Attractive invariant manifolds for maps: Existence, smoothness and continuous dependence on the map

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CH-8092 Zürich
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Abstract

A global invariant manifold result for maps is derived with conditions that are easy to verify for applications. The result supplies existence and smoothness of the attractive manifold as well as additional useful properties. It is also shown that a $C^{k,1}$-perturbation of the map yields a C^k-perturbation of the manifold. Moreover, it is proved that if there is an attractive invariant manifold for the time-T map of an ODE then this manifold is invariant for the flow as well. For an illustration, the results are applied to a system of two weakly coupled harmonic oscillators.

Keywords: attractive invariant manifold, centre-unstable manifold, smooth manifold

Subject Classification: 34A, 65L

Attractive invariant manifolds for maps: Existence, smoothness and continuous dependence on the map

The topic of this paper was initiated by the investigation of discrete dynamical systems occurring if an ODE is approximated by some numerical scheme. In many cases such a map admits an attractive invariant manifold. This strong geometric property implies that the dynamics of the system is essentially described by the dynamics on the manifold.

Theorems on invariant manifolds for maps have been proved many times for many different settings. The first results were obtained by Hadamard [2] and by Perron [6]. They consider the stable and unstable manifold of a fixed point. A very general treatment of the topic can be found in Hirsch, Pugh, Shub [3] (see also Shub [7]). Our aim was to derive a global invariant manifold result with conditions that are easy to verify for the applications in mind. Moreover, we wanted to show smoothness of the manifold as well as additional useful properties. We also tried to present a complete and transparent proof.

The global setting appropriate for our applications can be found in a result by U. Kirchgraber (cf. [4]). It supplies the existence and additional properties of an attractive global manifold without giving smoothness, however. We slightly generalized the situation to get our centre-unstable manifold result. An elegant and transparent approach for proving smoothness of an invariant manifold can be found in Lanford III [5]. He considered the local centre-stable and centre-unstable manifold of a fixed point. We closely follow the lines of his approach to establish our smoothness results. Global C^k centre-unstable manifolds are also treated in Chow and Lu [1] for the case of a fixed point. Their weak coupling condition requires both coupling terms to be small, however, and not only the product as in our condition.

As a corollary of the invariant manifold result we also show that a $C^{k,1}$-perturbation of the map yields a C^k-perturbation of the manifold.

The paper is organized as follows. In a first section we derive the existence of the invariant manifold together with additional properties. In Section 2 we prove the smoothness of the manifold. In Section 3 we show that if there is an attractive invariant manifold for the time-T map of an ODE then this manifold is also invariant for the flow. For illustration, in a last section we apply our results to a simple example describing two weakly coupled harmonic oscillators. Here, we do not have the case of a fixed point, in general, and only the product of the coupling terms is small enough.

1. Existence

We consider maps of the following form

$$(1)\ P : (X \times Y) \ni \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} f_0(x) + \hat{f}(x,y) \\ g(x,y) \end{pmatrix} \in (X \times Y)$$

where X, Y are Banach spaces.

We assume that P is contracting in y-direction and that f_0 is invertible and f_0^{-1} is Lipschitz continuous.

Assumption H1

a) *f_0 is invertible and $|f_0^{-1}(x_1) - f_0^{-1}(x_2)| < \alpha|x_1 - x_2|$ holds.*

b) The function g is bounded and the functions $\hat{f}$ and g satisfy the Lipschitz conditions

$$\begin{aligned} |\hat{f}(x_1,y_1) - \hat{f}(x_2,y_2)| &\leq L_{11}|x_1 - x_2| + L_{12}|y_1 - y_2| \\ |g(x_1,y_1) - g(x_2,y_2)| &\leq L_{21}|x_1 - x_2| + L_{22}|y_1 - y_2|\ . \end{aligned}$$

Here and in what follows, norms are denoted by $|\,.\,|$ independently of the spaces considered.

We want to find an invariant manifold of the form

$$M_s := \{(x,y) \,|\, x \in X,\ y = s(x)\}$$

for the map P where $s(x)$ is uniformly Lipschitz continuous.

Let $C_\mu := \{\sigma \in C_b^0(X,Y) \,|\, \sigma$ is Lipschitz continuous with uniform Lipschitz constant $\mu\}$ be the space of bounded μ-Lipschitz functions equipped with the sup norm $|\sigma| = \sup_{x\in X} |\sigma(x)|$. Note that C_μ is complete with respect to this norm.

Remark:

0) A frequent situation in applications is the following (see, e.g., Section 4): The function g is not bounded in $X \times Y$, the map P, however, is invariant in a strip $X \times Y_d$ with $Y_d := \{y \in Y \mid |y| \leq d\}$. All results of this paper hold and are proved in exactly the same way if in Assumption H1 the existence of such an invariant strip is required instead of the boundedness of g in $X \times Y$ and if in the spaces considered Y is replaced by Y_d (e.g., $C_b^0(X, Y_d)$ instead of $C_b^0(X,Y)$). There is one minor exception concerning the proof of Theorem 3 iii). There, the function $\sigma(x)$ has to be defined in a slightly different way: Let $\tilde{\sigma}(x) := s(x) + y_0 - s(x_0), x \in X$, and define

$$\sigma(x) := \begin{cases} \tilde{\sigma}(x)\ , & \text{for } x \in X \text{ with } \tilde{\sigma}(x) \leq d \\ \\ s(x) + \xi(x)(y_0 - s(x_0)) \text{ with } \xi(x) \in [0,1) \text{ such that } |\sigma(x)| = d\ , & \text{else}\ . \end{cases}$$

$\dashv$

For any $\sigma \in C_\mu$ consider the manifold $M_\sigma = \{(x,y)\,|\,x \in X,\ y = \sigma(x)\}$ and its image $P(M_\sigma)$. We first show that for appropriate μ the set $P(M_\sigma)$ may be described by some function $\overline{\sigma} \in C_\mu$, i.e.,

$$P(M_\sigma) = M_{\overline{\sigma}}\ .$$

Since $P(M_\sigma) = \{(\overline{x},\overline{y})\,|\,\overline{x} = f_0(x) + \hat{f}(x,\sigma(x)),\ \overline{y} = g(x,\sigma(x)),\, x \in X\}$ we look for a function $\overline{\sigma}$ such that $\overline{y} = g(x,\sigma(x)) = \overline{\sigma}(\overline{x})$ holds. As shown in Lemma 1 below the equation $\overline{x} = f_0(x) + \hat{f}(x,\sigma(x))$ may be solved for x provided

Condition B1

$$\beta(\mu) := \frac{1}{\alpha} - L_{11} - L_{12}\,\mu > 0$$

holds. This leads to $x = h_\sigma(\overline{x})$ and hence to

$$\overline{\sigma}(\overline{x}) = g(h_\sigma(\overline{x}), \sigma(h_\sigma(\overline{x})))\ . \tag{2}$$

Lemma 1 *Let $\sigma \in C_\mu$ and let Condition B1 be satisfied. Then the equation*

$$\overline{x} = f_0(x) + \hat{f}(x,\sigma(x)) \tag{3}$$

has a unique solution $x = h_\sigma(\overline{x})$ and h_σ is Lipschitz continuous with uniform Lipschitz constant $1/\beta(\mu)$.

Proof: We first fix $\overline{x}$. Suppose Eq.(3) has a solution $x = h$. Then h has to satisfy

$$h = E(h) := f_0^{-1}(\overline{x} - \hat{f}(h,\sigma(h))\ .$$

Since

$$|E(h_1) - E(h_2)| \le \alpha(L_{11} + L_{12}\,\mu)|h_1 - h_2|$$

the map $E(h)$ is a contraction in X provided condition B1 holds. We denote the unique fixed point of $E(h)$ by $h_\sigma(\overline{x})$. Obviously, $x = h_\sigma(\overline{x})$ is a solution of Eq.(3) for any $\overline{x}$. In order to show that h_σ is Lipschitz continuous we use the identity

$$h_\sigma(\overline{x}) = f_0^{-1}(\overline{x} - \hat{f}(h_\sigma(\overline{x}), \sigma(h_\sigma(\overline{x}))))$$

and find

$$|h_\sigma(\overline{x}_1) - h_\sigma(\overline{x}_2)| \le \alpha\Big[|\overline{x}_1 - \overline{x}_2\,| + (L_{11} + L_{12}\,\mu)\,|\,h_\sigma(\overline{x}_1) - h_\sigma(\overline{x}_2)|\Big]\ .$$

Due to condition B1 this may be written as

$$|h_\sigma(\overline{x}_1) - h_\sigma(\overline{x}_2)| \leq \frac{\alpha}{1 - \alpha(L_{11} + L_{12}\mu)} \, |\overline{x}_1 - \overline{x}_2|$$

which completes the proof of Lemma 1. $\perp$

Hence, the function

$$\overline{\sigma}(x) := g(h_\sigma(x), \sigma(h_\sigma(x)))$$

satisfies $P(M_\sigma) = M_{\overline{\sigma}}$. We define an operator $\mathcal{F}$ for functions $\sigma \in C_\mu$ by

$$(\mathcal{F}\sigma)(x) := g(h_\sigma(x), \sigma(h_\sigma(x))) \tag{4}$$

where $h_\sigma(x)$ satisfies

$$x = f_0(h_\sigma(x)) + \hat{f}(h_\sigma(x), \sigma(h_\sigma(x))) \, . \tag{5}$$

We have shown that

$$P(M_\sigma) = M_{\mathcal{F}\sigma} \, . \tag{6}$$

For the existence of an invariant manifold M_σ we need a function $\sigma(x) \in C_\mu$ for which $P(M_\sigma) = M_\sigma$. This is equivalent to the requirement that σ is a fixed point of the operator $\mathcal{F}$.

We want to show that under certain conditions $\mathcal{F}$ is a contraction in C_μ and hence has a unique fixed point. We first show that under Condition B1 and under

Condition B2

$$\frac{L_{21} + L_{22}\,\mu}{\beta(\mu)} \leq \mu$$

the operator $\mathcal{F}$ maps C_μ into itself. Since g is bounded, $\mathcal{F}\sigma$ is also bounded for $\sigma \in C_\mu$ by definition. We verify that $\mathcal{F}\sigma$ is μ-Lipschitz:

$$\begin{aligned} |(\mathcal{F}\sigma)(x_1) - (\mathcal{F}\sigma)(x_2)| &= |g(h_\sigma(x_1), \sigma(h_\sigma(x_1))) - g(h_\sigma(x_2), \sigma(h_\sigma(x_2)))| \\ &\leq (L_{21} + L_{22}\,\mu)\,|\,h_\sigma(x_1) - h_\sigma(x_2)| \, . \end{aligned}$$

By means of Lemma 1, where we have estimated the Lipschitz constant of h_σ, and by Condition B2 we find

$$|(\mathcal{F}\sigma)(x_1) - (\mathcal{F}\sigma)(x_2)\,| \leq \frac{L_{21} + L_{22}\,\mu}{\beta(\mu)} \, |x_1 - x_2| \leq \mu|x_1 - x_2| \, .$$

This implies $\mathcal{F} : C_\mu \to C_\mu$.

Next we show that under the additional

Condition B3

$$\chi(\mu) := L_{22} + \frac{L_{12}(L_{21} + L_{22}\,\mu)}{\beta(\mu)} < 1$$

the operator $\mathcal{F}$ is a contraction.

Lemma 2 *If there is $\mu > 0$ such that Conditions B1, B2, B3 are satisfied, then the operator $\mathcal{F}$ is a contraction in C_μ with contractivity constant $\chi(\mu)$.*

Proof: We first derive the estimate

$$|h_{\sigma_1}(x) - h_{\sigma_2}(x)| \le \frac{L_{12}}{\beta(\mu)}\,|\sigma_1 - \sigma_2|\,.$$

From the proof of Lemma 1 we know that

$$h_\sigma(x) = f_0^{-1}(x - \hat{f}(h_\sigma(x),\ \sigma(h_\sigma(x))))$$

and hence,

$$\begin{aligned}
|h_{\sigma_1}(x) - h_{\sigma_2}(x)| &\le |f_0^{-1}(x - \hat{f}(h_{\sigma_1}(x), \sigma_1(h_{\sigma_1}(x)))) - f_0^{-1}(x - \hat{f}(h_{\sigma_1}(x), \sigma_2(h_{\sigma_1}(x))))| \\
&+ |f_0^{-1}(x - \hat{f}(h_{\sigma_1}(x), \sigma_2(h_{\sigma_1}(x)))) - f_0^{-1}(x - \hat{f}(h_{\sigma_2}(x), \sigma_2(h_{\sigma_2}(x))))| \\
&\le \alpha L_{12}\,|\,\sigma_1 - \sigma_2\,| + \alpha(L_{11} + L_{12}\,\mu)\,|\,h_{\sigma_1} - h_{\sigma_2}|\,.
\end{aligned}$$

This implies the above estimate.

By the definition of $\mathcal{F}$ we get

$$\begin{aligned}
|(\mathcal{F}\sigma_1)(x) - (\mathcal{F}\sigma_2)(x)| &\le |g(h_{\sigma_1}(x), \sigma_1(h_{\sigma_1}(x))) - g(h_{\sigma_1}(x), \sigma_2(h_{\sigma_1}(x)))| \\
&+ |g(h_{\sigma_1}(x), \sigma_2(h_{\sigma_1}(x))) - g(h_{\sigma_2}(x), \sigma_2(h_{\sigma_2}(x)))| \\
&\le L_{22}|\sigma_1 - \sigma_2| + (L_{21} + L_{22}\,\mu)|h_{\sigma_1}(x) - h_{\sigma_2}(x)| \\
&\le \Big(L_{22} + \frac{L_{12}(L_{21} + L_{22}\,\mu)}{\beta(\mu)}\Big)\,|\sigma_1 - \sigma_2|\,.
\end{aligned}$$

Hence, under Condition B3 the operator $\mathcal{F}$ is contractive with contractivity constant $\chi(\mu)$.

$\perp$

Up to now we have shown: If there is $\mu > 0$ such that Conditions B1, B2, B3 hold the operator $\mathcal{F}$ has a unique fixed point $s \in C_\mu$. By construction it follows that the map P admits an invariant manifold $M_s = \{(x,y)\,|\, x \in X,\ y = s(x)\}$.

We now seek conditions on the Lipschitz constants $\alpha, L_{11}, L_{12}, L_{21}$ and L_{22} under which there exists $\mu > 0$ satisfying B1, B2, B3.

If Condition B1 is satisfied then Condition B2 is equivalent to

Condition B2'

$$L_{21} + L_{22}\,\mu \le \beta(\mu)\,\mu = (\frac{1}{\alpha} - L_{11} - L_{12}\,\mu)\mu\ .$$

We now consider Conditions B1, B2', B3. Condition B2' is equivalent to the requirement on μ that

$$q(\mu) := L_{12}\,\mu^2 - (\frac{1}{\alpha} - L_{11} - L_{22})\mu + L_{21} \le 0$$

holds. Such a μ exists if and only if

$$L_{11} + L_{22} + 2\sqrt{L_{12}L_{21}} \le \frac{1}{\alpha}\ .$$

Under this condition we have $q(\mu) \le 0$ for $\mu \in [\lambda, \nu]$ where

$$\lambda, \nu = \frac{\frac{1}{\alpha} - L_{11} - L_{22} \mp \sqrt{(\frac{1}{\alpha} - L_{11} - L_{22})^2 - 4L_{12}L_{21}}}{2L_{12}} \tag{7}$$

are the zeros of $q(\mu)$. Since under Condition B1 the quantity $\chi(\mu)$ is monotonically increasing we conclude: If there exists $\mu = \mu_0$ satisfying Conditions B1, B2', B3 then these conditions are also satisfied for $\mu = \lambda$. This means that λ is the lowest μ-value satisfying Conditions B1, B2, B3 and, therefore, is the best possible estimate for the Lipschitz constant of s.

Since for $\mu = \lambda$ Condition B2' and therefore B2 are satisfied with the equality sign we have

$$\frac{L_{21} + L_{22}\,\lambda}{\beta(\lambda)} = \lambda \tag{8}$$

and $\chi(\lambda)$ may be written as

$$\chi(\lambda) = L_{22} + L_{12}\lambda\ .$$

Conditions B1, B2, B3 for $\mu = \lambda$ are equivalent to

$$\begin{aligned} L_{12}\lambda &< \frac{1}{\alpha} - L_{11} \\ 2\sqrt{L_{12}L_{21}} &\le \frac{1}{\alpha} - L_{11} - L_{22} \\ L_{12}\lambda &< 1 - L_{22}\ . \end{aligned}$$

We are now able to state the existence result for the invariant manifold M_s together with additional properties of M_s.

Theorem 3 *Let the map P given in Eq.(1) satisfy Assumption H1. Moreover, assume that the Lipschitz constants $\alpha, L_{11}, L_{12}, L_{21}, L_{22}$ satisfy the conditions*

$$(9) \qquad \begin{aligned} &a) \qquad & 2\sqrt{L_{12}L_{21}} &\le \frac{1}{\alpha} - L_{11} - L_{22} \\ &b) \qquad & L_{12}\lambda &< 1 - L_{22} \\ &c) \qquad & L_{12}\lambda &< \frac{1}{\alpha} - L_{11} \end{aligned}$$

where

$$\lambda = \frac{2L_{21}}{\frac{1}{\alpha} - L_{11} - L_{22} + \sqrt{(\frac{1}{\alpha} - L_{11} - L_{22})^2 - 4L_{12}L_{21}}}\ .$$

Then there is a function $s : X \to Y$ such that the following assertions hold.

i) The set $M_s = \{(x,y)\,|\,x \in X,\ y = s(x)\}$ is invariant under the map P, i.e., if $(x,y) \in M_s$ then also $P(x,y) \in M_s$.

ii) The function s is uniformly λ-Lipschitz.

iii) The invariant manifold M_s is uniformly attractive with attractivity constant

$$\chi(\lambda) = L_{22} + L_{12}\lambda < 1\ ,$$

i.e., for $(x_0, y_0) \in X \times Y$ and $(x_1, y_1) := P(x_0, y_0)$ the inequality

$$|y_1 - s(x_1)| \le \chi(\lambda)|y_0 - s(x_0)|$$

holds.

iv) If instead of Condition (9) a) the slightly sharper condition

$$(9) \qquad a*) \qquad 2\sqrt{L_{12}L_{21}} < \frac{1}{\alpha} - L_{11} - L_{22}$$

holds then the invariant manifold M_s has the "property of asymptotic phase": For every $(x_0, y_0) \in X \times Y$ there is $(\tilde{x}_0, \tilde{y}_0) \in M_s$ such that for $(x_i, y_i) := P^i(x_0, y_0)$ and $(\tilde{x}_i, \tilde{y}_i) := P^i(\tilde{x}_0, \tilde{y}_0) \in M_s$, $i \in \mathbb{N}_0$,

$$\begin{aligned} |x_i - \tilde{x}_i| &\leq c\,\chi(\lambda)^i\, |y_0 - s(x_0)| \\ &\qquad\qquad \textit{with} \quad c = \frac{L_{12}}{\sqrt{(\frac{1}{\alpha} - L_{11} - L_{22})^2 - 4L_{12}L_{21}}} \\ |y_i - \tilde{y}_i| &\leq \chi(\lambda)^i (1 + \lambda c)\, |y_0 - s(x_0)| \end{aligned}$$

holds.

v) *If g has the form $g(x, y) = B(x, y)y + \hat{g}(x, y)$ with $|B(x, s(x))| \leq b < 1$ for all $x \in X$ then the estimate*

$$|s(x)| \leq \frac{1}{1-b} \sup_{x \in X} |\hat{g}(x, s(x))|$$

holds.

vi) *If $f(x, y) - x$ and $g(x, y)$ are z-periodic in x then s is also z-periodic, i.e., if there is a constant $z \in X$ such that $f(x + z, y) = f(x, y) + z$ and $g(x + z, y) = g(x, y)$ for all $x \in X$, $y \in Y$ then $s(x + z) = s(x)$ for $x \in X$.*

vii) *Every invariant set $\Omega \subset X \times Y$ of the map P is contained in M_s, i.e., $P(\Omega) = \Omega$ implies $\Omega \subset M_s$.*

Remarks:

1) Note that if instead of Condition (9) a) the slightly stronger condition (9) a*) is required then Condition (9) c) may be dropped. We show that (9) a*) implies (9) c):

$$\begin{aligned} L_{12}\,\lambda &= \frac{2L_{12}L_{21}}{\frac{1}{\alpha} - L_{11} - L_{22} + \sqrt{(\frac{1}{\alpha} - L_{11} - L_{22})^2 - 4L_{12}L_{21}}} < \frac{\frac{1}{2}(\frac{1}{\alpha} - L_{11} - L_{22})^2}{\frac{1}{\alpha} - L_{11} - L_{22} + \sqrt{\ldots}} \\ &\leq \frac{1}{2}(\frac{1}{\alpha} - L_{11} - L_{22}) < \frac{1}{\alpha} - L_{11} - L_{22} \leq \frac{1}{\alpha} - L_{11}\,. \end{aligned}$$

2) For the case $\alpha \geq 1$ which is important for our applications the three Conditions (9) a), b), c) may be replaced by the single Condition (9) a*). We show that Condition (9) a*) and $\alpha \geq 1$ imply Condition (9) b): As in Remark 1) we have

$$L_{12}\,\lambda < \frac{1}{\alpha} - L_{11} - L_{22}$$

and hence, for $\alpha \geq 1$ obviously Condition (9) b) is satisfied. $\dashv$

The following facts concerning the manifold M_s have already been shown and will be needed below. From now on we write $h(x)$ for $h_s(x)$ dropping the index s. We know from Lemma 1 that h satisfies the functional equation

$$x = f_0(h(x)) + \hat{f}(h(x), s(h(x))) \,, \tag{10}$$

i.e., h provides the preimage of $(x, s(x))$ in M_s. Moreover, h has the uniform Lipschitz constant

$$L_h := \frac{1}{\beta(\lambda)} = \frac{1}{\frac{1}{\alpha} - L_{11} - L_{12}\,\lambda} \,. \tag{11}$$

Note also that s satisfies the invariance equation

$$s(x) = g(h(x), s(h(x))) \,. \tag{12}$$

Proof of Theorem 3: We have already proved Assertions i) and ii).

iii) We use the contractivity of the operator $\mathcal{F}$. Let $(x_0, y_0) \in X \times Y$ be given. We choose $\sigma \in C_\lambda$ as $\sigma(x) := s(x) + y_0 - s(x_0)$. For this special choice we have $(x_0, y_0) \in M_\sigma = \{(x,y) \,|\, x \in X,\ y = \sigma(x)\}$ and $|\sigma - s| = |y_0 - s(x_0)|$. Eq.(6) implies $(x_1, y_1) = P(x_0, y_0) \in M_{\mathcal{F}\sigma}$ and hence,

$$|y_1 - s(x_1)| = |(\mathcal{F}\sigma)(x_1) - s(x_1)| \ \leq\ |\mathcal{F}\sigma - s| = |\mathcal{F}\sigma - \mathcal{F}s| \,.$$

From Lemma 2 we know that

$$|\mathcal{F}\sigma - \mathcal{F}s| \leq \chi(\lambda)|\sigma - s|$$

and therefore

$$|y_1 - s(x_1)| \leq \chi(\lambda)\,|\,y_0 - s(x_0)| \,.$$

$\chi(\lambda) < 1$ follows directly from Condition (9) b).

iv) For $k \in \mathbb{N}_0$ let $(x_k, y_k) := P^k(x_0, y_0)$ and consider the k preimages in M_s of $(x_k,\, s(x_k)) \in M_s$ under the map P, i.e., we define

$$\begin{aligned} \tilde{x}_k^{(k)} &:= x_k \\ \tilde{x}_{\ell-1}^{(k)} &:= h(\tilde{x}_\ell^{(k)}) \quad \text{for} \quad \ell = k,\ k-1, ..., 1 \,. \end{aligned}$$

We show that for fixed $i \geq 0$ the sequence $\left(\tilde{x}_i^{(k)}\right)$, $k \geq i$, is a Cauchy sequence.

Note that

$$x_{k+1} = f_0(x_k) + \hat{f}(x_k, s(x_k) + d_k) \quad \text{with} \quad d_k := y_k - s(x_k)$$

and by Eq.(10)

$$x_{k+1} = \widetilde{x}_{k+1}^{(k+1)} = f_0\Big(\widetilde{x}_k^{(k+1)}\Big) + \hat{f}\Big(\widetilde{x}_k^{(k+1)}, s\Big(\widetilde{x}_k^{(k+1)}\Big)\Big) \ .$$

Thus,

$$\begin{aligned}
\Big|x_k - \widetilde{x}_k^{(k+1)}\Big| \ &= \ \Big|f_0^{-1}\Big(x_{k+1} - \hat{f}(x_k, s(x_k) + d_k)\Big) - f_0^{-1}\Big(x_{k+1} - \hat{f}(x_k, s(x_k))\Big) \\
&+ \ f_0^{-1}\Big(x_{k+1} - \hat{f}(x_k, s(x_k))\Big) - f_0^{-1}\Big(x_{k+1} - \hat{f}(\widetilde{x}_k^{(k+1)}, s(\widetilde{x}_k^{(k+1)}))\Big)\Big| \\
&\leq \ \alpha\, L_{12}|d_k| + \alpha(L_{11} + L_{12}\,\lambda)|x_k - \widetilde{x}_k^{(k+1)}| \ .
\end{aligned}$$

With $x_k = \widetilde{x}_k^{(k)}$ we therefore have

$$\Big|\widetilde{x}_k^{(k)} - \widetilde{x}_k^{(k+1)}\Big| \leq \ \frac{L_{12}}{\beta(\lambda)}\ |d_k| \ .$$

Since by definition $\widetilde{x}_i^{(k)} = h^{\ell-i}\Big(\widetilde{x}_\ell^{(k)}\Big),\ i \leq \ell \leq k$, we obtain

$$\Big|\widetilde{x}_i^{(k)} - \widetilde{x}_i^{(k+1)}\Big| = |h^{k-i}\Big(\widetilde{x}_k^{(k)}\Big) - h^{k-i}\Big(\widetilde{x}_k^{(k+1)}\Big)\Big| \leq (L_h)^{k-i}\ \frac{L_{12}}{\beta(\lambda)}\ |d_k| \ .$$

Using Assertion iii) of Theorem 3 and Eq.(11) leads to

$$\Big|\widetilde{x}_i^{(k)} - \widetilde{x}_i^{(k+1)}\Big| \leq \ \frac{L_{12}}{\beta(\lambda)}\ \left(\frac{\chi(\lambda)}{\beta(\lambda)}\right)^{k-i}\ |d_i| \ .$$

We have

$$\beta(\lambda) - \chi(\lambda) = \frac{1}{\alpha} - L_{11} - L_{22} - 2L_{12}\,\lambda$$

and by Eq.(7)

$$\beta(\lambda) - \chi(\lambda) = \sqrt{(\frac{1}{\alpha} - L_{11} - L_{22})^2 - 4L_{12}L_{21}}$$

which is positive by means of Condition (9) a*). It follows that $\chi(\lambda)/\beta(\lambda) < 1$. Therefore, for any $\ell > k$

$$\begin{aligned}
\Big|\widetilde{x}_i^{(k)} - \widetilde{x}_i^{(\ell)}\Big| \ &\leq \ \Big|\widetilde{x}_i^{(k)} - \widetilde{x}_i^{(k+1)}\Big| + |\widetilde{x}_i^{(k+1)} - \widetilde{x}_i^{(k+2)}\Big| + \ldots \\
&\leq \ \frac{L_{12}}{\beta(\lambda)}\ \frac{1}{1 - \frac{\chi(\lambda)}{\beta(\lambda)}}\ \left(\frac{\chi(\lambda)}{\beta(\lambda)}\right)^{k-i}\ |d_i| \ .
\end{aligned}$$

Hence, for fixed $i \geq 0$ the sequence $\left(\tilde{x}_i^{(k)}\right), k \geq i$, is a Cauchy sequence. We denote its limit by $\tilde{x}_i$. The above estimate implies that

$$|x_i - \tilde{x}_i| \leq \frac{L_{12}}{\beta(\lambda) - \chi(\lambda)} \, |d_i| \; .$$

Since h and s are continuous, obviously, for $\tilde{y}_i := s(\tilde{x}_i)$ one has $(\tilde{x}_{i+1}, \tilde{y}_{i+1}) = P(\tilde{x}_i, \tilde{y}_i)$. Now, $|d_i| \leq \chi(\lambda)^i \, |d_0|$ yields the first estimate of Assertion iv).

For the second estimate we have

$$|y_i - \tilde{y}_i| = |y_i - s(\tilde{x}_i)| \leq |y_i - s(x_i)| + |s(x_i) - s(\tilde{x}_i)| \; .$$

Using Assertion iii) and the first estimate of Assertion iv) we therefore obtain

$$|y_i - \tilde{y}_i| \leq \chi(\lambda)^i \, |d_0| \; + \; \lambda c \, \chi(\lambda)^i \, |d_0| \; .$$

v) Due to Eq.(12) we have

$$s(x) = g(h, s(h)) = B(h, s(h)) \; s(h) + \hat{g}(h, s(h))$$

where we have dropped the argument x of h. Taking norms we get

$$\sup_x |s(x)| \leq b \; \sup_x |s(x)| + \sup_x |\, \hat{g}(x, s(x))|$$

and hence,

$$|s| \leq \frac{1}{1-b} \; \sup_x |\, \hat{g}(x, s(x))| \; .$$

vi) We restrict the operator $\mathcal{F}$ to the space

$$C_{\mu|z} := \{\sigma \in C_\mu \,|\, \sigma(x+z) = \sigma(x)\}$$

of z-periodic functions which is a closed subspace of C_μ. It remains to show that under the given conditions the operator $\mathcal{F}$ maps $C_{\mu|z}$ into itself.

By Eq.(5) we have

$$x + z = f(h_\sigma(x+z), \sigma(h_\sigma(x+z)))$$

for $\sigma \in C_{\mu|z}$. Due to the periodicity assumptions, this is equivalent to

$$x = f(h_\sigma(x+z) - z, \sigma(h_\sigma(x+z) - z)) \; .$$

Lemma 1 implies that the equation $x = f(u, \sigma(u))$ has the unique solution $u = h_\sigma(x)$ which yields

$$h_\sigma(x+z) = h_\sigma(x) + z \; .$$

We conclude (cf. Eq.(4))

$$(\mathcal{F}\sigma)(x+z) = g(h_\sigma(x+z),\ \sigma(h_\sigma(x+z))) = g(h_\sigma(x), \sigma(h_\sigma(x))) = (\mathcal{F}\sigma)(x)$$

which implies $\mathcal{F}\sigma \in C_{\mu|z}$.

vii) Define $D := \sup_{(x,y)\in\Omega} |y - s(x)|$. The invariance of Ω implies that for any $(x,y) \in \Omega$ there is $(\tilde{x}, \tilde{y}) \in \Omega$ with $P(\tilde{x},\tilde{y}) = (f(\tilde{x},\tilde{y}), g(\tilde{x},\tilde{y})) = (x,y)$. Since g is bounded it follows that $D < \infty$. Assume that $D > 0$. Hence, there is $(x,y) \in \Omega$ with

$$|y \;-\; s(x)| \;>\; \chi(\lambda)\, D\,. \tag{13}$$

Let $(\tilde{x},\tilde{y}) \in \Omega$ satisfy $P(\tilde{x},\tilde{y}) = (x,y)$. By Assertion iii), we have $|y - s(x)| \le \chi(\lambda)|\tilde{y} - s(\tilde{x})| \le \chi(\lambda)D$ contradicting (13). $\perp$

As a corollary of Theorem 3 we show that a perturbation of the given map leads to a corresponding perturbation of the invariant manifold. Let us consider the maps

$$\begin{aligned} P:\ (X\times Y) \ &\ni \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} f_0(x) + \hat{f}(x,y) \\ g(x,y) \end{pmatrix} \in X \times Y \\ \overline{P}:\ (X\times Y) \ &\ni \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} f_0(x) + \hat{\overline{f}}(x,y) \\ \overline{g}(x,y) \end{pmatrix} \in X \times Y \end{aligned} \tag{14}$$

and assume that they both satisfy the assumptions of Theorem 3 with the same Lipschitz constants L_{ij}. Then there are functions s and $\overline{s}$ such that the manifolds M_s and $M_{\overline{s}}$ are invariant under P and $\overline{P}$, respectively, with all the properties stated in Theorem 3. We define

$$\begin{aligned} \delta_f \ &:= \sup_{x\in X,\, y\in Y} |\hat{f}(x,y) - \hat{\overline{f}}(x,y)| \\ \delta_g \ &:= \sup_{x\in X,\, y\in Y} |g(x,y) - \overline{g}(x,y)|\,. \end{aligned}$$

Corollary 4 *Let the maps P and $\overline{P}$ defined in (14) satisfy Assumption H1 with common Lipschitz constants $\alpha, L_{11}, L_{12}, L_{21}, L_{22}$. Moreover, let Conditions (9) a), b) c) be satisfied. Then, for the functions s and $\overline{s}$ the estimate*

$$|s(x) - \overline{s}(x)| \le \frac{1}{1-\chi(\lambda)}\,(\lambda\delta_f + \delta_g)$$

holds.

Proof: The functions s and $\overline{s}$ satisfy the invariance equations (cf. Eq.(12))

$$s(x) = g(h, s(h)), \quad \overline{s}(x) = \overline{g}(\overline{h}, \overline{s}(\overline{h}))$$

where the functions h and $\overline{h}$ satisfy the functional equations (cf. Eq.(10))

$$x = f_0(h) + \hat{f}(h, s(h)), \quad x = f_0(\overline{h}) + \hat{\overline{f}}(\overline{h}, \overline{s}(\overline{h})) \ .$$

We therefore get

$$\begin{aligned} |s(x) - \overline{s}(x)| \ &\leq \ |g(h, s(h)) \ - \ g(\overline{h}, s(h))| \ + \ |g(\overline{h}, s(h)) \ - \ g(\overline{h}, s(\overline{h}))| \\ &\quad + |g(\overline{h}, s(\overline{h})) \ - \ g(\overline{h}, \overline{s}(\overline{h}))| \ + \ |g(\overline{h}, \overline{s}(\overline{h})) \ - \ \overline{g}(\overline{h}, \overline{s}(\overline{h}))| \\ &\leq \ (L_{21} + L_{22}\lambda) \ |h - \overline{h}| \ + \ L_{22} \, |s - \overline{s}| \ + \ \delta_g \ . \end{aligned} \tag{15}$$

In order to estimate $|h - \overline{h}|$ we use

$$h = f_0^{-1}\Big(x - \hat{f}(h, s(h))\Big), \quad \overline{h} = f_0^{-1}\Big(x - \hat{\overline{f}}(\overline{h}, \overline{s}(\overline{h}))\Big) \ .$$

We have

$$\begin{aligned} |h - \overline{h}| \ &\leq \ \Big|f_0^{-1}\Big(x - \hat{f}(h, s(h))\Big) \ - \ f_0^{-1}\Big(x - \hat{f}(\overline{h}, s(h))\Big)\Big| \\ &\quad + \Big|f_0^{-1}\Big(x - \hat{f}(\overline{h}, s(h))\Big) \ - \ f_0^{-1}\Big(x - \hat{f}(\overline{h}, s(\overline{h}))\Big)\Big| \\ &\quad + \Big|f_0^{-1}\Big(x - \hat{f}(\overline{h}, s(\overline{h}))\Big) \ - \ f_0^{-1}\Big(x - \hat{f}(\overline{h}, \overline{s}(\overline{h}))\Big)\Big| \\ &\quad + \Big|f_0^{-1}\Big(x - \hat{f}(\overline{h}, \overline{s}(\overline{h}))\Big) \ - \ f_0^{-1}\Big(x - \hat{\overline{f}}(\overline{h}, \overline{s}(\overline{h}))\Big)\Big| \\ &\leq \ \alpha\Big[(L_{11} + L_{12}\lambda) \, |h - \overline{h}| \ + \ L_{12} \, |s - \overline{s}| \ + \ \delta_f\Big] \ . \end{aligned}$$

Due to Condition (9) c) we obtain

$$|h - \overline{h}| \leq \frac{1}{\beta(\lambda)} \, (\delta_f + L_{12} \, |s - \overline{s}|) \ .$$

Inserting this estimate into Eq.(15) we find

$$|s - \overline{s}| \leq \left(\frac{L_{21} + L_{22}\,\lambda}{\beta(\lambda)} \ L_{12} + L_{22}\right) |s - \overline{s}| \ + \ \frac{L_{21} + L_{22}\,\lambda}{\beta(\lambda)} \, \delta_f \ + \ \delta_g \ .$$

By means of Eq.(8) this may be written as

$$|s - \overline{s}| \leq \frac{1}{1 - \chi(\lambda)} \left(\lambda\, \delta_f + \delta_g\right) .$$

This also implies the estimate

$$|h - \overline{h}| \leq \frac{1}{\beta(\lambda)\,(1 - \chi(\lambda))} \Big[(1 - L_{22})\, \delta_f + L_{12}\, \delta_g\Big] \tag{16}$$

needed in the proof of Corollary 9. $\perp$

2. Smoothness

We show that if the map P is smooth then the invariant manifold M_s has the same smoothness properties provided some additional conditions are satisfied. The essential ones are

Condition B4(k)

$$L_{22} + L_{12}\,\lambda < \min\Big\{1,\ \Big(\frac{1}{\alpha} - L_{11} - L_{12}\,\lambda\Big)^k\Big\} .$$

and

Condition B5 $\quad$ *$f_0'(x)$ is an isomorphism of X*

We shall need the space C_b^k, $k \in \mathbb{N}$, of functions in C^k with bounded derivatives as well as the space $C_b^{k,1}$, $k \in \mathbb{N}$, of functions in $C^{k,1}$ with bounded derivatives and with uniform Lipschitz constant of the k-th derivative.

Theorem 5 *Let the map P given in Eq.(1) satisfy Assumption H1. Moreover, let Conditions (9) a) and B4(1) be satisfied. Then there exists a function s such that*

i) all assertions of Theorem 3 hold;

ii) if f, g are of class C_b^k and if Conditions B4(k) and B5 hold then s is of class C_b^k as well;

iii) if f, g are of class $C_b^{k,1}$ and if Conditions B4(k+1) and B5 hold then s is also of class $C_b^{k,1}$.

Remarks:

3) Note that for $k \in \mathbb{N}$ Conditions (9) a) and B4(k) imply Condition (9) a*). Condition B4(k) implies Condition B4(k−1) for $k > 1$. In particular, Condition B4(k) implies Condition B4(1). Hence, from B4(k) it follows that

$$2L_{12}\,\lambda < \frac{1}{\alpha} - L_{11} - L_{22}\ .$$

By definition of λ (cf. Eq.(7)) this is equivalent to

$$\sqrt{(\frac{1}{\alpha} - L_{11} - L_{22})^2 - 4L_{12}L_{21}} > 0\ .$$

This proves the above claim.

4) If $\alpha \geq 1$ the single Condition (9) a*) yields the existence of s (cf. Remark 2)) as well as the differentiability of s provided f, g are of class C_b^1.
As stated in Remark 3) Condition (9) a*) is equivalent to

$$L_{22} + L_{12}\,\lambda < \frac{1}{\alpha} - L_{11} - L_{12}\,\lambda\ .$$

Since $1/\alpha - L_{11} - L_{12}\,\lambda \leq 1$ for $\alpha \geq 1$ this implies Condition B4(1). $\dashv$

Proof of Theorem 5 i): Obviously, Condition B4(1) implies Conditions (9) b) and (9) c) and hence, Theorem 3 applies. $\perp$

The proof of Assertions ii), iii) of Theorem 5 is by induction with respect to the order of differentiability k and is started with $k = 1$. We therefore prove the case $k = 1$ separately in the two subsequent lemmas. We also need that h is differentiable. This is stated in Lemma 8. In the following we suppose that the map P satisfies Assumption H1 and that Condition (9) a) holds.

Lemma 6 *If f and g are of class C_b^1 and if B4(1) holds then s is of class C_b^1.*

Proof: In order to get a functional equation for $s'(x)$ we differentiate formally the invariance equation Eq.(12) for $s(x)$:

(17) $$s'(x) = \Big[g_1(h(x), s(h(x))) + g_2(h(x), s(h(x)))\ s'(h(x))\Big]\ h'(x)\ .$$

We use the following notations:

$$g_1(x,y) = \frac{\partial}{\partial x}\, g(x,y), \quad g_2(x,y) = \frac{\partial}{\partial y}\, g(x,y)$$

$$f_0'(x) = \frac{d}{dx}\, f_0(x), \quad \text{etc.}$$

Moreover, we shall mostly suppress the argument x and we shall write v for $(h, s(h))$. We need express $h'(x)$ in Eq.(17) in terms of s and its derivative. Differentiating formally the functional equation (10) we obtain

$$Id = [f_0'(h) + \hat{f}_1(v) + \hat{f}_2(v)\; s'(h)]\; h'(x) \tag{18}$$

and together with Eq.(17)

$$s'(x) = [g_1(v) + g_2(v)\; s'(h)][f_0'(h) + \hat{f}_1(v) + \hat{f}_2(v)\; s'(h)]^{-1} \tag{19}$$

or for short,

$$s'(x) = G(h(x))\; F(h(x))^{-1}\; . \tag{19}$$

Although we do not yet know that h and s are differentiable we do know that, if they are differentiable, their derivatives satisfy Eqs.(18) and (19).

In the following we define an operator $\mathcal{K}$ acting on a space of functions from X to $\mathcal{L}(X,Y)$ by

$$(\mathcal{K}\sigma)(x) := [g_1(v) + g_2(v)\; \sigma(h)][f_0'(h) + \hat{f}_1(v) + \hat{f}_2(v)\; \sigma(h)]^{-1}\; . \tag{20}$$

Note that a fixed point of $\mathcal{K}$ is a candidate for the derivative of s. We first show that the operator $\mathcal{K}$ is well defined and is a contraction in the space

$$Z_\lambda := \{\sigma \in C^0(X, \mathcal{L}(X,Y)) \,\big|\, |\sigma| \le \lambda\}\; .$$

λ is given in Theorem 3 and Z_λ is equipped with the supremum norm. Hence, $\mathcal{K}$ has a unique fixed point in Z_λ denoted by σ.

In a next step we shall show that σ is in fact the derivative of s.

Assertion 5.1

$$\mathcal{K} : Z_\lambda \to Z_\lambda\; .$$

Take $\sigma \in Z_\lambda$ and write $\mathcal{K}\sigma$ in the form $(\mathcal{K}\sigma)(x) = G_\sigma(h) F_\sigma(h)^{-1}$ where

$$G_\sigma(h) := g_1(v) + g_2(v)\; \sigma(h)$$

$$F_\sigma(h) := f_0'(h) + \hat{f}_1(v) + \hat{f}_2(v)\; \sigma(h)\; .$$

Condition B5 implies that $f_0'(h)$ is invertible and from Assumption H1 it follows that $|f_0'(h)^{-1}| \leq \alpha$. Since

$$\left| f_0'(h)^{-1}(\hat{f}_1(v) + \hat{f}_2(v)\sigma(h)) \right| \leq \alpha(L_{11} + L_{12}\,\lambda) < 1$$

by Condition (9) c), $\left[Id + f_0'(h)^{-1}(\hat{f}_1(v) + \hat{f}_2(v)\sigma(h)) \right]$ is invertible and hence, $F_\sigma(h)$ is invertible and for each $x \in X$

$$F_\sigma(h)^{-1} = \left[Id + f_0'(h)^{-1}(\hat{f}_1(v) + \hat{f}_2(v)\sigma(h)) \right]^{-1} f_0'(h)^{-1} \; . \tag{21}$$

We may expand the right-hand side of Eq.(21) into a Neumann series and we get the estimate

$$|F_\sigma(h)^{-1}| \leq \frac{1}{\frac{1}{\alpha} - L_{11} - L_{12}\,\lambda} = \frac{1}{\beta(\lambda)} \; .$$

Moreover, we have

$$|G_\sigma(h)| \leq L_{21} + L_{22}\,\lambda \; .$$

By means of Eq.(8) we find

$$|(\mathcal{K}\sigma)(x)| \leq \frac{L_{21} + L_{22}\,\lambda}{\beta(\lambda)} = \lambda \; .$$

This proves Assertion 5.1.

Assertion 5.2 $\mathcal{K}$ is a contraction.

Take $\sigma_1, \sigma_2 \in Z_\lambda, x \in X$. Using the abbreviations $G_i := G_{\sigma_i}(h)$ and $F_i := F_{\sigma_i}(h), i = 1, 2$, we may write

$$\begin{aligned} (\mathcal{K}\sigma_1)(x) - (\mathcal{K}\sigma_2)(x) &= G_1\, F_1^{-1} - G_2\, F_2^{-1} \\ &= (G_1 - G_2)\, F_1^{-1} + G_2\, F_1^{-1}(F_2 - F_1)\, F_2^{-1} \; . \end{aligned} \tag{22}$$

Hence, we get

$$\begin{aligned} &|(\mathcal{K}\sigma_1)(x) - (\mathcal{K}\sigma_2)(x)| \leq |F_1^{-1}||G_1 - G_2| + |G_2||F_1^{-1}||F_2^{-1}||F_1 - F_2| \\ \leq\; & \frac{1}{\beta(\lambda)}\, L_{22}|\sigma_1(h) - \sigma_2(h)| + (L_{21} + L_{22}\,\lambda)\, \frac{1}{\beta(\lambda)^2}\, L_{12}|\sigma_1(h) - \sigma_2(h)| \\ \leq\; & \frac{1}{\beta(\lambda)} \left(L_{22} + \frac{L_{21} + L_{22}\,\lambda}{\beta(\lambda)}\, L_{12} \right) |\sigma_1 - \sigma_2| = \frac{1}{\beta(\lambda)}\, (L_{22} + L_{12}\,\lambda)|\sigma_1 - \sigma_2| \end{aligned}$$

where to obtain the last expression we again have used Eq.(8).

By Condition B4(1) the constant

$$\chi_1(\lambda) := \frac{L_{22} + L_{12}\,\lambda}{\beta(\lambda)} \tag{23}$$

is less than 1 and hence, $\mathcal{K}$ is a contraction. It follows that $\mathcal{K}$ has a unique fixed point in Z_λ which we denote by σ.

We show that indeed σ is the derivative of s. We have to verify

Assertion 5.3

$$\sup_x \limsup_{\delta x \to 0} \frac{|s(x+\delta x) - s(x) - \sigma(x)\delta x|}{|\delta x|} = 0\ .$$

Note that the left hand-side exists since the quotient is bounded by 2λ. We define $\delta h = h(x+\delta x) - h(x)$ and $\Delta(x, \delta x) = s(x+\delta x) - s(x) - \sigma(x)\delta x$. Take $x \in X$. By means of Eq.(12) we obtain expanding g at $v = (h, s(h))$

$$\begin{aligned} s(x+\delta x) - s(x) &= g_1(v)\delta h + g_2(v)\sigma(h)\delta h + g_2(v)\big(s(h+\delta h) - s(h) - \sigma(h)\delta h\big) \\ &\quad + o(|\delta h|) \quad \text{for } |\delta x| \to 0 \\ &= G_\sigma(h)\delta h + g_2(v)\ \Delta(h, \delta h) + o(|\delta h|) \quad \text{for } |\delta x| \to 0\ . \end{aligned} \tag{24}$$

Taking Eq.(10) with x replaced by $x + \delta x$ we have

$$x + \delta x = f_0(h(x) + \delta h) + \hat{f}(h + \delta h, s(h + \delta h))\ .$$

Subtracting Eq.(10) yields

$$\begin{aligned} \delta x &= f_0'(h)\delta h + \hat{f}_1(v)\delta h + \hat{f}_2(v)\sigma(h)\delta h \\ &\quad + \hat{f}_2(v)\big(s(h+\delta h) - s(h) - \sigma(h)\delta h\big) + o(|\delta h|) \\ &= F_\sigma(h)\delta h + \hat{f}_2(v)\ \Delta(h, \delta h) + o(|\delta h|)\ . \end{aligned} \tag{25}$$

By definition of $\mathcal{K}$ (cf. Eq.(20)) we have $\sigma(x)\delta x = G_\sigma(h)\,F_\sigma(h)^{-1}\delta x$. Hence, using Eqs. (24), (25) we obtain

$$\begin{aligned} &|s(x+\delta x) - s(x) - \sigma(x)\delta x| = |\Delta(x, \delta x)| \\ &= |g_2(v)\Delta(h, \delta h) - G_\sigma(h) F_\sigma(h)^{-1} \hat{f}_2(v)\ \Delta(h, \delta h)| + o(|\delta h|) \\ &\le \left(L_{22} + \frac{L_{21} + L_{22}\,\lambda}{\beta(\lambda)}\ L_{12}\right) |\Delta(h, \delta h)| + o(|\delta h|) \\ &= (L_{22} + L_{12}\,\lambda)\ |\Delta(h, \delta h)| + o(|\delta h|)\ . \end{aligned}$$

Dividing by $|\delta x|$ we get

$$\frac{|\Delta(x,\delta x)|}{|\delta x|} \leq (L_{22}+L_{12}\,\lambda)\,\frac{|\delta h|}{|\delta x|}\,\frac{|\Delta(h,\delta h)|}{|\delta h|}+\frac{o(|\delta h|)}{|\delta x|}\ .$$

Since $|\delta h| \leq L_h|\delta x|$ and $L_h = 1/\beta(\lambda)$ we have

$$\frac{|\Delta(x,\delta x)|}{|\delta x|} \leq \frac{L_{22}+L_{12}\,\lambda}{\beta(\lambda)}\,\frac{|\Delta(h,\delta h)|}{|\delta h|}+o(1) \quad \text{for } |\delta x| \to 0\ .$$

Defining $M(x) := \limsup\limits_{\delta x\to 0}\ \dfrac{|\Delta(x,\delta x)|}{|\delta x|}\ \leq\ 2\lambda$ and $M := \sup\limits_{x}\ M(x)$ we obtain

$$M(x) \leq \chi_1(\lambda) M(h) \leq \chi_1(\lambda) M\ .$$

and therefore,

$$M \leq \chi_1(\lambda) M\ .$$

Since $\chi_1(\lambda) < 1$, this implies $M = 0$ and hence Assertion 5.3 holds. This completes the proof of Lemma 6. $\perp$

Lemma 7 *If f and g are of class $C_b^{1,1}$ and if B4(2) holds then s is of class $C_b^{1,1}$.*

Proof: The existence of s' is established in Lemma 6.

We show that the operator $\mathcal{K}$ defined in Eq.(20) maps the space

$$Z_{\lambda,\gamma} := \{\sigma \in Z_\lambda \mid \sigma \ \text{ is } \ \gamma-\text{Lipschitz}\}$$

into itself if γ is chosen appropriately. The space $Z_{\lambda,\gamma}$ again equipped with the supremum norm is a closed subspace of Z_λ. Take $\sigma \in Z_{\lambda,\gamma}$ for some γ not yet determined. We have

$$(\mathcal{K}\sigma)(x_1)-(\mathcal{K}\sigma)(x_2) = G_\sigma(h(x_1))F_\sigma(h(x_1))^{-1}-G_\sigma(h(x_2))F_\sigma(h(x_2))^{-1} =: G_{(1)}F_{(1)}^{-1}-G_{(2)}F_{(2)}^{-1}\ .$$

If we rewrite this expression as in Eq.(22) we get

$$|(\mathcal{K}\sigma)(x_1)-(\mathcal{K}\sigma)(x_2)| \leq |F_{(1)}^{-1}|\ |G_{(1)}-G_{(2)}| + |G_{(2)}|\ |F_{(1)}^{-1}|\ |F_{(2)}^{-1}|\ |F_{(1)}-F_{(2)}|\ .$$

Using the abbreviations $v_i = v(x_i)$, $\ h_i = h(x_i)$ it follows that

$$\begin{aligned}
|(\mathcal{K}\sigma)(x_1)-(\mathcal{K}\sigma)(x_2)| \ &\leq\ \frac{1}{\beta(\lambda)}\ |\ g_1(v_1)-g_1(v_2)+g_2(v_1)\sigma(h_1)-g_2(v_2)\sigma(h_1)\\
&\qquad\qquad + g_2(v_2)\sigma(h_1)-g_2(v_2)\sigma(h_2)|\\
&+\ \frac{L_{21}+L_{22}\,\lambda}{\beta(\lambda)^2}\,\Big|\ f_0'(h_1)-f_0'(h_2)+\hat f_1(v_1)-\hat f_1(v_2)\\
&\qquad + \hat f_2(v_1)\sigma(h_1)-\hat f_2(v_2)\sigma(h_1)+\hat f_2(v_2)\sigma(h_1)-\hat f_2(v_2)\sigma(h_2)\Big|\ .
\end{aligned}$$

Hence, we obtain

$$
\begin{aligned}
|(\mathcal{K}\sigma)(x_1) - (\mathcal{K}\sigma)(x_2)| &\leq \frac{1}{\beta(\lambda)} \Big(L_{g_1}(1+\lambda)L_h + L_{g_2}(1+\lambda)L_h\,\lambda + L_{22}\,\gamma\,L_h \Big)|x_1 - x_2| \\
&+ \frac{\lambda}{\beta(\lambda)} \Big(L_{f_0'}\,L_h + L_{\hat{f}_1}(1+\lambda)L_h + L_{\hat{f}_2}(1+\lambda)L_h\,\lambda + L_{12}\,\gamma\,L_h \Big)|x_1 - x_2| \\
&= \left[\frac{1}{\beta(\lambda)^2} \Big(L_{22} + L_{12}\,\lambda \Big)\gamma + R \right] |x_1 - x_2| \; .
\end{aligned}
$$

Here $L_{g_1}, L_{g_2}, ...$ denote the Lipschitz constants of $g_1, g_2, ...$ and R contains all terms not involving γ.

Since $\chi_2(\lambda) := (L_{22} + L_{12}\,\lambda)/\beta(\lambda)^2 < 1$ by Condition B4(2) we may put $\gamma = R/(1 - \chi_2(\lambda))$. The above estimate shows that the operator $\mathcal{K}$ maps $Z_{\lambda,\gamma}$ into itself. Therefore, the fixed point s' is γ-Lipschitz. $\perp$

Lemma 8 *If f and g are of class C_b^1 and if B4(1) holds then h is of class C_b^1 and (cf. (18), (19))*

$$h'(x) = F(h(x))^{-1} \; .$$

Proof: From Lemma 1 we know that $h^{-1} : X \ni x \mapsto f(x, s(x)) \in X$ is bijective. Due to Lemma 6 it is of class C_b^1. Moreover, $(h^{-1})'(x) = F_{s'}(x)$ and $(h^{-1})'$ is an isomorphism of X (cf. Eq.(21)). With $\left| F_{s'}(x)^{-1} \right| \leq 1/\beta(\lambda)$ the claim of Lemma 8 follows from the local inverse theorem. $\perp$

Proof of Theorem 5 ii), iii): The proof is by induction with respect to the order of differentiability. We have already proved the case $k = 1$. We shall derive a functional equation for $s^{(j)}, j = 2, 3, ...$, by formally differentiating Eqs.(17) and (18). This functional equation is taken as a fixed point equation for some operator $\mathcal{K}^{(j)}$. The operator $\mathcal{K}^{(j)}$ is shown to be a contraction in some suitable function space. Finally, we shall verify that the unique fixed point of $\mathcal{K}^{(j)}$ is the j-th derivative of s.

First, we get for $j = 2$

$$
\begin{aligned}
s''(x) &= G'(h)[h']^2 + G(h)h'' \\
0 &= F'(h)[h']^2 + F(h)h''
\end{aligned}
\tag{26}
$$

where

$$
\begin{aligned}
G'(h) &= G_a'(h) + G_b'(h), \quad F'(h) = F_a'(h) + F_b'(h) \\
G_a(h) &= g_1(v),\ G_b(h) = g_2(v)s'(h), \quad F_a(h) = f_0'(h) + \hat{f}_1(v),\ F_b(h) = \hat{f}_2(v)s'(h) \\
G_a'(h) &= g_{11}(v) + g_{12}(v)s'(h) \\
G_b'(h) &= g_{21}(v)s'(h) + g_{22}(v)[s'(h)]^2 + g_2(v)s''(h) \\
F_a'(h) &= f_0''(h) + \hat{f}_{11}(v) + \hat{f}_{12}(v)s'(h) \\
F_b'(h) &= \hat{f}_{21}(v)s'(h) + \hat{f}_{22}(v)[s'(h)]^2 + \hat{f}_2(v)s''(h)\ .
\end{aligned} \tag{27}
$$

Note that, e.g., the two-linear form $\hat{f}_{22}(v)[s'(h)]^2[h']^2$ applied to ξ and η means $\hat{f}_{22}(v)[s'(h)h'(x)\xi, s'(h)h'(x)\eta]$.

Solving the second equation of (26) for h'' and inserting it in the first one we get the following functional equation for s''

$$
s''(x) = (G'(h) - G(h)F(h)^{-1}F'(h))[h']^2 \tag{28}
$$

where $G'(h)$ and $F'(h)$ contain terms involving $s''(h)$. Collecting the terms containing $s''(h)$ we have

$$
s''(x) = W(x)s''(h)[h']^2 + T_2(x)
$$

where

$$
W(x) := g_2(v) - G(h)F(h)^{-1}\hat{f}_2(v) \tag{29}
$$

and

$$
\begin{aligned}
T_2(x) := &\Big\{g_{11}(v) + 2g_{12}(v)s'(h) + g_{22}(v)[s'(h)]^2 \\
&- G(h)F(h)^{-1}(f_0''(h) + \hat{f}_{11}(v) + 2\hat{f}_{12}(v)s'(h) + \hat{f}_{22}[s'(h)]^2)\Big\}[h']^2\ .
\end{aligned}
$$

By now differentiating formally this equation we obtain a functional equation for $s^{(j)},\ j = 2, 3, ...,$

$$
s^{(j)}(x) = W(x)s^{(j)}(h)[h']^j + T_j(x) \tag{30}
$$

where for $j > 2$

$$
T_j(x) := W'(x)s^{(j-1)}(h)[h']^{j-1} + (j-1)W(x)s^{(j-1)}(h)[h']^{j-2}[h''] + T_{j-1}'(x)\ . \tag{31}
$$

Note that, e.g., $\Big(W'(x)s^{(j-1)}(h)[h']^{j-1}\Big)(\xi_1, ..., \xi_j) = W'(x)[\xi_1]s^{(j-1)}(h)[h'(x)\xi_2, ..., h'(x)\xi_j]$.

$T_j(x), j \geq 2$, contains derivatives of s and h up to order $j-1$ only.

Consider the spaces

$$Z_{\rho_j}^{(j)} := \{\sigma \in C^0(X, \mathcal{L}(X^j, Y)) \,\Big|\, |\sigma| \leq \rho_j\} \ , \quad j = 2, 3, ...,$$

equipped with the supremum norm. $\mathcal{L}(X^j, Y)$ denotes the space of multilinear functions from $X \times ... \times X$ into Y. For each j we define the operator $\mathcal{K}^{(j)}$ acting on $Z_{\rho_j}^{(j)}$ by

$$(\mathcal{K}^{(j)}\sigma)(x) := W(x)\,\sigma(h(x))[h'(x)]^j + T_j(x) \ . \tag{32}$$

We shall also need the spaces

$$Z_{\rho_j,\gamma_j}^{(j)} := \{\sigma \in Z_{\rho_j}^{(j)} \,\Big|\, \sigma \text{ is } \gamma_j - \text{Lipschitz}\} \ .$$

For $j = 1, 2, ...$ we state two assertions which we shall prove by induction.

Assertion A1(j): *If f and g are of class C_b^j and if Condition B4(j) holds then there is $\rho_j > 0$ such that*

I) $\mathcal{K}^{(j)} : Z_{\rho_j}^{(j)} \to Z_{\rho_j}^{(j)}$.

II) $\mathcal{K}^{(j)}$ *is a contraction with contractivity constant*

$$\chi_j(\lambda) := \frac{L_{22} + L_{12}\,\lambda}{\beta(\lambda)^j}$$

and hence has a unique fixed point $\sigma \in Z_{\rho_j}^{(j)}$.

III) $M := \sup\limits_x \limsup\limits_{\delta x \to 0} \dfrac{|s^{(j-1)}(x + \delta x) - s^{(j-1)}(x) - \sigma(x)\delta x|}{|\delta x|} = 0.$

Hence, s is of class C_b^j and $s^{(j)}$ satisfies $s^{(j)} = \mathcal{K}^{(j)}\, s^{(j)}$. Moreover, h is of class C_b^j.

Assertion A2(j): *If f and g are of class $C_b^{j,1}$ and if Condition B4(j+1) holds then there are constants ρ_j, γ_j such that*

$$\mathcal{K}^{(j)} : Z_{\rho_j,\gamma_j}^{(j)} \to Z_{\rho_j,\gamma_j}^{(j)} \ .$$

Thus, $s^{(j)} \in Z_{\rho_j,\gamma_j}^{(j)}$ and therefore, s is of class $C_b^{j,1}$.

We have already proved Assertions A1(1) and A2(1) (cf. Lemmas 5,6 and 7). Assume that Assertions A1(j) and A2(j) hold for $j = 1, 2, ..., k$. We now prove Assertion A1(k+1).

We first determine ρ_{k+1} such that I) holds. Assume $\sigma^{(k+1)} \in Z^{(k+1)}_{\rho_{k+1}}$ for some ρ_{k+1}. Taking norms in Eq.(29) and using the identity (8) we obtain

$$|W(x)| \leq L_{22} + \frac{L_{21} + L_{22}\,\lambda}{\beta(\lambda)}\, L_{12} = L_{22} + L_{12}\,\lambda\ . \tag{33}$$

Hence, Eq.(32) for $j = k+1$ yields

$$|(\mathcal{K}^{(k+1)}\,\sigma)(x)| \leq (L_{22} + L_{12}\,\lambda)(L_h)^{k+1}\rho_{k+1} + R_{k+1} = \chi_{k+1}(\lambda)\rho_{k+1} + R_{k+1}$$

where we have put $R_{k+1} := \sup_x\ |T_{k+1}(x)|$. The remainder T_{k+1} contains derivatives of s and h up to order k only. Since these derivatives are bounded by Assertion A1(k) we have $R_{k+1} < \infty$. Since $\chi_{k+1}(\lambda) < 1$ by Condition B4(k+1), Claim I) holds with $\rho_{k+1} = R_{k+1}/(1 - \chi_{k+1}(\lambda))$.

Claim II) follows directly from Eq.(32):

$$\begin{aligned} |(\mathcal{K}^{(k+1)}\sigma_1)(x) - (\mathcal{K}^{(k+1)}\sigma_2)(x)| &\leq |W(x)|(L_h)^{k+1}|\sigma_1(h) - \sigma_2(h)| \\ &\leq \chi_{k+1}(\lambda)|\sigma_1 - \sigma_2|\ . \end{aligned}$$

By Assertion A2(k) we know that $s^{(k)}$ is γ_k-Lipschitz. It follows that M in Claim III) exists and is bounded by $\gamma_k + \rho_k$. We again put $\delta h = h(x + \delta x) - h(x)$ and $\Delta(x, \delta x) = s^{(k)}(x + \delta x) - s^{(k)}(x) - \sigma(x)\delta x$.

As in Lemma 6 we need show the inequality

$$\frac{|\Delta(x, \delta x)|}{|\delta x|} \leq \chi_{k+1}(\lambda)\ \frac{|\Delta(h, \delta h)|}{|\delta h|} + o(1) \quad \text{for } |\delta x| \to 0\ . \tag{34}$$

Since the functional equations for s' and $s^{(j)}$, $j \geq 2$, are different (cf. Eqs.(19), (30)) we prove Eq.(34) for $k = 1$ and $k > 1$ separately. For $k = 1$ we have from Eqs.(19), (28)

$$\begin{aligned} \Delta(x, \delta x) &= G(h + \delta h)\ F(h + \delta h)^{-1} - G(h)\ F(h)^{-1} \\ &- G'_\sigma(h)[h'\delta x][h'] + G(h)\ F(h)^{-1}\ F'_\sigma(h)[h'\delta x][h'] \end{aligned}$$

where G'_σ, F'_σ are defined as are G', F' in Eq.(27) with s'' replaced by σ. It is easy to verify that $\Delta(x, \delta x)$ may be rewritten as

$$\begin{aligned} \Delta(x, \delta x) &= \{G(h + \delta h) - G(h) - G'_\sigma(h)[h'\delta x]\}\ F(h + \delta h)^{-1} \\ &- G(h)\ F(h + \delta h)^{-1}\ \{F(h + \delta h) - F(h) - F'_\sigma(h)[h'\delta x]\}\ F(h)^{-1} \\ &+ G'_\sigma(h)[h'\delta x]\ \{F(h + \delta h)^{-1} - F(h)^{-1}\} \\ &- G(h)\{F(h + \delta h)^{-1} - F(h)^{-1}\}\ F'_\sigma(h)[h'\delta x]\ F(h)^{-1} \end{aligned} \tag{35}$$

where we have used $h' = F(h)^{-1}$. We treat each term of Eq.(35) separately. Again using the notations introduced in Eq.(27) (s'' replaced by σ) we find

$$\begin{aligned} G(h+\delta h) - G(h) - G'_\sigma(h)[h'\delta x] &= G_a(h+\delta h) - G_a(h) - G'_a(h)[h'\delta x] \\ &+ G_b(h+\delta h) - G_b(h) - G'_{\sigma,b}(h)[h'\delta x] \end{aligned}$$

$$\begin{aligned} &= G'_a(h)[\delta h - h'\delta x] \\ &+ g_2(v)s'(h+\delta h) + g_{21}(v)[s'(h+\delta h)][\delta h] \\ &+ g_{22}(v)[s'(h+\delta h)][s(h+\delta h) - s(h)] \\ &- g_2(v)s'(h) - g_{21}(v)[s'(h)][h'\delta x] - g_{22}(v)[s'(h)][s'(h)h'\delta x] \\ &- g_2(v)[\sigma(h)[h'\delta x]] + o(|\delta h|) \\ &= g_2(v)\,\Delta(h,\delta h) + o(|\delta x|) \end{aligned}$$

since

$$\delta h = h'\delta x + o(|\delta x|), \quad s'(h+\delta h) = s'(h) + o(1) \quad \text{for } |\delta x| \to 0$$

and

$$s(h+\delta h) - s(h) = s'(h)h'\delta x + o(|\delta x|)\ .$$

Similarly, we find

$$F(h+\delta h) - F(h) - F'_\sigma(h)[h'\delta x] = \hat{f}_2(v)\Delta(h,\delta h) + o(|\delta x|)\ .$$

The last two terms in Eq.(35) are of order $o(|\delta x|)$. It follows that

$$\Delta(x,\delta x) = g_2(v)\Delta(h,\delta h)F(h+\delta h)^{-1} - G(h)F(h+\delta h)^{-1}\hat{f}_2(v)\Delta(h,\delta h)F(h)^{-1} + o(|\delta x|)\ .$$

Taking norms we get

$$\begin{aligned} |\Delta(x,\delta x)| &\le \frac{1}{\beta(\lambda)}\Big[L_{22} + \frac{L_{21}+L_{22}\,\lambda}{\beta(\lambda)}\,L_{12}\Big]\,|\Delta(h,\delta h)| \ + \ o(|\delta x|) \\ &= \frac{L_{22}+L_{12}\,\lambda}{\beta(\lambda)}\,|\Delta(h,\delta h)| \ + \ o(|\delta x|) \end{aligned}$$

and hence Eq.(34) for $k=1$.

For the case $k>1$ we use Eq.(30) for $j=k$ and the fixed point equation $\sigma = \mathcal{K}^{(k+1)}\sigma$ (cf.

Eq.(32). We find

$$
\begin{aligned}
\Delta(x,\delta x) &= W(x+\delta x)s^{(k)}(h+\delta h)[h'(x+\delta x)]^k - W(x)s^{(k)}(h+\delta h)[h'(x+\delta x)]^k \\
&+ W(x)s^{(k)}(h+\delta h)[h'(x+\delta x)]^k - W(x)s^{(k)}(h+\delta h)[h']^k \\
&+ W(x)s^{(k)}(h+\delta h)[h']^k - W(x)s^{(k)}(h)[h']^k - W(x)\sigma(h)[h']^k[h'\delta x] \\
&+ T_k(x+\delta x) - T_k(x) - T_{k+1}(x)\delta x \; .
\end{aligned}
$$

Using the fact that $W(x+\delta x) - W(x) = W'(x)\delta x + o(|\delta x|)$ and $\delta h = h'(x)\delta x + o(|\delta x|)$ and taking Eq.(31) for $j = k+1$ we get

$$
\begin{aligned}
\Delta(x,\delta x) &= W'(x)\delta x\left(s^{(k)}(h+\delta h)[h'(x+\delta x)]^k - s^{(k)}(h)[h']^k\right) \\
&+ W(x)\left(s^{(k)}(h+\delta h)[h'(x+\delta x)]^k - s^{(k)}(h+\delta h)[h']^k - ks^{(k)}(h)[h']^{k-1}[h''\delta x]\right) \\
&+ W(x)\Delta(h,\delta h)[h']^k + T_k(x+\delta x) - T_k(x) - T_k'(x)\delta x + o(|\delta x|) \; .
\end{aligned}
$$

Since

$$
s^{(k)}(h+\delta h)([h'(x+\delta x)]^k - [h']^k) = ks^{(k)}(h+\delta h)[h']^{k-1}[h''(x)\delta x] + o(|\delta x|)
$$

we obtain

$$
\Delta(x,\delta x) = W(x)\Delta(h,\delta h)[h']^k + o(|\delta x|)
$$

where we have also used Assertion A2(k) for estimating the T_k-terms. Now Eq.(34) for $k > 1$ follows from Eq.(33).

Again as in the proof of Lemma 6 we conclude $M \leq \chi_{k+1}(\lambda)M$ and therefore $M = 0$. This terminates the proof of Claim III).

It remains to show that h is of class C_b^{k+1}. By means of Lemma 8 and by differentiating Eq.(26) we find

$$
h^{(k)}(x) = \begin{cases} \; F(h)^{-1} & \text{for} \quad k = 1 \\ - \; F(h)^{-1}\left(\hat{f}_2(v)s^{(k)}(h) + H_k(x)\right) & \text{for} \quad k \geq 2 \end{cases} \tag{36}
$$

where H_k contains derivatives of s and h up to order $k-1$ only. Since $s \in C_b^{k+1}$ and by induction we know that the right-hand side is in C_b^1. This completes the proof of Assertion A1(k+1).

We prove Assertion A2(k+1). Take $\sigma \in Z^{(k+1)}_{\rho_{k+1},\gamma_{k+1}}$ for some γ_{k+1} not yet determined.

Consider

$$\begin{aligned}
|(\mathcal{K}^{(k+1)}\sigma)(x_1) - (\mathcal{K}^{(k+1)}\sigma)(x_2)| &= |W(x_1)\sigma(h_1)[h_1^{'}]^{k+1} - W(x_2)\sigma(h_1)[h'_1]^{k+1} \\
&+ W(x_2)\sigma(h_1)[h_1^{'}]^{k+1} - W(x_2)\sigma(h_2)[h_1^{'}]^{k+1} \\
&+ W(x_2)\sigma(h_2)[h_1^{'}]^{k+1} - W(x_2)\sigma(h_2)[h'_2]^{k+1} \\
&+ T_{k+1}(x_1) - T_{k+1}(x_2)| \\
&\leq |W(x_1) - W(x_2)|\,|\sigma(h_1)|\,(L_h)^{k+1} \\
&+ |W(x_2)|\,|\sigma(h_1) - \sigma(h_2)|\,(L_h)^{k+1} \\
&+ |W(x_2)|\,|\sigma(h_2)|\,(k+1)(L_h)^k|h_1^{'} - h_2^{'}| \\
&+ |T_{k+1}(x_1) - T_{k+1}(x_2)| \\
&\leq (L_{22} + L_{12}\,\lambda)(L_h)^{k+1}|\sigma(h_1) - \sigma(h_2)| + Q_{k+1}|x_1 - x_2|
\end{aligned}$$

for some $Q_{k+1} > 0$. We have used the fact that W and T_{k+1} contain derivatives of s and h up to order k only and hence are Lipschitz. Moreover, we have estimated $|W|$ using Eq.(33). Since

$$|\sigma(h_1) - \sigma(h_2)| \leq \gamma_{k+1}|h_1 - h_2| \leq \gamma_{k+1}L_h|x_1 - x_2|$$

we have

$$|(\mathcal{K}^{(k+1)}\sigma)(x_1) - (\mathcal{K}^{(k+1)}\sigma)(x_2)| \leq (\chi_{k+2}(\lambda)\gamma_{k+1} + Q_{k+1})|x_1 - x_2|\ .$$

Condition B4(k+2) implies $\chi_{k+2}(\lambda) < 1$ and hence, Assertion A2(k+1) holds with the choice $\gamma_{k+1} = Q_{k+1}/(1 - \chi_{k+2}(\lambda))$.

This completes the proof of Theorem 5 ii), iii). $\perp$

As a corollary of Theorem 5 we show that a $C_b^{k,1}$-perturbation of the map leads to a C_b^k-perturbation of the invariant manifold.

Corollary 9 *Let the maps P and $\overline{P}$ defined in Eq.(14) satisfy Assumption H1 with common Lipschitz constants $\alpha, L_{11}, L_{12}, L_{21}, L_{22}$. Assume that f, g and $\overline{f}, \overline{g}$ are of class $C_b^{k,1}$ and let Conditions (9) a) and B4(k+1), B5 be satisfied. Moreover, let the derivatives of f and g satisfy*

$$\left.\begin{aligned}
\left|D^j(f(x,y) - \overline{f}(x,y))\right| &\leq \delta \\
\left|D^j(g(x,y) - \overline{g}(x,y))\right| &\leq \delta
\end{aligned}\right\} \quad for\ 0 \leq j \leq k\,.$$

Then there is a constant c such that for the manifolds M_s of P and $M_{\overline{s}}$ of $\overline{P}$ the following holds:

$$|s^{(j)}(x) - \overline{s}^{(j)}(x)| \leq c\,\delta \qquad \textit{for } \; j = 1, ..., k \; .$$

Proof: The proof is by induction. We first show the case $k = 1$. Using the functional equation (19) for s' we get

$$\begin{aligned} |s'(x) - \overline{s}\,'(x)| &= \left|G(h)\,F(h)^{-1} - \overline{G}(\overline{h})\,\overline{F}(\overline{h})^{-1}\right| \\ &\leq \left|G(h)\,F(h)^{-1} - G(\overline{h})\,F(\overline{h})^{-1}\right| \\ &\quad + \left|G(\overline{h})\,F(\overline{h})^{-1} - \overline{G}(\overline{h})\,\overline{F}(\overline{h})^{-1}\right| \\ &=: D_1 + D_2 \; . \end{aligned}$$

We estimate the two terms on the right-hand side separately. We have $G(h) = g_1(h, s(h)) + g_2(h, s(h))s'(h)$ and $F(h) = f_0'(h) + \hat{f}_1(h, s(h)) + \hat{f}_2(h, s(h))s'(h)$. Using the identity in Eq.(22) our assumptions imply that $D_1 \leq c_1|h - \overline{h}|$ and by means of Eq.(16) $D_1 \leq c_2\delta$.

For estimating D_2 we again shall use the identity in Eq.(22). We then have to estimate $|G - \overline{G}|$ and $|F - \overline{F}|$. Using Corollary 4 we get

$$\begin{aligned} |G(\overline{h}) - \overline{G}(\overline{h})| &\leq |g_1(\overline{h}, s(\overline{h})) - \overline{g}_1(\overline{h}, \overline{s}(\overline{h}))| \\ &\quad + |g_2(\overline{h}, s(\overline{h}))s'(\overline{h}) - \overline{g}_2(\overline{h}, \overline{s}(\overline{h}))\overline{s}\,'(\overline{h})| \\ &\leq c_3\,\delta + L_{22}\,|s' - \overline{s}\,'| \end{aligned}$$

and analogously

$$|F(\overline{h}) - \overline{F}(\overline{h})| \leq c_4\,\delta + L_{12}\,|s' - \overline{s}\,'| \; . \tag{37}$$

Now, by the identity in Eq.(22) we obtain (compare with the estimate after Eq.(22))

$$\begin{aligned} D_2 &\leq c_5\,\delta \; + \left(\frac{1}{\beta(\lambda)}\,L_{22} + (L_{21} + L_{22}\,\lambda)\,\frac{1}{\beta(\lambda)^2}\,L_{12}\right)|s' - \overline{s}\,'| \\ &= c_5\,\delta \; + \frac{L_{22} + L_{12}\,\lambda}{\beta(\lambda)}\,|s' - \overline{s}\,'| \; . \end{aligned}$$

Since $\big(L_{22} + L_{12}\,\lambda)/\beta(\lambda) = \chi_1(\lambda) < 1$ we have

$$|s' - \overline{s}\,'| \leq \frac{c_2 + c_5}{1 - \chi_1(\lambda)}\,\delta$$

which proves the assertion for $k=1$.

Assume the assertion of Corollary 9 holds for $k-1$. We show that it also holds for k. From the functional equation (30) for $s^{(k)}$ we get

$$
\begin{aligned}
\left|s^{(k)}(x)-\overline{s}^{(k)}(x)\right| &= \left|W(x)\, s^{(k)}(h)[h']^k \;+\; T_k(x)-\overline{W}(x)\,\overline{s}^{(k)}(\overline{h})[\overline{h}\,']^k-\overline{T}_k(x)\right| \\
&\leq \left|W(x)\, s^{(k)}(h)[h\,']^k \;-\; W(x)\, s^{(k)}(\overline{h})[h\,']^k\right| \\
&\quad + \left|W(x)\, s^{(k)}(\overline{h})[h\,']^k \;-\; W(x)\, s^{(k)}(\overline{h})[\overline{h}\,']^k\right| \\
&\quad + \left|W(x)\, s^{(k)}(\overline{h})[\overline{h}\,']^k \;-\; W(x)\, \overline{s}^{(k)}(\overline{h})[\overline{h}\,']^k\right| \\
&\quad + \left|W(x)\, \overline{s}^{(k)}(\overline{h})[\overline{h}\,']^k \;-\; \overline{W}(x)\, \overline{s}^{(k)}(\overline{h})[\overline{h}\,']^k\right| \\
&\quad + \left|T_k(x)-\overline{T}_k(x)\right| \\
&=: d_1+d_2+d_3+d_4+d_5\ .
\end{aligned}
$$

We estimate each term separately. a) Using Eq.(16) our assumptions imply that $d_1 \leq C_1\,\delta$.

b) We have $d_2 \leq C_2|h'-\overline{h}\,'|$. By Lemma 7 we know that $h'(x)=F(h(x))^{-1}$. We use the identity $F(h)^{-1}-\overline{F}(\overline{h})^{-1}=F(h)^{-1}(\overline{F}(\overline{h})-F(h))\,\overline{F}(\overline{h})^{-1}$. By means of Eq.(37) we estimate

$$
\begin{aligned}
|\overline{F}(\overline{h})-F(h)| &\leq |\overline{F}(\overline{h})-\overline{F}(h)| \;+\; |\overline{F}(h)-F(h)| \\
&\leq C_3|h-\overline{h}| \;+\; c_4\,\delta+L_{12}\,|s'-\overline{s}'|\ .
\end{aligned}
$$

Now, Eq.(16) and the assertion for $k=1$ proved above yields $d_2 \leq C_4\,\delta$.

c) Using Eqs.(33) and (11) we obtain

$$
d_3 \leq (L_{22}+L_{12}\,\lambda)\,\frac{1}{\beta(\lambda)^k}\,|\,s^{(k)}-\overline{s}^{(k)}|=\chi_k(\lambda)|s^{(k)}-\overline{s}^{(k)}|\ .
$$

d) By means of Eq.(29) we have

$$
\begin{aligned}
\left|W(x)-\overline{W}(x)\right| = \Big|&g_2(h,s(h))-G(h)\,F(h)^{-1}\,\hat{f}_2(h,s(h)) \\
&-\overline{g}_2(\overline{h},\overline{s}(\overline{h}))+\overline{G}(\overline{h})\,\overline{F}(\overline{h})^{-1}\,\hat{\overline{f}}_2(\overline{h},\overline{s}(\overline{h}))\Big|
\end{aligned}
$$

As in b) we find $d_4 \leq C_5\,\delta$.

e) $T_k(x)$ contains derivatives of s and h up to order $k-1$ (cf. Eq.(31)). By Eq.(36) the derivatives $h^{(j)}(x)$, $j < k$, may be expressed in terms of derivatives of s up to order $k-1$. Therefore, our assumptions imply $\left| h^{(j)} - \overline{h}^{(j)} \right| \leq C_6\,\delta$ for $j < k$. Now it is easily seen that $d_5 \leq C_7\,\delta$.

Note that Condition B4(k+1) implies $\chi_k(\lambda) < 1$. Combining the above estimates we find

$$\left| s^{(k)} - \overline{s}^{(k)} \right| \leq \frac{C_1 + C_4 + C_5 + C_7}{1 - \chi_k(\lambda)}\,\delta$$

which proves the assertion of Corollary 9 for k. $\perp$

3. An application

Consider the autonomous system of ODEs

$$\begin{aligned} \dot{x} &= F(x,y) \\ \dot{y} &= G(x,y) \end{aligned} \tag{38}$$

where $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, $G : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ are bounded. For simplicity, we assume that F and G are C^1-functions with bounded derivatives. Let $(\varphi(t;x,y), \psi(t;x,y))$ be the solution of Eq.(38) with $\varphi(0;x,y) = x$ and $\psi(0;x,y) = y$. It exists for all $t \in \mathbb{R}$. Assume that there is $T > 0$ such that the time-T map P_T of (38)

$$P_T : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} := \begin{pmatrix} \varphi(T;x,y) \\ \psi(T;x,y) \end{pmatrix} \tag{39}$$

satisfies the assumptions of Theorem 3. It follows that P_T admits an attractive invariant manifold M_s. We want to show that this manifold M_s is invariant for the flow of Eq.(38).

Theorem 10 *Let F, G in Eq.(38) be bounded and of class C_b^k, $k \geq 1$. Assume that there is $T > 0$ such that the map P_T given in Eq.(39) satisfies Assumption H1 and Conditions (9) a), B4(k) and B5. Then there is a function $s : \mathbb{R}^m \to \mathbb{R}^n$ of class C_b^k such that the following assertions hold.*

I) The set $M_s = \{(x,y) \,|\, x \in X,\ y = s(x)\}$ is invariant under the differential equation (38), i.e., if $(x,y) \in M_s$ then also $(\varphi(t;x,y), \psi(t;x,y)) \in M_s$ for all $t \in \mathbb{R}$.

II) The manifold M_s satisfies the properties iii), iv), v), vi) and vii) of Theorem 3 for the map (39).

Proof: I) We show that $P_t(M_s) \subset M_s$ for all $t \in \mathbb{R}$ where P_t is defined as

$$P_t : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \varphi(t;x,y) \\ \psi(t;x,y) \end{pmatrix} .$$

For fixed t we define $\Omega := P_t(M_s)$. From the group property of the flow of Eq.(38) and from the invariance of M_s under P_T it follows that

$$P_T(\Omega) \;=\; P_T(P_t(M_s)) \;=\; P_t(P_T(M_s)) \;=\; P_t(M_s) \;=\; \Omega\,.$$

Using the maximality property vii) of Theorem 3 we conclude that $\Omega = P_t(M_s) \subset M_s$.

II) The assertions claimed are established in Theorem 3. $\perp$

Remark:

5) In applications it may occur that although the map P considered does not satisfy the assumptions of Theorem 5 there is $N > 0$ such that the N-th iterate P^N satisfies those assumptions. In a similar way as in the proof of Theorem 10 above it can be shown that the invariant manifold of P^N is also invariant under the map P. $\dashv$

4. An example

For illustration of the results in the previous sections we discuss a system of two weakly coupled harmonic oscillators. It is well known that under the assumptions made below such a system admits an attractive invariant torus. Consider the ODE

$$\begin{aligned} \dot{\tilde{\varphi}} &= w + \epsilon\,\widetilde{R}(\tilde{\varphi},\tilde{a},\epsilon) \\ \dot{\tilde{a}} &= \qquad \epsilon\,\widetilde{T}(\tilde{\varphi},\tilde{a},\epsilon) \end{aligned} \tag{40}$$

where $\tilde{\varphi},\tilde{a} \in \mathbb{R}^2$, $\epsilon \in (0,\epsilon_0)$ and $\widetilde{R}$ and $\widetilde{T}$ are 2π-periodic in $\tilde{\varphi}$ and of class C_b^k. For simplicity we assume that $\widetilde{R}$ and $\widetilde{T}$ have a finite Fourier series in $\tilde{\varphi}$

$$\begin{aligned} \widetilde{R}(\tilde{\varphi},\tilde{a},\epsilon) &= \sum_{n\in\mathbb{Z}^2} \widetilde{R}_n(\tilde{a},\epsilon)\, e^{i(n,\tilde{\varphi})} \\ \widetilde{T}(\tilde{\varphi},\tilde{a},\epsilon) &= \sum_{n\in\mathbb{Z}^2} \widetilde{T}_n(\tilde{a},\epsilon)\, e^{i(n,\tilde{\varphi})} \end{aligned}$$

and we suppose that $(n,w) = n_1w_1 + n_2w_2 \neq 0$ for all $n \in \mathbb{Z}^2$ with the property that $\widetilde{R}_n \not\equiv 0$ or $\widetilde{T}_n \not\equiv 0$. Such a system describes two weakly coupled harmonic oscillators in action-angle variables. By means of the method of averaging Eq.(40) may be transformed into the system

$$\begin{aligned} \dot{\varphi} &= w + \epsilon\,R(a) + \epsilon^2\,R^2(\varphi,a,\epsilon) \\ \dot{a} &= \qquad \epsilon\,T(a) + \epsilon^2\,T^2(\varphi,a,\epsilon) \end{aligned} \tag{41}$$

where the right-hand side is again of class C_b^k.

We consider the time-1 map of Eq.(41)

$$P : \begin{pmatrix} \varphi \\ a \end{pmatrix} \longmapsto \begin{pmatrix} \overline{\varphi} \\ \overline{a} \end{pmatrix} = \begin{pmatrix} \varphi + w & + & \epsilon\, R(a) + \epsilon^2\, \hat{R}(\varphi, a, \epsilon) \\ a & + & \epsilon\, T(a) + \epsilon^2\, \hat{T}(\varphi, a, \epsilon) \end{pmatrix} .$$

Starting with a dissipative perturbation of the oscillators, we assume that $T(0) = 0,\ DT(0)$ has eigenvalues with negative real parts. Therefore, for a small, the map P has the form

$$\begin{array}{rcl} \overline{\varphi} & = & \varphi + w + \epsilon\, R(a) + \epsilon^2\, \hat{R}(\varphi, a, \epsilon) \\ \overline{a} & = & [Id + \epsilon(DT(0) + \Delta(a))]\, a + \epsilon^2\, \hat{T}(\varphi, a, \epsilon) \end{array}$$

where $|\Delta(a)| \leq c|a|$. In the a-space we take a norm such that

$$|Id + \epsilon(DT(0) + \Delta(a))| < 1 - c_0\epsilon$$

for all a in a neighborhood of 0 and ϵ sufficiently small. In the φ-a-space we consider the strip $|a| \leq \rho,\ \varphi \in \mathbb{R}^2$, ρ sufficiently small. We verify the conditions of Theorem 5. According to Remark 0) Assumption H1 has to be verified in the strip $|a| \leq \rho$ only. We have

$$\begin{array}{rclcrcl} \alpha & = & 1 & & & & \\ L_{11} & = & c_1\epsilon^2 & , & L_{12} & = & c_2\epsilon \\ L_{21} & = & c_3\epsilon^2 & , & L_{22} & = & 1 - c_4\epsilon \end{array}$$

for $\epsilon \leq \epsilon_1,\ |a| \leq \rho$. Condition (9) a) takes the form

$$\sqrt{c_2\, c_3}\, \epsilon^{3/2} \leq c_4\epsilon - c_1\epsilon^2 .$$

Since

$$L_{12}\lambda \leq \frac{2L_{12}\, L_{21}}{\frac{1}{\alpha} - L_{11} - L_{22}} = \frac{2c_2c_3}{c_4 - c_1\epsilon}\, \epsilon^2 ,$$

Condition B4(k) follows from the condition

$$1 - c_4\epsilon + \frac{2c_2c_3}{c_4 - c_1\epsilon}\epsilon^2 < \left(1 - c_1\epsilon^2 - \frac{2c_2c_3}{c_4 - c_1\epsilon}\, \epsilon^2\right)^k .$$

Obviously, there is $\epsilon_2 > 0$ such that these two conditions are satisfied for $\epsilon \in (0, \epsilon_2),\ |a| \leq \rho,\ \varphi \in \mathbb{R}^2$.

Now, Theorem 5 implies that there is a function $s : \mathbb{R}^2 \ni \varphi \longmapsto a = s(\varphi) \in \mathbb{R}^2$, of class C_b^k, 2π-periodic in both components of φ, λ-Lipschitz with $\lambda = \frac{2c_3}{c_4 - c_1\epsilon}\, \epsilon$ and $|s| \leq \frac{|\hat{T}|}{c_4}\, \epsilon$. Moreover, the set

$$M_s = \{(\varphi, a) \mid \varphi \in \mathbb{R}^2,\ a = s(\varphi)\}$$

is invariant for the map P. It is attractive with attractivity constant $1-c_4\epsilon+\frac{2c_2c_3}{c_4-c_1\epsilon}\,\epsilon^2<1$ and the property of asymptotic phase holds. Applying Theorem 10 it follows that M_s is also invariant under the flow of Eq.(41). The transformation back to the original variables $\tilde{\varphi},\tilde{a}$ gives analogous conclusions for Eq.(40) and its time-1 map with slightly different constants. This means that Eq.(40) admits an attractive invariant torus.

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