If $E[\tilde{\sigma}_0^2] > 0$ then the quadratic variation

$$\langle \tilde{M} \rangle_t = \int_0^t \tilde{\sigma}_s^2 ds \qquad (t \geq 0) \tag{4.10}$$

of $\tilde{M}$ satisfies

$$\lim_{t\uparrow\infty} \frac{1}{t} \langle \tilde{M} \rangle_t \;=\; E[\tilde{\sigma}_0^2] \; > 0 \quad P-a.s. \tag{4.11}$$

by the ergodic theorem. In particular we have $\langle \tilde{M} \rangle_\infty = \infty$, hence

$$\lim_{t\uparrow\infty} \frac{\tilde{M}_t}{\langle \tilde{M} \rangle_t} \;=\; 0 \qquad P-a.s. \tag{4.12}$$

by the law of large numbers for continuous martingales. But (4.12) together with (4.11) implies (4.9).

2) By the ergodic theorem, the process

$$\tilde{C}_t \;=\; \int_0^t (\tilde{m}_s - \frac{1}{2}\tilde{\sigma}_s^2) ds \tag{4.13}$$

satisfies

$$\lim_{t\uparrow\infty} \frac{1}{t} \tilde{C}_t \;=\; \tilde{c} \qquad P-a.s. \tag{4.14}$$

We have

$$X_t = (X_0 + M_t + C_t) \exp(\tilde{M}_t + \tilde{C}_t) \tag{4.15}$$

with

$$M_t = \int_0^t e^{-(\tilde{M}_s+\tilde{C}_s)}\sigma_s dW_s \tag{4.16}$$

and

$$C_t = \int_0^t e^{-(\tilde{M}_s+\tilde{C}_s)}(m_s-\gamma_s)ds. \tag{4.17}$$

Due to (4.9) and (4.14), it is enough to show that both M and C converge P-a.s. to a finite limit. Again by (4.9) and (4.14), the total variation $|C|$ of C satisfies

$$|C|_t \;\le\; |C|_{t(\varepsilon)} + \int_0^\infty e^{-(\tilde{c}-\varepsilon)s}|m_s-\gamma_s|ds \tag{4.18}$$

for $0<\varepsilon<\tilde{c}$, and the second term has finite expectation due to assumption (4.4). This implies the convergence of C to some finite limit, P-a.s. Similarly, the local martingale M has quadratic variation

$$\langle M\rangle_t \;\le\; \langle M\rangle_{t(\varepsilon)} + \int_0^\infty e^{-2(\tilde{c}-\varepsilon)s}\sigma_s^2 ds, \tag{4.19}$$

where the second term has finite expectation. Thus M converges P-a.s. to some finite limit.

Theorem (4.5) says that for $\tilde{c}>0$, stock prices are not stable in any sense: The trajectories either tend to 0 or go off to infinity. Note that this is quite different from the behavior of geometric Brownian motion (3.12) where

$$\lim_{t\uparrow\infty}\frac{1}{t}\log S_t = \mu-\frac{1}{2}\sigma^2 \qquad P-a.s. \tag{4.20}$$

According to the sign of $\mu-\frac{1}{2}\sigma^2$, the geometric Brownian motion either goes to 0 P-a.s. or to $+\infty$ P-a.s., in both cases at an exponential rate. In contrast, growth and decay in Theorem (4.5) can both occur with positive probability, and the situation is much more instable since the convergence rate is higher than exponential.

We have seen in the previous section that a recurrent geometric Ornstein-Uhlenbeck process may be viewed as a canonical *stationary* model for price fluctuations. Let us now show in our general situation that we get recurrent behavior if the environment is on average not too destabilizing, i.e., if $\tilde{c}<0$. The following result is a continuous-time version of a theorem of Brandt (1986) for discrete-time processes of the form (2.22).

In addition to (4.4), we need some further integrability assumptions in order to guarantee that

$$\log^+\Big(\int_0^1 e^{-(\tilde{Z}_s-\frac{1}{2}\langle\tilde{Z}\rangle_s)}d(Z-\langle\tilde{Z},Z\rangle)_s\Big)\in\mathcal{L}^1. \tag{4.21}$$

By standard estimates on the moments of stochastic integrals, condition (4.21) follows if, e.g., $m_0, \sigma_0^2, \gamma_0 \in \mathcal{L}^2$ and $E[e^{p\tilde{\sigma}_0^2}] < \infty$ for $p = 16$.

(4.22) **Theorem.** *If*

$$\tilde{c} = E[\tilde{m}_0 - \frac{1}{2}\tilde{\sigma}_0^2] < 0, \tag{4.23}$$

the logarithmic price process X converges to an ergodic process:

$$\lim_{t\uparrow\infty} |X_t - \hat{X}_0 \circ \theta_t| = 0 \qquad P - a.s. \tag{4.24}$$

where

$$\hat{X}_0 = \int_{-\infty}^{0} \exp\Big(- \int_s^0 d(\tilde{Z} - \frac{1}{2}\langle\tilde{Z}\rangle)_u\Big) d(Z - \langle\tilde{Z}, Z\rangle)_s \tag{4.25}$$

is P-a.s. well-defined. In particular, the distribution of X_t converges to the distribution of $\hat{X}_0$.

Proof. 1) The formal definition (4.25) of $\hat{X}_0$ should be read as

$$\hat{X}_0 = \lim_{n\to-\infty} e^{\tilde{V}_0 - \tilde{V}_n} \int_n^0 e^{-(\tilde{V}_s - \tilde{V}_n)} dV_s, \tag{4.26}$$

where

$$dV_s = dZ_s - d\langle\tilde{Z}, Z\rangle_s, \qquad d\tilde{V}_s = d\tilde{Z}_s - \frac{1}{2} d\langle\tilde{Z}\rangle_s. \tag{4.27}$$

In order to show that this limit exists P-a.s., we write

$$\hat{X}_0 = \lim_{n\to-\infty} \sum_{n\le k<0} \Big(\prod_{k\le\ell<0} e^{\tilde{V}_{\ell+1} - \tilde{V}_\ell} \Big) \int_k^{k+1} e^{-(\tilde{V}_s - \tilde{V}_k)} dV_s. \tag{4.28}$$

We have

$$\begin{aligned} &\frac{1}{|k|} \log \left(\Big(\prod_{k\le\ell<0} e^{\tilde{V}_{\ell+1} - \tilde{V}_\ell} \Big) \int_k^{k+1} e^{-(\tilde{V}_s - \tilde{V}_k)} dV_s \right) \\ &\le \frac{1}{|k|} \sum_{k\le\ell<0} (\tilde{V}_1 - \tilde{V}_0) \circ \theta^\ell + \frac{1}{|k|} (\log^+ B) \circ \theta^k \end{aligned} \tag{4.29}$$

with

$$B = \int_0^1 e^{-(\tilde{V}_s - \tilde{V}_0)} dV_s. \tag{4.30}$$

Since we have assumed $\log^+ B \in \mathcal{L}^1$, the ergodic theorem implies the P-a.s. convergence of the right-hand side of (4.29) to

$$E[\tilde{V}_1 - \tilde{V}_0] = \tilde{c} < 0. \tag{4.31}$$

This shows that the limit on the right-hand side of (4.28) does exist as a finite random variable.

2) The process $\hat{X}_t = \hat{X}_0 \circ \theta_t \quad (t \geq 0)$ is stationary and ergodic and satisfies

$$\hat{X}_t = e^{\tilde{V}_t - \tilde{V}_0}(\hat{X}_0 + \int_0^t e^{-(\tilde{V}_s - \tilde{V}_0)} dV_s). \tag{4.32}$$

Thus,

$$\hat{X}_t - X_t = e^{\tilde{V}_t - \tilde{V}_0}(\hat{X}_0 - X_0), \tag{4.33}$$

and since

$$\lim_{t\uparrow\infty} \frac{1}{t}(\tilde{V}_t - \tilde{V}_0) = \lim_{t\uparrow\infty} \frac{1}{t}(\tilde{M}_t + \int_0^t (\tilde{m}_s - \frac{1}{2}\tilde{\sigma}_s^2) ds) = \tilde{c} < 0, \tag{4.34}$$

we obtain (4.24). The ergodic theorem for $(\hat{X}_t)$ together with (4.34) implies

$$\lim_{t\uparrow\infty} \frac{1}{t} \int_0^t f(X_s) ds = E[f(\hat{X}_0)] \quad P - a.s. \tag{4.35}$$

for any bounded Lipschitz function f, hence weak convergence of the distribution of X_t to the distribution of $\hat{X}_0$.

In the recurrent case (4.23) we have seen that the invariant distribution of the logarithmic price process is given by the distribution μ of the random variable

$$\hat{X}_0 = \int_{-\infty}^0 \exp\left(-\int_s^0 (\tilde{\sigma}_u d\tilde{W}_u + (\tilde{m}_u - \frac{1}{2}\tilde{\sigma}_u^2) du)\right)(\sigma_s dW_s + (m_s - \gamma_s) ds). \tag{4.36}$$

If the Wiener process W is independent of e and $\tilde{W}$, then the conditional distribution of $\hat{X}_0$, given e and $\tilde{W}$, is Gaussian, and so the invariant distribution μ is a mixture of normal distributions. If, moreover, $\tilde{\sigma}$ and $\tilde{m}$ are deterministic, it is easy to compute the moments of μ.

In the next section we consider the case where X is an ergodic diffusion process and derive explicit formulae for the density of the invariant distribution μ.

5. The invariant distribution

Let us now consider the special case of equation (4.1) where $\tilde\sigma, \tilde m, \sigma, m, \varrho$ are deterministic constants. Thus, the process X is a Markovian diffusion with stochastic differential equation

$$dX_t = X_t(\tilde\sigma\, d\tilde W_t + \tilde m\, dt) + \sigma\, dW_t + m\, dt \tag{5.1}$$

where W, $\tilde W$ are Wiener processes with constant correlation ϱ. Condition (4.23) reduces to

$$\tilde m < \frac{1}{2}\tilde\sigma^2, \tag{5.2}$$

and in this case X is an ergodic diffusion whose invariant distribution μ can be computed explicitly.

(5.3) **Theorem.** *Assume condition (5.2). If $\tilde\sigma^2 = 0$ then $\tilde m < 0$, and the invariant distribution is a normal distribution:*

$$\mu = \mathcal{N}\left(-\frac{m}{\tilde m}, -\frac{\sigma^2}{2\tilde m}\right). \tag{5.4}$$

If $\tilde\sigma^2 > 0$ and $|\varrho| < 1$ then the invariant distribution is given by the density

$$\text{const}\cdot \big(\sigma^2(1-\varrho^2) + (\tilde\sigma x + \varrho\sigma)^2\big)^{-\left(1-\frac{\tilde m}{\tilde\sigma^2}\right)} \exp\left(-\frac{2(\varrho\tilde m\sigma - m\tilde\sigma)}{\sigma\tilde\sigma^2\sqrt{1-\varrho^2}} \arctan\frac{\tilde\sigma x + \varrho\sigma}{\sigma\sqrt{1-\varrho^2}}\right); \tag{5.5}$$

for $\varrho = m = 0$ this reduces to

$$\text{const}\cdot \left(1 + \frac{\tilde\sigma^2}{\sigma^2}x^2\right)^{-\left(1-\frac{\tilde m}{\tilde\sigma^2}\right)} \tag{5.6}$$

For $\tilde\sigma > 0$ and $|\varrho| = 1$ the density is

$$\text{const}\cdot (\tilde\sigma x + \varrho\sigma)^{-2\left(1-\frac{\tilde m}{\tilde\sigma^2}\right)} \exp\left(\frac{2(\varrho\tilde m\sigma - m\tilde\sigma)}{\tilde\sigma^2(\tilde\sigma x + \varrho\sigma)}\right). \tag{5.7}$$

Proof. By (5.1), X satisfies the stochastic differential equation

$$dX_t = (\tilde m X_t + m)\, dt + \sqrt{\sigma^2 + 2\varrho\sigma\tilde\sigma X_t + \tilde\sigma^2 X_t^2}\, dB_t \tag{5.8}$$

for some Wiener process B. By Kolmogorov's formula (see for instance Theorem IV.7 of Mandl (1968)), the density h of the invariant measure μ is given by

$$\big(\log h(x)\big)' = -2\frac{\varrho\sigma\tilde\sigma - m + (\tilde\sigma^2 - \tilde m)x}{\sigma^2 + 2\varrho\sigma\tilde\sigma x + \tilde\sigma^2 x^2} \tag{5.9}$$

and therefore belongs to the family of Pearson type distributions. Integration leads to (5.4) – (5.7); see Johnson/Kotz (1970).

(5.10) **Remark.** The density in (5.5) is a Pearson type IV distribution; this is interesting since it seems that so far, "no common statistical distributions are of type IV" (Johnson/Kotz (1970)).

Let us now consider the case where $m = 0$ and $\varrho = 0$ so that

$$dX_t = X_t(\tilde{\sigma}\, d\tilde{W}_t + \tilde{m}\, dt) + \sigma\, dW_t \tag{5.11}$$

with independent Wiener processes W and $\tilde{W}$. We can then think of $\tilde{W}$ as an environment of the diffusion which depends on some independent variable η.

(5.12) **Remark.** Equation (5.11) can be viewed as a special limiting case of the equation

$$dX_t = \tilde{m}_t X_t\, dt + \sigma\, dW_t \tag{5.13}$$

of an Ornstein-Uhlenbeck process in a random environment; cf. (3.14). To see this, suppose that the process $(\tilde{m}_t)$ in (5.13) is itself a recurrent Ornstein-Uhlenbeck process around the level $\tilde{m}$ with variance v^2 and drift parameter $\alpha < 0$, defined as the pathwise solution of

$$d\tilde{m}_t = \alpha(\tilde{m}_t - \tilde{m})dt + v\, d\tilde{W}_t \tag{5.14}$$

with respect to the independent Wiener process $\tilde{W}$. If $v \to \infty$ and $\frac{v^2}{2\alpha^2} = \tilde{\sigma}^2$ is fixed, then it is easy to check the pathwise convergence

$$\lim_{v\to\infty} \int_0^t \tilde{m}_s(\eta)\, ds = \tilde{\sigma}\tilde{W}_t(\eta) + \tilde{m}t. \tag{5.15}$$

Thus, equation (5.13) is transformed into equation (5.11).

The strong solution (3.6) of equation (5.11) takes the form

$$X_t = \exp\big(\tilde{\sigma}\tilde{W}_t + (\tilde{m} - \frac{1}{2}\tilde{\sigma}^2)t\big)\Big(X_0 + \int_0^t \exp\big(-\tilde{\sigma}\tilde{W}_u - (\tilde{m} - \frac{1}{2}\tilde{\sigma}^2)u\big)\sigma\, dW_u\Big) \tag{5.16}$$

For a given environment η, X is therefore a Gaussian process with distribution

$$\mathcal{N}\big(E_\eta[X_0]\exp\big(\tilde{\sigma}\tilde{W}_t(\eta) + (\tilde{m} - \frac{1}{2}\tilde{\sigma}^2)t\big), V_t(\eta, \mathrm{Var}_\eta[X_0])\big) \tag{5.17}$$

at time t, where

$$V_t(\eta, v^2) = \exp\big(2\tilde{\sigma}\tilde{W}_t(\eta) + (2\tilde{m} - \tilde{\sigma}^2)t\big)\big(v^2 + \int_0^t \exp\big(-2\tilde{\sigma}\tilde{W}_s(\eta) - (2\tilde{m} - \tilde{\sigma}^2)s\big)\, ds\big). \tag{5.18}$$

The study of the diffusion (V_t) leads to an alternative proof of (5.5) and to an explicit representation of μ in terms of a mixing measure on the variances.

(5.19) **Theorem.** *Under condition (5.2) the process $V = (V_t)_{t\geq 0}$ is a recurrent diffusion on $[0,\infty)$ whose invariant distribution ϑ is given by the density*

$$g(x) = \frac{1}{\Gamma(\frac{1}{2} - \frac{\tilde{m}}{\tilde{\sigma}^2})} (2\tilde{\sigma}^2 x)^{-(\frac{1}{2} - \frac{\tilde{m}}{\tilde{\sigma}^2})} \frac{1}{x} \exp\left(-\frac{\sigma^2}{2\tilde{\sigma}^2 x}\right). \tag{5.20}$$

The invariant distribution of X is given by

$$\mu = \int_0^\infty \mathcal{N}(0, v^2)\, \vartheta(dv^2). \tag{5.21}$$

Proof. By (5.18), V satisfies the stochastic differential equation

$$dV_t(\eta, v^2) = 2V_t(\eta, v^2)\tilde{\sigma}\, d\tilde{W}_t(\eta) + \big(\sigma^2 + V_t(\eta, v^2)(2\tilde{m} + \tilde{\sigma}^2)\big)\ dt. \tag{5.22}$$

By Kolmogorov's formula, the density of the invariant measure for V is therefore given by

$$\big(\log g(x)\big)' = -(\log 4\tilde{\sigma}^2 x^2)' + \frac{2\sigma^2 + 2x(2\tilde{m} + \tilde{\sigma}^2)}{4\tilde{\sigma}^2 x^2} \tag{5.23}$$

which yields

$$g(x) = \text{const}\cdot (4\tilde{\sigma}^2 x^2)^{\frac{2\tilde{m} - 3\tilde{\sigma}^2}{4\tilde{\sigma}^2}} \exp\left(-\frac{\sigma^2}{2\tilde{\sigma}^2 x}\right),$$

where the norming constant can be found by integration. Simplifying then leads to (5.20). To prove that μ in (5.21) is invariant for X, we consider the transition kernels

$$P_t^\eta : \mathcal{N}(c, v^2) \longmapsto \mathcal{N}\left(ce^{\tilde{\sigma}\tilde{W}_t(\eta) + (\tilde{m} - \frac{1}{2}\tilde{\sigma}^2)t}, V_t(\eta, v^2)\right),$$

$$\tilde{P}_t = \int P_t^\eta\, \nu(d\eta),$$

$$Q_t^\eta : v^2 \longmapsto V_t(\eta, v^2),$$

$$\tilde{Q}_t = \int Q_t^\eta\, \nu(d\eta),$$

where ν denotes the distribution of the variable η. Then $\vartheta\tilde{Q}_t = \vartheta$ for each t, and we want to show that $\mu\tilde{P}_t = \mu$ for each t. For any bounded function f on $I\!R$, let

$$\hat{f}(v^2) = \int_{I\!R} f\, d\mathcal{N}(0, v^2).$$

Then we obtain

$$
\begin{aligned}
\int_{\mathbb{R}} f\,d(\mu\tilde{P}_t) &= \int_0^\infty \vartheta(dv^2)\int_{\mathbb{R}} f\,d\Big(\mathcal{N}(0,v^2)\tilde{P}_t\Big)\\
&= \int_0^\infty \vartheta(dv^2)\int \nu(d\eta)\int_{\mathbb{R}} f\,d\left(\mathcal{N}(0,v^2)P_t^\eta\right)\\
&= \int_0^\infty \vartheta(dv^2)\int \nu(d\eta)\hat{f}\left(V_t(\eta,v^2)\right)\\
&= \int_0^\infty \vartheta(dv^2)\int \nu(d\eta)\int_0^\infty \hat{f}\,dQ_t^\eta(v^2)\\
&= \int_0^\infty \vartheta(dv^2)\int_0^\infty \hat{f}\,d\tilde{Q}_t(v^2)\\
&= \int_0^\infty \hat{f}\,d(\vartheta\tilde{Q}_t)\\
&= \int_0^\infty \hat{f}\,d\vartheta\\
&= \int_0^\infty \vartheta(dv^2)\int_{\mathbb{R}} f\,d\mathcal{N}(0,v^2)\\
&= \int_{\mathbb{R}} f\,d\mu
\end{aligned}
$$

which shows the invariance of μ.

Let us summarize the characteristic features of the stock price process corresponding to equation (5.11) under condition (5.2). First of all, $X = \log S$ admits a stationary distribution whose density is explicitly given by (5.5). Furthermore, this distribution is a mixture of normal distributions, where the variances are mixed according to the inverse of a Gamma distribution with parameters $\frac{1}{2} - \frac{\tilde{m}}{\tilde{\sigma}^2}$ and $\frac{\sigma^2}{2\tilde{\sigma}^2}$; see Nelson (1990) for similar results. It is easy to see from (5.5) that $X_t = \log S_t$ has a finite p-th moment if and only if

$$
p < 1 - \frac{2\tilde{m}}{\tilde{\sigma}^2},
$$

while S_t does not have any finite moments at all. This instability is also illustrated by the fact that the mixing measure for the variances in (5.21) has an unbounded support. By (5.20), the average variance in the mixture is given by

$$
\int_0^\infty xg(x)\,dx = -\frac{1}{2\tilde{m} + \tilde{\sigma}^2}.
$$

The preceding case study exhibits two typical features of our class of models: The invariant distribution is a mixture of normal distributions, and the mixing measure has unbounded support as soon as a transient component is involved. Let us now consider another case which illustrates these points and also shows that the mixing measure may become *singular*. The situation we shall examine arises if the random environment of the Ornstein-Uhlenbeck process in (3.14) and (5.13) is piecewise constant. More precisely, we consider the following situation: $(\beta_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables independent of W with values in $\{b_1, ..., b_m\}$ and distribution $(p_1, \ldots, p_m)$. The process $(X_n)_{n=0,1,\ldots}$ is the Markov chain whose transition kernel is given by

$$P(x, \cdot) = \sum_{j=1}^{m} p_j P_j(x, \cdot),$$

where

$$P_i(x, \cdot) = \mathcal{N}\left(xe^{\beta_i}, \sigma^2 \frac{e^{2\beta_i} - 1}{2\beta_i}\right)$$

is the transition kernel of a discrete-time Ornstein-Uhlenbeck process. If $\beta_i = 0$, we set $\frac{e^{2\beta_i}-1}{2\beta_i} = 1$.

In order to find an invariant measure μ for the transition kernel P, we define a transition kernel $Q(u, dv)$ on $(0, \infty)$ by

$$Q(u, dv) = \sum_{j=1}^{m} p_j \delta_{A_j u + B_j}(dv)$$

with

$$A_j = e^{2b_j}, \qquad B_j = \sigma^2 \frac{e^{2b_j} - 1}{2b_j}.$$

Intuitively, this corresponds to picking at random (according to $p_1, \ldots, p_m$) one of the m affine linear mappings $u \mapsto T_j u = A_j u + B_j$ and then jumping from u to the image of u under the chosen mapping. The question of existence of an invariant measure for random iterations of affine linear maps which are contractive on average has been studied in detail by Barnsley/Elton (1988), and hence we obtain the following result.

(5.24) **Theorem.** *Suppose that $b_1, \ldots, b_m$ and $p_1, \ldots, p_m$ satisfy the condition*

$$\bar{b} = \sum_{j=1}^{m} p_j b_j < 0. \tag{5.25}$$

Then the process X has a unique invariant distribution μ given by

$$\mu = \int_0^\infty \mathcal{N}(0, c^2)\, \vartheta(dc^2), \tag{5.26}$$

where ϑ is the unique invariant measure for Q on $(0, \infty)$.

Proof. The existence and uniqueness of ϑ immediately follows from Theorem 1 of Barnsley/Elton (1988) since (5.25) is equivalent to their condition of average contractivity, i.e.,

$$\prod_{j=1}^{m}(A_j)^{p_j} < 1.$$

Thus it only remains to show that μ in (5.26) is invariant for P. But in analogy to the proof of (5.21),

$$\begin{aligned}
\mu P &= \sum_{j=1}^{m} p_j(\mu P_j) \\
&= \sum_{j=1}^{m} p_j \left(\int_0^\infty \vartheta(du)\mathcal{N}(0,u)P_j \right) \\
&= \int_0^\infty \vartheta(du) \sum_{j=1}^{m} p_j \mathcal{N}\left(0, e^{2b_j}u + \sigma^2 \frac{e^{2b_j}-1}{2b_j}\right) \\
&= \int_0^\infty \vartheta(du) \sum_{j=1}^{m} p_j \int_0^\infty \mathcal{N}(0,v)\delta_{A_j u + B_j}(dv) \\
&= \int_0^\infty \vartheta(du) \int_0^\infty \mathcal{N}(0,v)\, Q(u,dv) \\
&= \int_0^\infty \mathcal{N}(0,v)\, (\vartheta Q)(dv) \\
&= \int_0^\infty \mathcal{N}(0,v)\, \vartheta(dv) \\
&= \mu,
\end{aligned}$$

and this completes the proof.

If we have additional information about the values $b_1, \ldots, b_m$, we can also say more about the mixing measure ϑ. Let us suppose that (5.25) holds. If all b_j are equal to some $b < 0$, then ϑ is a point mass at $-\frac{\sigma^2}{2b}$; otherwise, ϑ is a continuous measure, and ϑ is either absolutely continuous or singular. In fact, this is just Proposition 1 of Barnsley/Elton (1988). Furthermore, Theorem 3 of Barnsley/Elton (1988) tells us that supp $\vartheta = [d, \infty)$ for some $d \geq 0$, if there is at least one $b_\ell \geq 0$. Thus we see again that the existence of at least one transient component is sufficient to destabilize the situation in the sense that the mixing measure over the variances has unbounded support.

(5.27) **Theorem.** *Suppose that $b_1, \ldots, b_m$ are all < 0.*

1) *If $b_1 < \ldots < b_m < 0$, and if there exists some $c > 0$ such that*

(5.28) $$0 < B_1 < A_1 c + B_1 < B_2 < A_2 c + B_2 < \ldots < B_m < A_m c + B_m < c,$$

then supp $\vartheta \subseteq [0, c]$, and ϑ is singular.

2) *Let* $f_j = -\dfrac{\sigma^2}{2b_j}$ *so that* $\mathcal{N}(0, f_j)$ *is the invariant distribution of the recurrent Ornstein-Uhlenbeck process with parameter* $b_j < 0$. *Then*

$$\text{(5.29)} \qquad \textit{supp } \vartheta \subseteq \left[\min_{1 \le j \le m} f_j, \max_{1 \le j \le m} f_j \right].$$

Proof. **1)** If we define the sets $U = (0, c)$ and $V_j = [B_j, A_j c + B_j]$, then U is open, the compact sets V_j are pairwise disjoint and satisfy $T_j(U) \subseteq V_j$ and $\bigcup_{j=1}^{m} V_j \subseteq U$ by (5.28). Thus 1) follows from Diaconis/Shahshahani (1986) since their condition (SC) is satisfied.
2) We may assume that $b_1 \le \ldots \le b_m < 0$ so that $f_1 \le \ldots \le f_m$. By Theorem 3.1 of Diaconis/Shahshahani (1986), supp ϑ is the closure of the set

$$F = \left\{ x \middle| x \text{ is fixed point of } T_{i_1} \ldots T_{i_n} \text{ for some } n \text{ and } i_1, \ldots, i_n \in \{1, \ldots, m\} \right\}.$$

Thus it is enough to show that all fixed points of all $T_{i_1} \ldots T_{i_n}$ lie in the interval $[f_1, f_m]$. By an explicit computation, the fixed point of $T_{i_1} \ldots T_{i_n}$ is given by

$$f^{(n)} = \frac{\sum_{k=0}^{n-1} A_{i_1} \ldots A_{i_k} B_{i_{k+1}}}{1 - A_{i_1} \ldots A_{i_n}}$$

so that

$$f_m - f^{(n)} = \frac{\sigma^2 D_n}{1 - A_{i_1} \ldots A_{i_n}}$$

where

$$D_n = -\sum_{k=0}^{n-1} A_{i_1} \ldots A_{i_k} \frac{A_{i_{k+1}} - 1}{2b_{i_{k+1}}} + \frac{1}{2b_m} \left(\prod_{k=1}^{n} A_{i_k} - 1 \right).$$

We now show by induction that $D_n \ge 0$ for all n. For $n = 1$,

$$D_1 = \frac{1 - A_{i_1}}{2} \left(\frac{1}{b_{i_1}} - \frac{1}{b_m} \right) \ge 0$$

since $A_{i_1} < 1$ and $b_m \ge b_{i_1}$. For $n \ge 1$,

$$\begin{aligned} D_{n+1} &= D_n - A_{i_1} \ldots A_{i_n} \frac{A_{i_{n+1}} - 1}{2b_{i_{n+1}}} + \frac{1}{2b_m} \left(\prod_{k=1}^{n+1} A_{i_k} - \prod_{k=1}^{n} A_{i_k} \right) \\ &= D_n + \prod_{k=1}^{n} A_{i_k} \frac{1 - A_{i_{n+1}}}{2} \left(\frac{1}{b_{i_{n+1}}} - \frac{1}{b_m} \right) \ge 0 \end{aligned}$$

since $D_n \geq 0$, $0 < A_{i_k} < 1$ for all k and $b_m \geq b_{i_{n+1}}$. This implies that $f_m \geq f^{(n)}$ for all n, and an analogous argument shows that $f_1 \leq f^{(n)}$ for all n.

Finally we remark that using 1) of Theorem (5.27), one can easily construct examples where the measure ϑ is singular. For instance, we could take $m = 2$, $b_1 = -1$, $b_2 = -0.5$ and any $\sigma^2 \leq 0.5$.

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