A Minimality Property of the Minimal Martingale Measure

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Abstract: Let $X$ be a continuous adapted process for which there exists an equivalent local martingale measure (ELMM). The minimal martingale measure $\hat{P}$ is the unique ELMM for $X$ with the property that local $P$-martingales strongly orthogonal to the $P$-martingale part of $X$ are also local $\hat{P}$-martingales. We prove that if $\hat{P}$ exists, it minimizes the reverse relative entropy $H(P|Q)$ over all ELMMs $Q$ for $X$. A counterexample shows that the assumption of continuity cannot be dropped.

Key words: minimal martingale measure, relative entropy, equivalent martingale measures

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1. The result

In this section, we introduce the framework for our problem and present our main result. Let \((\Omega, \mathcal{F}, P)\) be a probability space with a filtration \(\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\) satisfying the usual conditions of right-continuity and completeness, where \(T \in (0, \infty)\) is a fixed time horizon. For all unexplained terminology from stochastic analysis, we refer to Protter (1990). We consider an \(\mathbb{R}^d\)-valued \(\mathcal{F}\)-adapted process \(X = (X_t)_{0 \leq t \leq T}\) and assume that \(X\) has \(P\)-a.s. continuous trajectories. Intuitively, \(X\) represents the discounted price evolution of \(d\) risky assets in a financial market, and we want to exclude the possibility of having arbitrage (“money-pumps”) in this market. We therefore assume that \(X\) admits an \emph{equivalent local martingale measure (ELMM)}, i.e., there exists a probability measure \(Q \equiv P\) with \(Q = P\) on \(\mathcal{F}_0\) such that \(X\) is a local \(Q\)-martingale; see for instance Delbaen/Schachermayer (1994) for a more detailed discussion of the economic significance of such a condition. Together with the continuity of \(X\), it implies by Theorem 2.2 of Choulli/Stricker (1996) that \(X\) is a special semimartingale satisfying the \emph{structure condition (SC)}: In the canonical decomposition \(X = X_0 + M + A\), the process \(M\) is an \(\mathbb{R}^d\)-valued locally square-integrable local \(P\)-martingale, and the \(\mathbb{R}^d\)-valued process \(A\) of finite variation has the form

\[
A_t = \int_0^t d\langle M \rangle_s \lambda_s, \quad 0 \leq t \leq T
\]

for an \(\mathbb{R}^d\)-valued predictable process \(\lambda\) such that

\[
K_t := \int_0^t \lambda_s^\text{tr} d\langle M \rangle_s \lambda_s = \sum_{i,j=1}^d \int_0^t \lambda_s^i \lambda_s^j d\langle M^i, M^j \rangle_s < \infty \quad P\text{-a.s. for all } t \in [0, T].
\]

The process \(K\) is called the \emph{mean-variance tradeoff process} of \(X\).

Since \(X\) admits at least one ELMM, one can ask about ELMMs having some special properties. One possibility is the \emph{minimal martingale measure} \(\hat{P}\) introduced by Föllmer/Schweizer (1991) and generalized by Ansel/Stricker (1992, 1993). This is defined by

\[
\frac{d\hat{P}}{dP} := \hat{Z}_T \quad \text{with } \hat{Z} := \mathcal{E} (-\int \lambda dM),
\]

where we assume that the exponential local \(P\)-martingale \(\hat{Z}\) is strictly positive and a true \(P\)-martingale so that \(E[\hat{Z}_T] = 1\). If in addition \(\hat{Z}_T \in L^2(P)\), then Theorem (3.5) of Föllmer/Schweizer (1991) shows that every square-integrable \(P\)-martingale \(L\) strongly \(P\)-orthogonal to \(M\) is also a \(\hat{P}\)-martingale (and strongly \(\hat{P}\)-orthogonal to \(X\)). Thus \(\hat{P}\) is minimal in the sense that it preserves the martingale structure as far as possible under the constraint
of turning $X$ into a martingale. Moreover, $\hat{P}$ is also the natural candidate for an ELMM for $X$ by Girsanov’s theorem.

Because the preceding description of minimality is somewhat awkward, there have been several attempts to characterize $\hat{P}$ in a different way. An economic characterization in a multidimensional diffusion framework has been given in Hofmann/Platen/Schweizer (1992). Föllmer/Schweizer (1991) and Schweizer (1995a) have shown that for $X$ continuous, $\hat{P}$ minimizes the “free energy” $H(Q|P) - \frac{1}{2} E_Q[K_T]$ over all ELMMs $Q$ for $X$ satisfying $E_Q[K_T] < \infty$. Here we recall that for two probability measures $P, Q$ and a $\sigma$-algebra $\mathcal{G} \subseteq \mathcal{F}$, the relative entropy of $Q$ with respect to $P$ on $\mathcal{G}$ is

$$H_\mathcal{G}(Q|P) := \left\{ \begin{array}{ll} E_Q \left[ \log \frac{dQ}{dP} \right] & \text{if } Q \ll P \text{ on } \mathcal{G} \\ +\infty & \text{otherwise.} \end{array} \right.$$ 

We also recall that $H_\mathcal{G}(Q|P)$ is always nonnegative, increasing in $\mathcal{G}$, and that $H(Q|P) := H_{\mathcal{F}}(Q|P)$ is 0 if and only if $Q = P$. In particular, the above characterization of $\hat{P}$ implies that $\hat{P}$ minimizes the relative entropy $H(Q|P)$ over all ELMMs $Q$ for $X$ if $X$ is continuous and the final value $K_T$ of the mean-variance tradeoff process is deterministic. Under the same conditions, $\hat{P}$ also minimizes $\text{Var} \left[ \frac{dQ}{dP} \right]$ or $\left\| \frac{dQ}{dP} \right\|_{L^2(P)}$ over all ELMMs $Q$ for $X$; see Theorem 7 of Schweizer (1995a). Miyahara (1996) has shown that $\hat{P}$ also minimizes $H(Q|P)$ over all ELMMs $Q$ if $X$ is a Markovian diffusion given by the multidimensional stochastic differential equation

$$dX_t = b(t, X_t) \, dt + \sigma(t, X_t) \, dW_t.$$ 

But all these results either use a very specific structure for $X$ or impose the very restrictive condition that $K_T$ should be deterministic. In contrast, the main result of this paper is completely general.

**Theorem 1.** Suppose that $X$ is a continuous adapted process admitting at least one equivalent local martingale measure $Q$. If $\hat{P}$ defined by (1.1) is a probability measure equivalent to $P$, then $\hat{P}$ minimizes the reverse relative entropy $H(P|Q)$ over all ELMMs $Q$ for $X$.

We remark that the idea of considering $H(P|Q)$ instead of $H(Q|P)$ first appeared in Platen/Rebolledo (1996). The assumption about $\hat{P}$ of course just states that the minimal martingale measure $\hat{P}$ should exist; it is thus a minimal requirement for the theorem’s assertion. Theorem 1 is only true for a continuous process $X$; we shall show by a counterexample in the next section that the conclusion fails in general if $X$ has jumps.

The next result is a preparation for the proof of Theorem 1. It does not really need any martingale structure; we could replace $N_x$ by any positive random variable with expectation 1. The present formulation just makes clear how we apply the lemma later on.
Lemma 2. Suppose that $N$ is a strictly positive local $P$-martingale with $N_0 = 1$. For any stopping time $\tau$ such that the stopped process $N^\tau$ is a $P$-martingale, we then have $E[\log N_\tau] \in [-\infty, 0]$.

Proof. We cannot use Jensen’s inequality because we do not know whether $\log N_\tau$ is integrable. But since $N^\tau$ is a strictly positive $P$-martingale starting from 1, $N_\tau$ is strictly positive and has expectation 1. Thus we can define a probability measure $R := P$ by $\frac{dR}{dP} := N_\tau$, and so we obtain

$$E_P[- \log N_\tau] = E_P \left[ \log \frac{dP}{dR} \right] = H(P|R) \in [0, \infty].$$

q.e.d.

Proof of Theorem 1: Let $Q$ be any ELMM for $X$ and denote by $Z$ its density process with respect to $P$. We may also assume that $H(P|Q) < \infty$ since there is nothing to prove otherwise. Because $X$ is continuous, we can write $Z$ as $Z = \hat{Z} \mathcal{E}(L)$ for a local $P$-martingale $L$ with $L_0 = 0$; see Theorem 1 of Schweizer (1995a) or Corollary 2.3 of Choulli/Stricker (1996). Let $(\tau_n)_{n \in \mathbb{N}}$ be a localizing sequence for $\mathcal{E}(L)$ and $\int \lambda dM$ and fix $n \in \mathbb{N}$. Then

$$\frac{dP}{dQ}_{\mathcal{F}_{\tau_n}} = \frac{1}{Z_{\tau_n}} \frac{1}{Z_{\tau_n} \mathcal{E}(L)_{\tau_n}} = \frac{dP}{dP}_{\mathcal{F}_{\tau_n}} \frac{1}{\mathcal{E}(L)_{\tau_n}},$$

and so Lemma 2 with $N := \mathcal{E}(L)$ implies that

$$H_{\mathcal{F}_{\tau_n}}(P|Q) = H_{\mathcal{F}_{\tau_n}}(P|\hat{P}) \leq E_P[\log \mathcal{E}(L)_{\tau_n}] \geq H_{\mathcal{F}_{\tau_n}}(P|\hat{P})$$

and therefore

$$(1.2) \quad \sup_{n \in \mathbb{N}} H_{\mathcal{F}_{\tau_n}}(P|\hat{P}) \leq \sup_{n \in \mathbb{N}} H_{\mathcal{F}_{\tau_n}}(P|Q) \leq H(P|Q) < \infty,$$

since $H_G(P|Q)$ is increasing in $G$. From Lemma 2 of Barron (1985), we thus obtain

$$\sup_{n \in \mathbb{N}} \log \frac{1}{Z_{\tau_n}} = \sup_{n \in \mathbb{N}} \log \frac{dP}{dP}_{\mathcal{F}_{\tau_n}} \in L^1(P),$$

and since $\hat{Z}_{\tau_n} \to \hat{Z}_T$ $P$-a.s. because $\tau_n$ increases stationarily to $T$, the dominated convergence theorem yields

$$H(P|\hat{P}) = E_P \left[ \log \frac{1}{Z_T} \right] = \lim_{n \to \infty} E_P \left[ \log \frac{1}{Z_{\tau_n}} \right] = \lim_{n \to \infty} H_{\mathcal{F}_{\tau_n}}(P|\hat{P}) \leq H(P|Q)$$

by (1.2). As $Q$ was arbitrary, the proof is complete.

q.e.d.
Remark. A closer look at the above proof shows that we only need continuity of $X$ to write the density process $Z$ of an arbitrary ELMM as $Z = \hat{Z}E(L)$ for some local $P$-martingale $L$ null at 0. One can ask if this is also possible for a general semimartingale $X$ satisfying the structure condition (SC), but the answer is negative. An explicit counterexample can be obtained by taking for $X$ the sum of a Brownian motion with drift and a compensated Poisson process. Alternatively, this is a consequence of the counterexample in the next section.

2. The counterexample

If the process $X$ is not continuous, the assertion of Theorem 1 is no longer true: We present here a counterexample with an ELMM $Q^*$ such that $H(P|Q^*) < H(P|\hat{P})$. It uses a bounded process in finite discrete time and basically consists of a number of elementary computations.

Fix some $U > 1$ and consider for $X$ a trinomial tree with time horizon 2 and parameters $U, 1, \frac{1}{U}$. Formally, let $Y_1, Y_2$ be i.i.d. under $P$ taking the values $U, 1, \frac{1}{U}$ with probability $\frac{1}{3}$ each. The process $X = (X_k)_{k=0,1,2}$ is then given by $X_0 := 1$, $X_1 := Y_1$ and $X_2 := Y_1Y_2$, and $\mathcal{F}$ is the filtration generated by $X$. We use the notation $\Delta X_k := X_k - X_{k-1}$ for the increments of $X$.

Any equivalent martingale measure (EMM) $Q$ for $X$ can be identified with a vector $q \in (0,1)^4$ via its transition probabilities

$$
q_1 := Q[X_1 = U], \quad q_2 := Q[X_2 = U|X_1 = U],
$$

$$
q_3 := Q[X_2 = U|X_1 = 1], \quad q_4 := Q[X_2 = U|X_1 = \frac{1}{U}].
$$

The other transition probabilities are then determined by the martingale property of $X$ under $Q$ and the fact that they add to 1 at each node in the tree. An elementary computation yields

$$
(2.1) \quad H(P|Q) = E_P \left[ -\log \frac{dQ}{dP} \right]
$$

$$
= -\frac{2}{3} \log q_1 - \frac{1}{3} \log (1 - (U + 1)q_1) - \frac{1}{9} \sum_{i=2}^{4} \left(2 \log q_i + \log (1 - (U + 1)q_i) \right)
$$

$$
+ \log 9 - \frac{2}{3} \log U,
$$

and setting the gradient with respect to $q$ equal to 0 gives an EMM $Q^*$ with

$$
q^*_i = \frac{2}{3(U + 1)} \quad \text{for } i = 1, \ldots, 4
$$

as a candidate for the entropy-optimal EMM. Under $Q^*$, the random variables $Y_1, Y_2$ are still i.i.d. and take the values $U, 1, \frac{1}{U}$ with probability $\frac{2}{3(U + 1)}$, $\frac{1}{3}$ and $\frac{2U}{3(U + 1)}$, respectively, so that
$Q^*$ is clearly equivalent to $P$. Inserting into (2.1) yields after some simplification

$$H(P|Q^*) = \log \frac{81}{\sqrt{16}} + \frac{2}{3} \log \frac{(U + 1)^2}{U}.$$  

To compute the minimal EMM $\hat{P}$ for $X$, we use the results of Schweizer (1995b). According to equations (2.21) and (1.2) in that paper, $\hat{P}$ is given by the density

$$\frac{d\hat{P}}{dP} = \hat{Z}_2 = \prod_{k=1}^{2} \frac{1 - \alpha_k \Delta X_k}{1 - \alpha_k \Delta A_k} = \prod_{k=1}^{2} \frac{E[\Delta X_k^{2} | F_{k-1}] - \Delta X_k E[\Delta X_k | F_{k-1}]}{E[\Delta X_k^{2} | F_{k-1}] - (E[\Delta X_k | F_{k-1}])^2}.$$  

Computing this explicitly shows that $\hat{P}$ can be identified with the vector $\hat{q}$ given by

$$\hat{q}_i = \frac{U + 1}{2(U^2 + U + 1)} \quad \text{for } i = 1, \ldots, 4.$$  

This means that under $\hat{P}$, $Y_1$ and $Y_2$ are again i.i.d. and take the values $U, 1, \frac{1}{U}$ with probability $rac{U+1}{2(U^2+U+1)}$, $rac{U^2+1}{2(U^2+U+1)}$, and $rac{U^2+U}{2(U^2+U+1)}$, respectively. Inserting into (2.1) now yields

$$H(P|\hat{P}) = \log 36 - \frac{2}{3} \log \frac{U(U^2 + 1)(U + 1)^2}{(U^2 + U + 1)^3}.$$  

If we take for instance $U = 2$, we obtain

$$q_i^* = \frac{2}{9}, \quad \hat{q}_i = \frac{3}{14} \quad \text{for } i = 1, \ldots, 4$$  

and

$$H(P|Q^*) = 4.473 < 4.475 = H(P|\hat{P}).$$  

This shows that $\hat{P}$ need not minimize the reverse relative entropy if $X$ is not continuous so that we have indeed a counterexample. Numerical evidence suggests that $H(P|Q^*) < H(P|\hat{P})$ for every $U > 1$, but we have not bothered to check this theoretically.

References


