Smooth attractive invariant manifolds of singularly perturbed ODE's

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Abstract

Under hypotheses suitable for applications an invariant manifold result for singularly perturbed ODE's is proved with sharp smoothness properties of the manifold.

Keywords: singular perturbations, attractive invariant manifold, smoothness

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Smooth attractive invariant manifolds of singularly perturbed ODE's

The aim of this paper is to improve a result stated in Nipp [4]. There, the existence of an invariant manifold for a singularly perturbed system of ODE's was derived without sharp smoothness properties, however. We now show that a C^k vector field yields a C^k manifold. A similar result given in Sakamoto [7] obtains a C^{k-1} manifold only. In Knobloch/Aulbach [3] the C^∞-case is considered with rather special hypotheses and with an outline of the proof only. Our results are based on assumptions appropriate for applications, and we tried to present a transparent proof. The proof is based on applying an invariant manifold result for maps established in Nipp/Stoffer [5] to an appropriate time map of the singularly perturbed system. Purfürst [6] also uses a time map approach. But he only shows C^1 smoothness. A very general treatment of the subject and sharp results can be found in Fenichel [2]. His abstract setting is not very transparent, however, and seems not well suited for applications. In order to improve applicability we have stated our result for bounded domains D in phase space. The hypothesis on D is weakend compared to Nipp [4]. We also prove the smooth dependence of the invariant manifold with respect to the perturbation parameter ϵ.

The paper is organized as follows. In Section 1 the invariant manifold result is stated for bounded domains. The proof is done by extending the vector field to the unbounded domain and by applying the corresponding invariant manifold result of Section 2. The invariant manifold result on the unbounded domain is stated in Section 2 and proved in Section 3.

1. A manifold result on bounded domains

Consider the singularly perturbed autonomous system

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \epsilon\,\frac{dy}{dt} &= g(x,y) \end{aligned} \tag{1}$$

where $\epsilon \in (0, \epsilon_0)$.

Let $D = D_1 \times D_2$ be a domain in $\mathbb{R}^m \times \mathbb{R}^n$. By C_b^k we denote spaces of functions of class C^k with bounded derivatives.

We make the following *assumptions:*

1) $k \geq 2$.

2) D is bounded and D_1 has a C^k-boundary.

3) $f \in C_b^k(D, \mathbb{R}^m)$, $g \in C_b^k(D, \mathbb{R}^n)$ and f and g are bounded in D.

4) There is a bounded function $s^0 \in C_b^k(D_1, D_2)$ such that $g(x, s^0(x)) = 0$ for $x \in D_1$.

5) There is a positive constant b_0 such that all eigenvalues of the Jacobian $B(x) := g_y(x, s^0(x))$ have real parts smaller than $-b_0$ for all $x \in D_1$.

Remark:

0) For simplicity only, we have omitted the dependence of the functions f and g on the parameter ϵ. If f and g depend on ϵ and are of class C_b^k also with respect to $\epsilon \in (-\epsilon_0, \epsilon_0)$ all the results of this paper hold identically as can easily be checked in the proof in Section 3. $\dashv$

Under the above assumptions it can be shown that for all $\epsilon > 0$ small enough Eq.(1) has a smooth attractive invariant manifold M_ϵ which is $O(\epsilon)$-close to the so called *reduced manifold* $M_0 := \{(x, y) \mid \ x \in D_1, \ y = s^0(x)\}$. The precise result is stated as

Theorem 1 *For every subdomain D_1' with $\overline{D_1'} \subset D_1$ and for every $\beta \in (0, b_0)$ there are positive constants ϵ^*, δ, K and a function $s \in C_b^k(D_1' \times (0, \epsilon^*), D_2)$ such that the following assertions hold for $\epsilon \in (0, \epsilon^*)$.*

i) (Invariance) *The set $M_\epsilon = \{(x, y) \mid x \in D_1', \, y = s(x, \epsilon)\} \subset D$ is invariant under Eq.(1); i.e., if $(x^0, y^0) \in M_\epsilon$ then also $(x(t), y(t)) \in M_\epsilon$ for all t as long as $x(t) \in D_1'$, $(x(t), y(t))$ being the solution of Eq.(1) with $(x(0), y(0)) = (x^0, y^0)$.*

ii) (Attractivity) *Every solution $(x(t), y(t))$ of Eq.(1) with $|y(0) - s^0(x(0))| \leq \delta$ satisfies*

$$|y(t) - s(x(t), \epsilon)| \leq K \, e^{-\beta t/\epsilon} \, |\, y(0) - s(x(0), \epsilon)|$$

for all $t \geq 0$ as long as $x(t) \in D_1'$.

iii) ("Asymptotic phase") *For every solution $(x(t), y(t))$ of Eq.(1) with initial values at $t = 0$ satisfying $|y^0 - s^0(x^0)| \leq \delta$ there is $(\tilde{x}^0, \tilde{y}^0) \in M_\epsilon$ such that for $(\tilde{x}(t), \tilde{y}(t))$ being the solution of Eq.(1) with $(\tilde{x}(0), \tilde{y}(0)) = (\tilde{x}^0, \tilde{y}^0)$*

$$|x(t) - \tilde{x}(t)| \leq K \, e^{-\beta t/\epsilon} \, |\, y^0 - s(x^0, \epsilon)|$$

$$|y(t) - \tilde{y}(t)| \leq K \, e^{-\beta t/\epsilon} \, |\, y^0 - s(x^0, \epsilon)|$$

holds for $t \geq 0$ as long as $x(t)$ and $\tilde{x}(t)$ are in D_1'.

iv) (Closeness to M_0)
$$s(x,0) = s^0(x) \quad \textit{for all } x \in D_1' \ .$$

v) (Maximality) *Every solution* $(x(t), y(t))$ *of Eq.(1) satisfying* $x(t) \in D_1'$ *and* $|y(t) - s^0(x(t))| \leq \delta$ *for all* $t \in \mathbb{R}$ *lies in* M_ϵ*, i.e.,* $y(t) = s(x(t), \epsilon)$ *for all* t.

Remarks:

1) As can be seen in the proofs in Section 3, the results of this paper also hold for the case $k = 1$, if the derivatives of g and s^0 have uniform Lipschitz constants. If in addition f is of class $C_b^{1,1}$ as well then the invariant manifold is also of class $C_b^{1,1}$ (cf. Theorem 5 of Nipp/Stoffer [5]).

2) The larger the order of differentiability k of the invariant manifold the smaller ϵ^* has to be taken; the constant δ, however, does not depend on k (see Section 3). Assume, e.g., that f and g are of class C_b^∞ then $s(x, \epsilon)$ is the smoother the smaller ϵ^* is taken.

3) Since, as stated in ii), the invariant manifold M_ϵ is exponentially attractive with an exponent $O(\epsilon^{-1})$, it makes sense to consider a bounded x-domain D_1.

4) In the case $D_1 = \mathbb{R}^m$ (Theorem 2 of Section 2) the invariant manifold $\overline{M}_\epsilon$ is unique in a neighbourhood of the reduced manifold $\overline{M}_0$. This follows from the maximality property v). If $D_1 \neq \mathbb{R}^m$, M_ϵ is not necessarily unique. This is due to the fact that the extension of the vectorfield to $x \in \mathbb{R}^m$ is not unique. (A simple example where such a manifold is not unique is given in Purfürst [6]). However, two invariant manifolds of Eq.(1) with properties ii), iv) are exponentially close with an exponent $O(\epsilon^{-1})$. Moreover, let $(x(t), y(t))$ be a solution of Eq.(1) with the properties required in v) (e.g., an equilibrium solution or a periodic solution) then all invariant manifolds of Eq.(1) with property iv) have to intersect in the trajectory of such a solution. $\dashv$

Proof of Theorem 1: Consider the system

$$\begin{aligned} \frac{dx}{d\tau} &= \epsilon\, f(x,y) \\ \frac{dy}{d\tau} &= \quad g(x,y) \end{aligned} \tag{2}$$

where $\epsilon \in (-\epsilon_0, \epsilon_0)$. Note that if $(x(\tau, \epsilon), y(\tau, \epsilon))$ is a solution of Eq.(2) then for non-positive ϵ-values excluded $(\hat{x}(t, \epsilon), \hat{y}(t, \epsilon)) := (x(t/\epsilon, \epsilon), y(t/\epsilon, \epsilon)$ is a solution of Eq.(1). Hence, if we can show the results corresponding to Theorem 1 for Eq.(2) they also hold for Eq.(1). Theorem 1 (for Eq.(2)) is proved by first extending the right-hand side of Eq.(2) to $x \in \mathbb{R}^m$ such that Assumptions 1), 3), 4), 5) hold for all $x \in \mathbb{R}^m$ and then applying Theorem 2 of Section 2 which deals with the case $D_1 = \mathbb{R}^m$.

We show how a scalar function defined in D_1 may be extended to an appropriate function defined in $\mathbb{R}^m$. For every $x \in D_1$ let $\Theta(x)$ be defined as $\Theta(x) := \min_{u \in \partial D_1} |x-u|$ and consider, for $\Theta_0 > 0$ small, the set (see Fig. 1)

$$\Omega_1^{\Theta_0} := \{x \in D_1 \mid \Theta(x) < \Theta_0\} \subset D_1 \ .$$

Then, for Θ_0 small enough the following statement holds:

$$\Theta(x) \in C_b^k(\Omega_1^{\Theta_0}, \mathbb{R}) \ \textit{and the domain} \ D_1^{\Theta_0} := \overline{D_1} \backslash \overline{\Omega_1^{\Theta_0}} \ \textit{has a} \ C^k\textit{-boundary.}$$

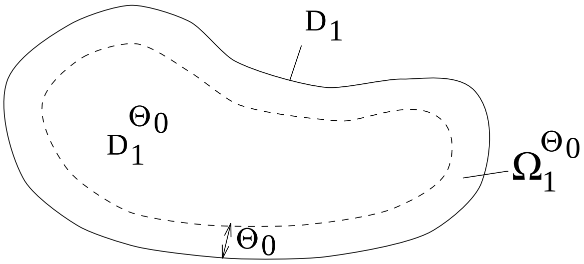

Fig. 1

Next, consider the scalar C^∞-function ρ defined as

$$\rho(a) := \begin{cases} 0 & , \ a \le 0 \\ \exp\Big(1 - \frac{1}{a}\exp(a-1)\Big) & , \ 0 < a < 1 \\ 1 & , \ 1 \le a \end{cases}$$

and sketched in Fig. 2. With

$$\overline{\overline{\Theta}}(x) := \begin{cases} 0 & , \ x \in \mathbb{R}^m \backslash D_1 \\ \Theta(x) & , \ x \in \Omega_1^{\Theta_0} \\ \Theta_0 & , \ x \in \overline{D_1^{\Theta_0}} \end{cases}$$

define

$$\overline{\Theta}(x) := \rho\left(\frac{\overline{\overline{\Theta}}(x)}{\Theta_0}\right) \ .$$

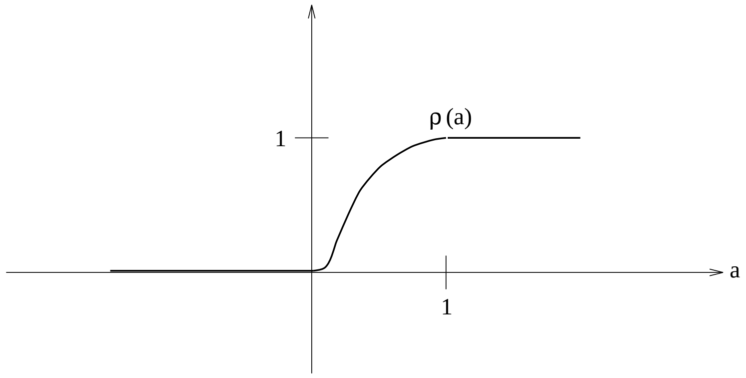

Fig. 2

Finally, for any $q \in C_b^r(D_1, \mathbb{R}),\ 1 \le r \le k$, and q bounded define

$$\overline{\overline{q}}(x) \ := \ \begin{cases} q(x) & , \quad x \in D_1 \\ 0 & , \quad x \in \mathbb{R}^m \backslash D_1 \end{cases}$$

$$\overline{q}(x) \ := \ \overline{\Theta}(x)\, \overline{\overline{q}}(x), \quad x \in \mathbb{R}^m \ .$$

Then, it holds that

$$\overline{q}(x) \in C_b^r(\mathbb{R}^m, \mathbb{R}) \ \textit{ and } \ \overline{q} \ \textit{ is uniformly bounded for } \ x \in \mathbb{R}^m.$$

We introduce the vector functions $F(x,z) := f(x, s^0(x)+z),\ G(x,z) := g(x, s^0(x)+z)$ and the matrix function $\hat{G}(x,z)$ by means of $[B(x) + \hat{G}(x,z)]\, z := G(x,z)$, and we consider the system

$$\text{(3)} \qquad \begin{aligned} \frac{dx}{d\tau} \ &= \ \epsilon\, F(x,z) \\ \frac{dz}{d\tau} \ &= \ \left[B(x) + \hat{G}(x,z)\right] z - \epsilon\, s_x^0(x)\, F(x,z) \end{aligned}$$

for $|\epsilon| < \epsilon_0,\ x \in D_1,\ |z| \le d_0$ with $d_0 > 0$ such that $\{(x,y) \,|\, x \in D_1,\ |y - s^0(x)| \le d_0\} \subset D$ (if $d_0 > 0$ is not possible, we redefine D_1 as $D_1^{\Theta_0/2}$) and such that $z = 0$ is the only solution of $G(x,z) = 0$ in $D_1 \times \{|z| \le d_0\}$. And we extend the components of the vector functions F, s^0 and the elements of the matrix function $\hat{G}$ with respect to x in the above way to

$$\overline{F}(x,z) \ := \ \overline{\Theta}(x)\, \overline{\overline{F}}(x,z), \quad \overline{s^0}(x) \ := \ \overline{\Theta}(x)\, \overline{\overline{s^0}}(x)$$

$$\overline{\hat{G}}(x,z) \ := \ \overline{\Theta}(x)\, \overline{\overline{\hat{G}}}(x,z)\ .$$

If we in addition define

$$\overline{B}(x) := \overline{\Theta}(x)\,\overline{\overline{B}}(x) - b_0[1-\overline{\Theta}(x)]I_n$$

and

$$\overline{G}(x,z) := \Big[\overline{B}(x) + \overline{\hat{G}}(x,z)\Big]z$$

then the vector field of the "extended" system

$$(\overline{3}) \qquad \begin{aligned} x' &= \epsilon\,\overline{F}(x,z) \\ z' &= \Big[\overline{B}(x) + \overline{\hat{G}}(x,z)\Big]\,z - \epsilon\,\overline{s^0_x}(x)\,\overline{F}(x,z) \end{aligned}$$

$\left({}' := \frac{d}{d\tau}\right)$ coincides with the one of Eq.(3) for $x \in D_1^{\Theta_0}$. Moreover, all functions are uniformly bounded for $x \in \mathbb{R}^m$ and $z \in D^n_{d_0} := \{z \in \mathbb{R}^n \mid |z| \le d_0\}$ and satisfy

$\overline{F} \in C_b^k(\mathbb{R}^m \times D^n_{d_0}, \mathbb{R}^m)$, $\overline{G} \in C_b^k(\mathbb{R}^m \times D^n_{d_0}, \mathbb{R}^n)$, $\overline{B} \in C_b^{k-1}\left(\mathbb{R}^m, \mathbb{R}^{n\times n}\right)$,
$\overline{\hat{G}} \in C_b^{k-1}\left(\mathbb{R}^m \times D^n_{d_0}, \mathbb{R}^{n\times n}\right)$ with $\overline{\hat{G}} = O(|z|)$ uniformly for $x \in \mathbb{R}^m$, $z \in D^n_{d_0}$,
$\overline{s^0} \in C_b^k\left(\mathbb{R}^m, \mathbb{R}^n\right)$, $\overline{s^0_x} \in C_b^{k-1}\left(\mathbb{R}^m, \mathbb{R}^{m\times n}\right)$.

The real parts of the eigenvalues of the matrix $\overline{B}(x)$ are smaller than $-b_0 < 0$ for all $x \in \mathbb{R}^m$.

Now, introducing the transformation

$$z = y - \overline{s^0}(x)$$

into Eq.$(\overline{3})$ we obtain the system

$$(\overline{2}) \qquad \begin{aligned} x' &= \epsilon\,\overline{F}(x, y - \overline{s^0}(x)) &&=: \epsilon\,\overline{f}(x,y) \\ y' &= \Big[\overline{B}(x) + \overline{\hat{G}}(x, y - \overline{s^0}(x))\Big](y - \overline{s^0}(x)) &&=: \overline{g}(x,y) \end{aligned}$$

where we have used the identity $\overline{s^0}(x)' = \epsilon\,\overline{s^0_x}(x)\,\overline{F}(x, y - \overline{s^0}(x))$. Obviously, the functions $\overline{f}$ and $\overline{g}$ are uniformly bounded in the space

$$\Omega^{m+n}_{d_0} = \{(x,y) \mid\; x \in \mathbb{R}^m,\; |y - \overline{s^0}(x)| \le d_0\} \subset \mathbb{R}^{m+n}$$

and are of class C_b^k there, and in addition Assumptions 4) and 5) are satisfied for $x \in \mathbb{R}^m$. (Note that the functions $\overline{f}, \overline{g}, \overline{B}, \overline{s^0}$ coincide with the functions f, g, B, s^0 for $x \in D_1^{\Theta_0}$).

Hence, Theorem 2 applies to the system $(\overline{2})$. The results of Theorem 2 carry over to the system (2) if we take $x \in D_1^{\Theta_0}$ and τ such that $x(\tau)$ stays in $D_1^{\Theta_0}$ and finally to the system (1) if we put $\tau = t/\epsilon$ and exclude non-positive ϵ-values.

Thus, the proof of Theorem 1 is reduced to the proof of Theorem 2. $\perp$

2. The result on the unbounded domain

Consider the system

$$
(4) \qquad \begin{aligned} \frac{dx}{d\tau} &= \epsilon\, f(x,y) \\ \frac{dy}{d\tau} &= g(x,y)\,, \end{aligned}
$$

$|\epsilon| < \epsilon_0$, with the following properties for $k \geq 2$: There is a function $s^0 \in C_b^k(\mathbb{R}^m, \mathbb{R}^n)$ such that $g(x, s^0(x)) = 0$ for $x \in \mathbb{R}^m$ and such that $B(x) := g_y(x, s^0(x))$ has eigenvalues with real parts smaller than $-b_0 < 0$ for all $x \in \mathbb{R}^m$. With respect to the space $\Omega_{d_0}^{m+n} := \{(x,y) \mid x \in \mathbb{R}^m,\ |\, y - s^0(x)\,| \leq d_0\} \subset \mathbb{R}^{m+n}$ the function $s^0(x)$ is the unique solution of $g(x,y) = 0$ and the functions f and g satisfy $f \in C_b^k(\Omega_{d_0}^{m+n}, \mathbb{R}^m)$, $g \in C_b^k(\Omega_{d_0}^{m+n}, \mathbb{R}^n)$ and they are bounded there.

Under the above conditions and for ϵ small enough Eq.(4) has a smooth attractive invariant manifold $\overline{M}_\epsilon$ which is $O(\epsilon)$-close to the reduced manifold $\overline{M}_0 := \{(x,y) \mid x \in \mathbb{R}^m,\ y = s^0(x)\}$. The result ist given in

Theorem 2 *For every $\beta \in (0, b_0)$ there are positive constants ϵ^*, δ, K and a function $s \in C_b^k(\mathbb{R}^m \times (-\epsilon^*, \epsilon^*), \mathbb{R}^n)$ such that the following assertions hold for $|\epsilon| < \epsilon^*$.*

i) (Invariance) *The set $\overline{M}_\epsilon = \{(x,y) \mid x \in \mathbb{R}^m,\ y = s(x,\epsilon)\} \subset \mathbb{R}^{m+n}$ is invariant under Eq.(4), i.e., if $(x^0, y^0) \in \overline{M}_\epsilon$ then also $(x(\tau),\, y(\tau)) \in \overline{M}_\epsilon$ for all $\tau \in \mathbb{R}$, $(x(\tau),\, y(\tau))$ being the solution of Eq.(4) with $(x(0),\, y(0)) = (x^0, y^0)$. More precisely, $P_\tau(\overline{M}_\epsilon) = \overline{M}_\epsilon$, $\tau \in \mathbb{R}$, for the map $P_\tau : (x^0, y^0) \longmapsto (x(\tau), y(\tau))$.*

ii) (Attractivity) *Every solution $(x(\tau),\, y(\tau))$ of Eq.(4) with $|y(0) - s^0(x(0))\,| \leq \delta$ satisfies*

$$|y(\tau) - s(x(\tau), \epsilon)\,| \leq K\, e^{-\beta\tau}\, |\, y(0) - s(x(0), \epsilon)|$$

for all $\tau \geq 0$.

iii) ("Asymptotic phase") *For every solution $(x(\tau),\, y(\tau))$ of Eq.(4) with initial conditions (x^0, y^0) at $\tau = 0$ satisfying $|y^0 - s^0(x^0)\,| \leq \delta$ there is $(\tilde{x}^0, \tilde{y}^0) \in \overline{M}_\epsilon$ such that for $(\tilde{x}(\tau),\, \tilde{y}(\tau))$ being the solution of Eq.(4) with $(\tilde{x}(0),\, \tilde{y}(0)) = (\tilde{x}^0, \tilde{y}^0)$*

$$|\, x(\tau) - \tilde{x}(\tau)\,| \leq K\, e^{-\beta\tau}\, |\, y^0 - s(x^0, \epsilon)\,|$$

$$|\, y(\tau) - \tilde{y}(\tau)\,| \leq K\, e^{-\beta\tau}\, |\, y^0 - s(x^0, \epsilon)\,|$$

holds for $\tau \geq 0$.

iv) (Closeness to $\overline{M}_0$)

$$s(x,0) = s^0(x) \quad \textit{for } x \in \mathbb{R}^m\,.$$

v) (Maximality) *Every solution* $(x(\tau),\, y(\tau))$ *of Eq.(4) satisfying* $|\, y(\tau) - s^0(x(\tau))\,| \le \delta$ *for all* $\tau \in \mathbb{R}$ *lies in* $\overline{M}_\epsilon$*, i.e.,* $y(\tau) = s(x(\tau),\epsilon)$ *for all* τ*.*

The proof of Theorem 2 is given in Section 3. It is mainly achieved by applying a general invariant manifold result for maps to an appropriate time τ-map of the flow.

3. Proof of Theorem 2

With the change of variables

$$y = s^0(x) + z$$

the system (4) can be written as

$$\text{(5)} \qquad \begin{array}{rcl} x' &=& \epsilon\, f(x, s^0(x) + z) \\ z' &=& \;\; g(x, s^0(x) + z) \quad - s^0(x)' \end{array}$$

$\left({}' := \frac{d}{d\tau}\right)$ or in a more appropriate form as

$$\text{(5)} \qquad \begin{array}{rcl} x' &=& \epsilon\, F(x,z) \\ z' &=& G(x,z) - \epsilon\, s^0_x(x)\, F(x,z) =: [B(x) + \hat{G}(x,z)]\, z - \epsilon\, s^0_x(x)\, F(x,z)\,. \end{array}$$

The functions on the right-hand side satisfy

$$\begin{array}{rcl} F &\in& C_b^k(\mathbb{R}^m \times D^n_{d_0}, \mathbb{R}^m) \\ G &\in& C_b^k(\mathbb{R}^m \times D^n_{d_0}, \mathbb{R}^n) \\ s^0 &\in& C_b^k(\mathbb{R}^m, \mathbb{R}^n)\,, \;\; s^0_x \in C_b^{k-1}(\mathbb{R}^m, \mathbb{R}^{m\times n}) \\ B &\in& C_b^{k-1}(\mathbb{R}^m, \mathbb{R}^{n\times n}) \\ \hat{G} &\in& C_b^{k-1}(\mathbb{R}^m \times D^n_{d_0}, \mathbb{R}^{n\times n}) \;\; \text{with} \;\; \hat{G} = O(|z|) \;\; \text{uniformly for} \;\; x \in \mathbb{R}^m,\; z \in D^n_{d_0} \end{array}$$

for $k \ge 2$, $D^n_{d_0} := \{z \in \mathbb{R}^n \mid |z| \le d_0\}$, and $F, G, s^0_x, B, \hat{G}$ are bounded in the domains considered. Moreover, the eigenvalues λ^B_j of the $n \times n$-matrix B satisfy

$$Re\, \lambda^B_j < -b_0 < 0\,, \quad j = 1, ..., n\,.$$

Let $\delta < d \le d_0$ where δ and d will be specified more precisely later, and let $\Omega_d := \{(x,z) \mid\, x \in \mathbb{R}^m,\, z \in D^n_d\} \subset \mathbb{R}^{m+n}$. We consider the solution $(\varphi(\tau; x, z, \epsilon),\, \psi(\tau; x, z, \epsilon))$ of Eq.(5) with initial values (x,z) at $\tau = 0$, $x \in \mathbb{R}^m$, $|z| \le \delta$. To save writing we denote this solution also by $(\varphi(\tau),\, \psi(\tau))$. Moreover, let (m_-, m_+) be its maximal interval of existence with respect to the space Ω_d.

Since $|\varphi'| \le \epsilon N$ for $\tau \in [0, m_+)$ holds for some positive constant N we have

$$| \varphi(\tau^1) - \varphi(\tau^2) | \le \epsilon N |\tau^1 - \tau^2|$$

for all $\tau^1, \tau^2 \in [0, m_+)$. Define $A_0(\tau; x, z, \epsilon) := B(\varphi(\tau; x, z, \epsilon))$ and consider the linear system

$$u' = A_0(\tau; x, z, \epsilon) u \; . \tag{6}$$

We have $|A_0| \le N_0$ for $\tau \in [0, m_+)$ and

$$| A_0(\tau^1) - A_0(\tau^2) | \le L_B |\varphi(\tau^1) - \varphi(\tau^2) | \le \epsilon L_B N | \tau^1 - \tau^2 |$$

where L_B is the Lipschitz constant of $B(x)$. Moreover, the eigenvalues of A_0 have negative real parts smaller than $-b_0$. Hence, we may apply Proposition 1.5 of Coppel [1] and obtain for the fundamental matrix $\Psi_0(\tau, \sigma; x, z, \epsilon)$ of Eq.(6) with $\Psi_0(\tau, \tau; x, z, \epsilon) = I_n$ the following

Assertion 1: For every $\mu > 0$ there is $\overline{\epsilon}(\mu) > 0$ such that

$$| \Psi_0 | \le K_\mu \, e^{(-b_0 + \mu)(\tau - \sigma)} \qquad \text{for} \;\; \tau \ge \sigma \, ,$$

$\tau, \sigma \in [0, m_+)$, $|\epsilon| < \overline{\epsilon}(\mu)$, where $K_\mu := \max\{(4 N_0/\mu)^{n-1}, 1\}$. $\dashv$

Now, define $A_1(\tau; x, z, \epsilon) := \hat{G}(\varphi(\tau; x, z, \epsilon), \psi(\tau; x, z, \epsilon))$. For $\tau \in [0, m_+)$ A_1 satisfies $|A_1| \le N_1 d$. Using Assertion 1 we may apply Proposition 1.1 of Coppel [1] to the fundamental matrix $\Psi(\tau, \sigma; x, z, \epsilon)$ with $\Psi(\tau, \tau; x, z, \epsilon) = I_n$ of the equation

$$v' = [A_0(\tau; x, z, \epsilon) + A_1(\tau; x, z, \epsilon)] \, v \tag{7}$$

and we get

Assertion 2: For every $\mu > 0$ there is $\overline{\epsilon}(\mu) > 0$ such that

$$|\Psi| \le K_\mu \, e^{(-b_0 + \mu + d N_1 K_\mu)(\tau - \sigma)} \qquad \text{for} \;\; \tau \ge \sigma \, ,$$

$\tau, \sigma \in [0, m_+)$, $|\epsilon| \le \overline{\epsilon}(\mu)$, with K_μ as in Assertion 1. $\dashv$

Combining the two results we have shown

Assertion 3: For every $\beta_0 \in (0, b_0)$ there are positive constants d, $K_0 \geq 1$ and $\epsilon_1 > 0$ such that

$$|\Psi| \leq K_0 \, e^{-\beta_0 (\tau - \sigma)} \qquad \text{for} \;\; \tau \geq \sigma; \; \tau, \sigma \in [0, m_+); \; |\epsilon| < \epsilon_1 \, . \qquad \dashv$$

Note that the constants β_0, d, K_0 and ϵ_1 are independent of the solution $(\varphi(\tau), \psi(\tau))$ considered.

Writing Eq.(5) as an integral equation and by means of the variation of constants formula we have

$$\begin{aligned} \varphi(\tau) &= \varphi(\nu) + \epsilon \int_{\nu}^{\tau} F(\varphi(\sigma), \, \psi(\sigma)) d\sigma \\ \psi(\tau) &= \Psi(\tau, \nu) \, \psi(\nu) - \epsilon \int_{\nu}^{\tau} \Psi(\tau, \sigma) \, s_x^0(\varphi(\sigma)) \, F(\varphi(\sigma), \, \psi(\sigma)) d\sigma \end{aligned} \tag{8}$$

for $\tau, \nu \in [0, m_+)$, $\tau \geq \nu$. We have also introduced the short notation $\Psi(\tau, \sigma)$ for $\Psi(\tau, \sigma; x, z, \epsilon)$.

We are now able to prove the following

Claim 1 *For every $\beta_0 \in (0, b_0)$ there is $d > 0$, $\delta > 0$ and $\epsilon_2 > 0$ such that for $|z| \leq \delta$ and $|\epsilon| < \epsilon_2$ the solution $(\varphi(\tau; x, z, \epsilon), \, \psi(\tau; x, z, \epsilon))$ of Eq.(5) exists for all $\tau \geq 0$ with respect to Ω_d (i.e., $m_+ = +\infty$).*

Proof: From Eq.(8) with $\nu = 0$ and using Assertion 3 we find that for $\delta = \frac{d}{4K_0}$

$$\begin{aligned} |\psi(\tau)| &\leq e^{-\beta_0 \tau} \, \frac{d}{4} + \epsilon \, K^0 \int_0^{\tau} e^{-\beta_0 (\tau - \sigma)} d\sigma \\ &= e^{-\beta_0 \tau} \, \frac{d}{4} + \frac{\epsilon \, K^0}{\beta_0} \left[1 - e^{-\beta_0 \tau} \right] \\ &\leq e^{-\beta_0 \tau} \, \frac{d}{4} + \epsilon \, \frac{K^0}{\beta_0} \, . \end{aligned} \tag{9}$$

Hence, there is $\epsilon_2 \in (0, \epsilon_1]$ such that for $|\epsilon| < \epsilon_2$

$$|\psi(\tau)| \leq \frac{d}{2} \qquad \text{for} \;\; \tau \in [0, m_+) \, .$$

From this estimate and from the fact that for $\tau \in [0, m_+)$ $|\varphi(\tau)| \leq |x| + \epsilon \, N \tau < |x| + \epsilon \, N m_+$ we conclude by means of the "global existence theorem for ODE's" that $m_+ = +\infty$ which completes the proof of Claim 1. $\perp$

An easy corollary of Eq.(9) is

Claim 2 *For $\beta_0 \in (0, b_0)$ let d, δ and $(\varphi(\tau), \psi(\tau))$ be chosen according to Claim 1. For every $\gamma \in (0, d/2)$ there is $\epsilon_3 > 0$ such that*

$$|\psi(\tau)| \leq \gamma \qquad \text{for } \tau \geq \frac{1}{\beta_0} \log \frac{d}{2\gamma}\,, \quad |\epsilon| < \epsilon_3.$$

We shall need the Lipschitz constant of the fundamental matrix Ψ of Eq.(7) with respect to x, z.

Claim 3 *Let $\beta_0 \in (0, b_0)$ and let $\delta > 0$ be as in Claim 1. For every $\widetilde{\beta} \in (0, \beta_0)$ there is $\widetilde{L} > 0$ and $\epsilon_4 > 0$ such that*

$$\left|\Psi(\tau, \sigma; x^1, z^1, \epsilon) - \Psi(\tau, \sigma; x^2, z^2, \epsilon)\right| \;\leq\; \widetilde{L}\,(\tau - \sigma)\, e^{-\widetilde{\beta}(\tau-\sigma)}\, e^{\epsilon \widetilde{L} \sigma} \left(\left|x^1 - x^2\right| + \left|z^1 - z^2\right|\right)$$

for $\tau \geq \sigma \geq 0$, $|\epsilon| < \epsilon_4$; $x^i \in \mathbb{R}^m$, $|z^i| \leq \delta$, $i = 1, 2$.

Proof: We introduce the notations $\Psi_i(\tau, \sigma) := \Psi(\tau, \sigma; x^i, z^i, \epsilon)$, $i = 1, 2$, $\Delta(\tau, \sigma) := \Psi_1(\tau, \sigma) - \Psi_2(\tau, \sigma)$ and $C_i(\tau) := A_0(\tau; x^i, z^i, \epsilon) + A_1(\tau; x^i, z^i, \epsilon)$, $i = 1, 2$. The matrix Δ satisfies the differential equation

$$\Delta' = C_1(\tau)\Delta + (C_1(\tau) - C_2(\tau))\, \Psi_2(\tau, \sigma)$$

and $\Delta(\tau, \tau) = 0$. Applying the variation of constants formula we have

$$\Delta(\tau, \sigma) = \int_\sigma^\tau \Psi_1(\tau, r)(C_1(r) - C_2(r))\, \Psi_2(r, \sigma) dr\,.$$

Taking norms and applying the estimate of Assertion 3 we get

$$|\Delta(\tau, \sigma)| \leq {K_0}^2\, e^{-\beta_0(\tau-\sigma)} \int_\sigma^\tau |C_1(r) - C_2(r)| dr\,.$$

With the notations $\varphi_i(\tau) := \varphi(\tau; x^i, z^i, \epsilon)$ and $\psi_i(\tau) := \psi(\tau; x^i, z^i, \epsilon)$ and since $C_i(\tau) = B(\varphi_i(\tau)) + \hat{G}(\varphi_i(\tau), \psi_i(\tau))$ we therefore obtain

$$(10) \qquad |\Delta(\tau, \sigma)| \leq K_1\, e^{-\beta_0(\tau-\sigma)} \int_\sigma^\tau \Big[|\varphi_1(r) - \varphi_2(r)| \,+\, |\psi_1(r) - \psi_2(r)|\Big] dr\,.$$

We need the Lipschitz constant of $(\varphi(\tau; x, z, \epsilon), \psi(\tau; x, z, \epsilon))$ with respect to x, z. Using Eq.(8) with $\nu = 0$ and introducing the notation $\overline{G} := -s_x^0\, F$ we obtain

$$(11) \qquad |\varphi_1(\tau)-\varphi_2(\tau)| \le |x^1-x^2| + \epsilon\, L_F \int_0^\tau [|\varphi_1(\sigma)-\varphi_2(\sigma)| + |\psi_1(\sigma)-\psi_2(\sigma)|]d\sigma$$

and

$$\begin{aligned} |\psi_1(\tau)-\psi_2(\tau)| \quad \le \quad & |\Psi_1(\tau,0)|\,|z^1-z^2| + |\Psi_1(\tau,0)-\Psi_2(\tau,0)|\,|z^2| \\ & + \epsilon\, L_{\overline{G}} \int_0^\tau |\Psi_1(\tau,\sigma)|\,[|\varphi_1(\sigma)-\varphi_2(\sigma)| + |\psi_1(\sigma)-\psi_2(\sigma)|]d\sigma \\ & + \epsilon\, K_{\overline{G}} \int_0^\tau |\Psi_1(\tau,\sigma)-\Psi_2(\tau,\sigma)|d\sigma \end{aligned}$$

where L_F and $L_{\overline{G}}$ are the Lipschitz constants of F and $\overline{G}$, respectively, and $K_{\overline{G}}$ is a bound for $\overline{G}$. Applying the estimates of Assertion 3 and of Eq.(10) we find for the second equation above

$$\begin{aligned} |\psi_1(\tau)-\psi_2(\tau)| \quad \le \quad & K_0\, e^{-\beta_0\tau}|z^1-z^2| \\ & + \delta\, K_1\, e^{-\beta_0\tau} \int_0^\tau [|\varphi_1(\sigma)-\varphi_2(\sigma)| + |\psi_1(\sigma)-\psi_2(\sigma)|]d\sigma \\ & + \epsilon\, L_{\overline{G}}\, K_0 \int_0^\tau e^{-\beta_0(\tau-\sigma)}[|\varphi_1(\sigma)-\varphi_2(\sigma)| + |\psi_1(\sigma)-\psi_2(\sigma)|]d\sigma \\ & + \epsilon\, K_{\overline{G}}\, K_1 \int_0^\tau e^{-\beta_0(\tau-\sigma)}\Big\{ \int_\sigma^\tau [|\varphi_1(r)-\varphi_2(r)| + |\psi_1(r)-\psi_2(r)|]dr \Big\} d\sigma\ . \end{aligned}$$
(12)

Adding Eqs.(11) and (12) and defining the functions $\rho_\varphi(\tau) := \max\limits_{0\le r\le\tau} (|\varphi_1(r)-\varphi_2(r)|)$ and $\rho_\psi(\tau) := \max\limits_{0\le r\le\tau} (|\psi_1(r)-\psi_2(r)|)$ we may derive the following inequality for $[\rho_\varphi+\rho_\psi]$:

$$\begin{aligned} \rho_\varphi(\tau)+\rho_\psi(\tau) \quad \le \quad & K\,[|x^1-x^2| + |z^1-z^2|] + (\epsilon + \delta\, e^{-\beta_0\tau})\, K \int_0^\tau [\rho_\varphi(\sigma)+\rho_\psi(\sigma)]d\sigma \\ & + \epsilon\, K\,[\rho_\varphi(\tau)+\rho_\psi(\tau)]\ . \end{aligned}$$

For ϵ small enough such that $1-\epsilon\, K \ge 1/2$ we therefore have

$$\rho_\varphi(\tau)+\rho_\psi(\tau) \le 2K\,[|x^1-x^2| + |z^1-z^2|] + 2(\epsilon + \delta\, e^{-\beta_0\tau})\, K \int_0^\tau [\rho_\varphi(\sigma)+\rho_\psi(\sigma)]d\sigma\ .$$

Now, applying the generalized version of Gronwall's lemma and going back to the functions φ_i, ψ_i yields

$$(13) \qquad |\varphi_1(\tau) - \varphi_2(\tau)| + |\psi_1(\tau) - \psi_2(\tau)| \leq \overline{K}\, e^{2\epsilon K \tau}\, [|x^1 - x^2| + |z^1 - z^2|]\ .$$

Inserting Eq.(13) into Eq.(10) and estimating the integral as

$$\int_\sigma^\tau e^{2\epsilon K r}\, dr \leq (\tau - \sigma)\, e^{2\epsilon K \tau}$$

we thus obtain

$$|\Delta(\tau,\sigma)| \leq K_2(\tau - \sigma) e^{-[\beta_0 - 2\epsilon K]\,(\tau-\sigma)}\, e^{2\epsilon K \sigma} [|x^1 - x^2| + |z^1 - z^2|]\ . \qquad \perp$$

Let $\beta_0 \in (0, b_0)$, $\delta > 0$ from Claim 1 with corresponding solution $(\varphi(\tau; x, z, \epsilon),\ \psi(\tau; x, z, \epsilon))$ of Eq.(5) and let $\tau^* > 0$ be fixed (it will be specified more precisely later). Moreover, let $\epsilon^* > 0$ be small enough and let $\xi \in C^\infty(\mathbb{R}, (-\epsilon^*, \epsilon^*))$ satisfy (see Fig. 3)

$$\xi(\epsilon) = \begin{cases} \epsilon & , \quad |\epsilon| \leq 4\epsilon^*/5 \\ \mathrm{sign}(\epsilon)\, 9\epsilon^*/10 & , \quad |\epsilon| \geq \epsilon^* \end{cases}$$

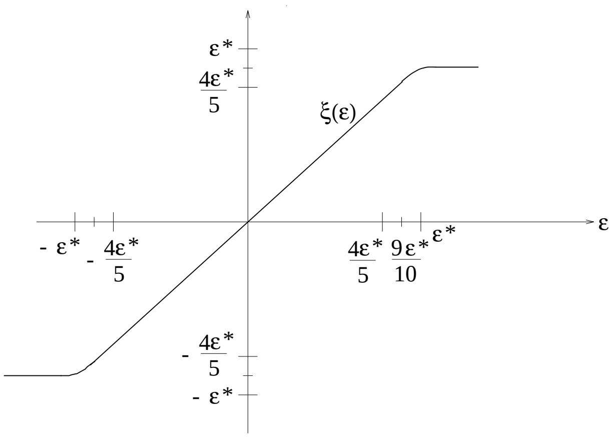

Fig. 3

We consider the time τ^*-map

$$(14) \qquad P_\epsilon :\ \mathbb{R}^m \times D^n_\delta \ni \begin{pmatrix} x \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \overline{x} \\ \overline{z} \end{pmatrix} = \begin{pmatrix} \varphi(\tau^*; x, z, \xi(\epsilon)) \\ \psi(\tau^*; x, z, \xi(\epsilon)) \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n$$

defined for $\epsilon \in \mathbb{R}$. We want to show that P_ϵ admits a smooth attractive invariant manifold. The extension to all ϵ in $\mathbb{R}$ will be needed for showing smoothness also with respect to the parameter ϵ. From the integral equation of Eq.(5) we know that the map P_ϵ has the representation

$$\begin{aligned}
\overline{x} &= x + \xi(\epsilon) \int_0^{\tau^*} f\Big(\varphi(\sigma; x, z, \xi(\epsilon)),\, s^0(\varphi(\sigma; x, z, \xi(\epsilon))) + \psi(\sigma; x, z, \xi(\epsilon))\Big) d\sigma \\
\overline{z} &= z + \int_0^{\tau^*} g\Big(\varphi(\sigma; x, z, \xi(\epsilon)),\, s^0(\varphi(\sigma; x, z, \xi(\epsilon))) + \psi(\sigma; x, z, \xi(\epsilon))\Big) d\sigma \\
&\quad + s^0(x) - s^0(\varphi(\tau^*; x, z, \xi(\epsilon)))\ .
\end{aligned}$$

From this form it is seen that the map P_ϵ is of class C_b^k with respect to x and z (with bounds depending on τ^*, cf. Eq.(13)).

The map P_ϵ can also be written as (cf. Eq.(8) for $\nu = 0$):

$$\begin{aligned}
\overline{x} &= x + \xi(\epsilon) \int_0^{\tau^*} F\Big(\varphi(\sigma; x, z, \xi(\epsilon)),\, \psi(\sigma; x, z, \xi(\epsilon))\Big) d\sigma \\
\overline{z} &= \Psi\Big(\tau^*, 0; x, z, \xi(\epsilon)\Big)\, z - \xi(\epsilon) \int_0^{\tau^*} \Psi\Big(\tau^*, \sigma; x, z, \xi(\epsilon)\Big)\, s_x^0\Big(\varphi(\sigma; x, z, \xi(\epsilon))\Big) \cdot \\
&\qquad \cdot F\Big(\varphi(\sigma; x, z, \xi(\epsilon)),\, \psi(\sigma; x, z, \xi(\epsilon))\Big) d\sigma
\end{aligned} \tag{15}$$

which is of the form

$$\begin{aligned}
\overline{x} &= x + U(x, z, \epsilon) \\
\overline{z} &= V(x, z, \epsilon)\ .
\end{aligned} \tag{16}$$

From Claim 2 it follows that for ϵ^* small enough and τ^* large enough the "strip" $\mathbb{R}^m \times D_\delta^n$ is mapped into itself by the map P_ϵ. Moreover, the functions U and V have the following Lipschitz constants with respect to x and z:

$$\begin{aligned}
L_{11} &= c^*\, \xi(\epsilon), \quad L_{12} = c^*\, \xi(\epsilon) \\
L_{21} &= \widetilde{L}\, \tau^*\, e^{-\widetilde{\beta}\tau^*} \delta + c^*\, \xi(\epsilon) < L_{22} \\
L_{22} &= \widetilde{L}\, \tau^*\, e^{-\widetilde{\beta}\tau^*} \delta + K_0\, e^{-\beta_0 \tau^*} + c^*\, \xi(\epsilon)
\end{aligned} \tag{17}$$

where we have used Assertion 3 and Claim 3 in Eq.(15). (Note that c^* depends on τ^*.)

We want to apply the invariant manifold result proved in Nipp/Stoffer [5] (Theorem 5). We have to verify the following two conditions for the Lipschitz constants $L_{11}, L_{12}, L_{21}, L_{22}$:

$$
\begin{array}{llll}
(I) & 2\sqrt{L_{12}\,L_{21}} & < & 1 - L_{11} - L_{22} \\
(II) & L_{22} + L_{12}\,\lambda & < & (1 - L_{11} - L_{12}\,\lambda)^k \\
 & \text{with} \qquad \lambda & := & \dfrac{2\,L_{21}}{1 - L_{11} - L_{22} + \sqrt{(1 - L_{11} - L_{22})^2 - 4\,L_{12}L_{21}}} \, .
\end{array}
$$

For every $\beta^* \in (0, \widetilde{\beta})$ the quantity L_{22} may be written as

$$L_{22} = e^{-\beta^*\tau^*}\left[\widetilde{L}\,\tau^*\,\delta\,e^{-(\widetilde{\beta}-\beta^*)\tau^*} + K_0\,e^{-(\beta_0-\beta^*)\tau^*} + c^*\,\xi(\epsilon)e^{\beta^*\tau^*}\right] .$$

Choosing first τ^* large enough and then ϵ^* small enough we can achieve that

$$L_{22} \le \frac{1}{4}\,e^{-\beta^*\tau^*} < \frac{1}{4}\, . \tag{18}$$

Condition (I) is satisfied if

$$c^*\,\xi(\epsilon) + \sqrt{c^*\,\xi(\epsilon)e^{-\beta^*\tau^*}} < 1 - L_{22}$$

holds. Since $1 - L_{22} \in \left(\frac{3}{4}, 1\right)$ this requirement can be satisfied for ϵ^* small enough.

Now, using Condition (I) note that

$$\lambda < \frac{2\,L_{21}}{1 - L_{11} - L_{22}}\, .$$

Since by Eqs.(17) and (18)

$$1 - L_{11} - L_{22} > \frac{3}{4} - c^*\,\xi(\epsilon)$$

we may achieve for ϵ^* small enough that $\lambda < 4\,L_{21}$ and also that

$$L_{22} + L_{12}\,\lambda < L_{22}(1 + 4c^*\,\xi(\epsilon)) < 2\,L_{22}\, . \tag{19}$$

Hence, from Eqs.(17), (18) and (19) we have that Condition (II) is satisfied if

$$\frac{1}{2}\,e^{-\beta^*\tau^*} < \left(1 - 2c^*\,\xi(\epsilon)\right)^k$$

holds. This requirement can be fulfilled for ϵ^* small enough. (Note that ϵ^* depends on the order of differentiability k).

Summarizing, we have shown the following facts concerning the map P_ϵ defined in (14). For every $\beta^* \in (0, b_0)$ there is $\delta > 0$, $\tau^* > 0$ and $\epsilon^* > 0$ such that P_ϵ maps the "strip" $\mathbb{R}^m \times D^n_\delta$ into itself; P_ϵ is of class C^k_b; the Lipschitz constants $L_{11}, L_{12}, L_{21}, L_{22}$ satisfy Conditions (I) and (II) and the estimates

$$\lambda < e^{-\beta^* \tau^*} < 1$$

$$L_{22} + L_{12}\,\lambda < \frac{1}{2}\, e^{-\beta^* \tau^*}$$

and

$$|\overline{z}| \leq \frac{1}{4}\,|z| + C\,\xi(\epsilon)\,\tau^*$$

with $C\,\xi(\epsilon)\,\tau^* \leq \delta/4$ hold, λ defined in Condition (II).

Hence, Theorem 5 of Nipp/Stoffer [5] implies the existence of a smooth attractive invariant manifold $\overline{M}^1_\epsilon$ of the map P_ϵ with properties given in

Lemma 3 *There is a function $s^1(x, \epsilon) : \mathbb{R}^m \times \mathbb{R} \longrightarrow D^n_\delta$, of class C^k_b with respext to x, such that the following assertions hold.*

i) The set $\overline{M}^1_\epsilon = \{(x, z) \mid \; x \in \mathbb{R}^m,\ z = s^1(x, \epsilon)\}$ is invariant under the map P_ϵ, i.e., $P_\epsilon(\overline{M}^1_\epsilon) = \overline{M}^1_\epsilon$.

ii) $\overline{M}^1_\epsilon$ is uniformly attractive for P_ϵ with attractivity constant

$$\chi(\lambda) = L_{22} + L_{12}\lambda < \frac{1}{2}\, e^{-\beta^* \tau^*} < 1\ .$$

iii) $\overline{M}^1_\epsilon$ has the "property of asymptotic phase":
For every $(x_0, z_0) \in \mathbb{R}^m \times D^n_\delta$ there is $(\tilde{x}_0, \tilde{z}_0) \in \overline{M}^1_\epsilon$ such that for $(x_j, z_j) := P^j_\epsilon(x_0, z_0)$ and $(\tilde{x}_j, \tilde{z}_j) := P^j_\epsilon(\tilde{x}_0, \tilde{z}_0) \in \overline{M}^1_\epsilon$, $j \in \mathbb{N}$,

$$\begin{aligned} |x_j - \tilde{x}_j| &\leq c^*\,\xi(\epsilon)\, e^{-\beta^* j \tau^*}\, |z_0 - s^1(x_0, \epsilon)| \\ |z_j - \tilde{z}_j| &\leq e^{-\beta^* j \tau^*}\, |z_0 - s^1(x_0, \epsilon)|\ . \end{aligned}$$

iv) $|s^1(x, \epsilon)| < 2C\xi(\epsilon)\tau^* \leq \frac{\delta}{2}$.

v) Maximality: Every invariant set Ω of P^l_ϵ, $l \in \mathbb{N}$, is contained in $\overline{M}^1_\epsilon$, i.e., $P^l_\epsilon(\Omega) = \Omega$ implies $\Omega \subset \overline{M}^1_\epsilon$.

vi) The function s^1 is uniformly λ-Lipschitz with respect to x with $\lambda < e^{-\beta^* \tau^*} < 1$.

We first show that the manifold $\overline{M}^1_\epsilon$ established in Lemma 3 is also smooth with respect to the parameter ϵ.

Smoothness with respect to ϵ: There is $\epsilon^* > 0$ such that $s^1(x, \epsilon)$ is of class C^k_b also with respect to ϵ.

Proof: We consider the augmented map

$$P:\ \mathbb{R}\times\mathbb{R}^m\times D^n_{\tilde\delta} \ni \begin{pmatrix}\epsilon\\ x\\ z\end{pmatrix} \longmapsto \begin{pmatrix}\overline{\epsilon}\\ \overline{x}\\ \overline{z}\end{pmatrix} = \begin{pmatrix}\epsilon\\ \varphi(\tilde\tau; x, z, \xi(\epsilon))\\ \psi(\tilde\tau; x, z, \xi(\epsilon))\end{pmatrix} \in \mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^n$$

and again apply Theorem 5 of Nipp/Stoffer [5]. Here, $\tilde\delta \in (0, \delta]$ will be specified more precisely later. The time step $\tilde\tau$ will be chosen appropriately as $l\tau^*$ for some $l \in \mathbb{N}$. This implies that $P(\epsilon, x, z) = (\epsilon, P^l_\epsilon(x, z))$ with P_ϵ of Lemma 3.

We write P as

$$\begin{aligned}
\overline{\epsilon} &= \epsilon\\
\overline{x} &= x + \xi(\epsilon)\int_0^{\tilde\tau} F\Big(\varphi(\sigma; x, 0, \xi(\epsilon)), 0\Big)\,d\sigma\\
&\quad + \xi(\epsilon)\int_0^{\tilde\tau}\Big[F\Big(\varphi(\sigma; x, z, \xi(\epsilon)),\ \psi(\sigma; x, z, \xi(\epsilon))\Big) - F\Big(\varphi(\sigma; x, 0, \xi(\epsilon)), 0\Big)\Big]\,d\sigma\\
\overline{z} &= \Psi\big(\tilde\tau, 0; x, z, \xi(\epsilon)\big)z - \xi(\epsilon)\int_0^{\tilde\tau}\Psi\Big(\tilde\tau, \sigma; x, z, \xi(\epsilon)\Big)\, s^0_x\Big(\varphi(\sigma; x, z, \xi(\epsilon))\Big)\cdot\\
&\qquad\qquad \cdot F\Big(\varphi(\sigma; x, z, \xi(\epsilon)),\ \psi(\sigma; x, z, \xi(\epsilon))\Big)\,d\sigma\ .
\end{aligned}$$

Defining $w := (\epsilon, x)$ the map P is of the form

$$\begin{aligned}
\overline{w} &= \tilde U_0(w) + \tilde U(w, z)\\
\overline{z} &= \tilde V(w, z)\ .
\end{aligned}$$

If we couple $\tilde\tau = l\tau^*$ and $\tilde\delta$ such that $\tilde\tau \in \Big[\frac{1}{\beta^*}\log\frac{\delta}{2\tilde\delta}, \frac{1}{\beta^*}\log\frac{\delta}{2\tilde\delta} + \tau^*\Big)$ we know from Claim 2 that for ϵ^* small enough $|\psi(\tau)| \le \tilde\delta$ for $\tau \ge \tilde\tau$. Hence, the "strip" $\mathbb{R}\times\mathbb{R}^m\times D^n_{\tilde\delta}$ is mapped into itself by the map P. Moreover, for $\tilde\delta$ small enough $\tilde\tau$ may be estimated as $\tilde\tau \le \frac{2}{\beta^*}\log\frac{1}{\tilde\delta}$ which is equivalent to $\tilde\delta \le e^{-\beta^*\tilde\tau/2}$. For ϵ^* small enough, $\tilde U_0$ is invertible and $\tilde U_0^{-1}$ is Lipschitz continuous with Lipschitz constant $\alpha = 1 + \tilde c\,\epsilon^* + N\,\tilde\tau$ where $\tilde c$ depends on $\tilde\tau$. Note that the solution $(\varphi(\tau), \psi(\tau))$ has Lipschitz constant $\overline{L}\,\tau\, e^{\epsilon\overline{L}\tau}$ with respect to ϵ (which can be derived in the same way as Eq.(13)). Moreover, for the Lipschitz continuity of Ψ with respect to ϵ Claim 3 similarly holds with an additional factor τ. Hence, we find that for ϵ^* small enough the functions $\widetilde{U}$ and $\widetilde{V}$ have the following Lipschitz constants with respect to w and z:

$$\widetilde{L}_{11} = \tilde{c}\,\epsilon^* + c\sqrt{\tilde{\delta}}\,, \qquad \widetilde{L}_{12} = \tilde{c}\,\epsilon^*$$

$$\widetilde{L}_{21} = \widetilde{L}\,\tilde{\tau}^2\, e^{-\beta^*\tilde{\tau}}\,\tilde{\delta} + \tilde{c}\,\epsilon^* + c\,, \qquad \widetilde{L}_{22} = \widetilde{L}\,\tilde{\tau}\, e^{-\beta^*\tilde{\tau}}\,\tilde{\delta} + K_0^*\, e^{-\beta^*\tilde{\tau}} + \tilde{c}\,\epsilon^*\,.$$

Note that the constants N and c do not depend on $\tilde{\tau}$. We have to verify the two conditions

$$(\widetilde{I}) \quad 2\sqrt{\widetilde{L}_{12}\,\widetilde{L}_{21}} \;<\; \frac{1}{\alpha} - \widetilde{L}_{11} - \widetilde{L}_{22}$$

$$(\widetilde{II}) \quad \widetilde{L}_{22} + \widetilde{L}_{12}\,\tilde{\lambda} \;<\; \Big(\frac{1}{\alpha} - \widetilde{L}_{11} - \widetilde{L}_{12}\,\tilde{\lambda}\Big)^k$$

$$\text{with} \qquad \tilde{\lambda} \;:=\; \frac{2\,\widetilde{L}_{21}}{\frac{1}{\alpha} - \widetilde{L}_{11} - \widetilde{L}_{22} + \sqrt{\Big(\frac{1}{\alpha} - \widetilde{L}_{11} - \widetilde{L}_{22}\Big)^2 - 4\,\widetilde{L}_{12}\,\widetilde{L}_{21}}}\,.$$

We require ϵ^* also to satisfy

$$\epsilon^* \;\leq\; \frac{1}{2\,N\,\tilde{c}\,\tilde{\tau}^{3k}}\,.$$

This implies that for $\tilde{\tau}$ large enough

$$\frac{1}{\alpha} \;>\; \frac{1}{2\,N\,\tilde{\tau}}\,.$$

For $\tilde{\tau}$ large enough we can also achieve that

$$\widetilde{L}_{21} \;<\; 2\,c\,, \qquad \widetilde{L}_{22} < \frac{1}{N\,\tilde{\tau}^{3k}}\,.$$

Hence, Condition $(\widetilde{I})$ is satisfied if

$$\frac{3}{2\,N\,\tilde{\tau}^{3k}} + c\,e^{-\beta^*\tilde{\tau}/4} + \frac{2\,c}{\sqrt{N}\,\tilde{\tau}^{3k/2}} \;<\; \frac{1}{2\,N\,\tilde{\tau}}$$

holds. This can again be satisfied for $\tilde{\tau}$ large enough.

Using Condition $(\widetilde{I})$ we find that

$$\tilde{\lambda} \;<\; \frac{2\,\widetilde{L}_{21}}{\frac{1}{\alpha} - \widetilde{L}_{11} - \widetilde{L}_{22}}$$

and since we may achieve that

$$\frac{1}{\alpha} - \widetilde{L}_{11} - \widetilde{L}_{22} \;>\; \frac{1}{4\,N\,\tilde{\tau}}$$

we have $\widetilde{\lambda} < 8\,N\,\widetilde{\tau}\,\widetilde{L}_{21}$. Thus, since

$$\widetilde{L}_{12}\,\widetilde{\lambda} \;<\; 8\,N\,\widetilde{\tau}\,\widetilde{L}_{12}\,\widetilde{L}_{21} \;<\; \frac{8\,c}{4\widetilde{\tau}^{3k-1}}$$

we have for $\widetilde{\tau}$ large enough

$$\widetilde{L}_{22} + \widetilde{L}_{12}\,\widetilde{\lambda} \;<\; \frac{9\,c}{\widetilde{\tau}^{3k-1}}$$

and

$$\frac{1}{\alpha} - \widetilde{L}_{11} - \widetilde{L}_{12}\,\widetilde{\lambda} \;>\; \frac{1}{4\,N\,\widetilde{\tau}}\,.$$

Hence, Condition $(\widetilde{II})$ is satisfied if

$$\frac{9\,c}{\widetilde{\tau}^{3k-1}} \;<\; \left(\frac{1}{4\,N\,\widetilde{\tau}}\right)^{k}$$

holds. This can be satisfied for $\widetilde{\tau}$ large enough.

Theorem 5 of Nipp/Stoffer [5] implies the existence of a function $\widetilde{s}^1 \in C_b^k(\mathbb{R}^{1+m}, D^n_{\widetilde{\delta}})$ such that the set $\{(\epsilon, x, z) \,|\, \epsilon \in \mathbb{R},\ x \in \mathbb{R}^m,\ z = \widetilde{s}^1(x,\epsilon)\}$ is an invariant set of the map P. Now, for every $\epsilon \in \mathbb{R}$ consider the set

$$\widetilde{\overline{M}}{}^1_\epsilon := \{(x,z) \,|\, x \in \mathbb{R}^m,\ z = \widetilde{s}^1(x,\epsilon)\}\ .$$

Since $P(\epsilon, x, z) = (\epsilon, P^l_\epsilon(x,z))$ this set is an invariant set of the map P^l_ϵ. Hence, from the maximality property v) of Lemma 3 it follows that $\widetilde{\overline{M}}{}^1_\epsilon \subset \overline{M}{}^1_\epsilon$. The special structure of the two sets finally implies $\widetilde{\overline{M}}{}^1_\epsilon = \overline{M}{}^1_\epsilon$ and therefore $\widetilde{s}^1 = s^1$. $\perp$

Thus, we have shown that for ϵ^* small enough the invariant manifold $\overline{M}{}^1_\epsilon$ of the map P_ϵ established in Lemma 3 is also smooth with respect to ϵ. The quantity ϵ^* depends on the order of differentiability k. However, the "thickness" δ of the domain D^n_δ in Lemma 3 does not depend on k. To show this was the reason for proving the smoothness with respect to ϵ seperately.

We now restrict ϵ to $|\epsilon| < \epsilon^{**} := 4\epsilon^*/5$ (see Fig. 3). The properties i), ii), iii) and v) of Lemma 3 hold for the time τ^*-map (14) of Eq.(5). It remains to show that corresponding properties also hold for the flow.

i) *Invariance:* $\overline{M}{}^1_\epsilon$ is also invariant under the differential equation (5), i.e., if $(x,z) \in \overline{M}{}^1_\epsilon$ then also $(\varphi(\tau; x, z, \epsilon), \psi(\tau; x, z, \epsilon)) \in \overline{M}{}^1_\epsilon$ for all $\tau \in \mathbb{R}$.

Proof: Let β^*, δ, τ^*, ϵ^{**}, $\overline{M}{}^1_\epsilon$ be according to Lemma 3. There is $\delta_* \in (0,\delta)$ such that the solution $(\varphi(\tau; x, z, \epsilon), \psi(\tau; x, z, \epsilon))$ of Eq.(5) with $|z| \le \delta_*$ exists with respect to $\mathbb{R}^m \times D^n_\delta$ for $\tau \in (\tau_-, \infty)$ (cf. Claim 1).

Let $\epsilon_* \in (0, \epsilon^{**}]$ be such that $\overline{M}^1_\epsilon \subset \mathbb{R}^m \times D^n_{\delta_*}$ for $|\epsilon| < \epsilon_*$ and define for $\tau \in (\tau_-, \infty)$, $|\epsilon| < \epsilon_*$ the map

$$P^\tau_\epsilon : \mathbb{R}^m \times D^n_\delta \ni \begin{pmatrix} x \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \varphi(\tau; x, z, \epsilon) \\ \psi(\tau; x, z, \epsilon) \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n .$$

For fixed τ define the set $\Omega := P^\tau_\epsilon(\overline{M}^1_\epsilon) \subset \mathbb{R}^m \times D^n_\delta$. The group property of the flow of Eq.(5) and the invariance of $\overline{M}^1_\epsilon$ under $P^{\tau^*}_\epsilon$ imply that

$$P^{\tau^*}_\epsilon(\Omega) = P^{\tau^*}_\epsilon(P^\tau_\epsilon(\overline{M}^1_\epsilon)) = P^\tau_\epsilon(P^{\tau^*}_\epsilon(\overline{M}^1_\epsilon)) = P^\tau_\epsilon(\overline{M}^1_\epsilon) = \Omega .$$

Using the maximality property v) it follows that $\Omega = P^\tau_\epsilon(\overline{M}^1_\epsilon) \subset \overline{M}^1_\epsilon$ for $\tau \in (\tau_-, \infty)$.

Since $|s^1(x, \epsilon)| \leq \frac{\delta}{2}$ for $x \in \mathbb{R}^m$ and since for $\tau \in (\tau_-, 0]$ we have $|\varphi(\tau)| \leq |x| + \epsilon N |\tau| < |x| + \epsilon N |\tau_-|$ we conclude from "the global existence theorem for ODE's" that $\tau_- = -\infty$. If $\epsilon_* < \epsilon^{**}$ we redefine ϵ^{**} as $\epsilon^{**} := \epsilon_*$. $\perp$

ii) *Attractivity:* We again denote the solution $(\varphi(\tau; x, z, \epsilon), \psi(\tau; x, z, \epsilon))$ of Eq.(5) with $|z| \leq \delta$ by $(\varphi(\tau), \psi(\tau))$ for short and also write $s^1(x)$ instead of $s^1(x, \epsilon)$. We want to estimate $|\psi(\tau) - s^1(\varphi(\tau))|$.

For $\tau > 0$ arbitrary but fixed choose $j \in \mathbb{N}_0$ such that $j\,\tau^* \leq \tau < (j+1)\tau^*$ and let $\tau_j := j\,\tau^*$, $\tau_{j+1} := (j+1)\tau^*$. From Lemma 3 ii) we know that

$$|\psi(\tau_j) - s^1(\varphi(\tau_j))| \leq e^{-\beta^* \tau_j}\, |z - s^1(x)| . \tag{20}$$

Consider the solution $(X(\tau), Z(\tau))$ of Eq.(5) with $X(\tau_j) = \varphi(\tau_j)$ and $Z(\tau_j) = s^1(\varphi(\tau_j))$. It lies in the invariant manifold $\overline{M}^1_\epsilon$ for all τ, i.e., $Z(\tau) = s^1(X(\tau))$ for all τ.

Our aim is to estimate

$$\begin{aligned} |\psi(\tau) - s^1(\varphi(\tau)| &\leq |\psi(\tau) - Z(\tau)| + |Z(\tau) - s^1(\varphi(\tau))| \\ &\leq |\psi(\tau) - Z(\tau)| + |\varphi(\tau) - X(\tau)| . \end{aligned} \tag{21}$$

Here we have used that s^1 has Lipschitz constant $\lambda < 1$. The two solutions of Eq.(5) satisfy the integral equations

$$\left.\begin{aligned} \varphi(\tau) &= \varphi(\tau_j) + \epsilon \int_{\tau_j}^{\tau} F(\varphi(\sigma), \psi(\sigma)) d\sigma \\ \psi(\tau) &= \psi(\tau_j) + \int_{\tau_j}^{\tau} \widetilde{G}(\varphi(\sigma), \psi(\sigma), \epsilon) d\sigma \end{aligned}\right. \quad , \; \tau \in [\tau_j, \tau_{j+1})$$

and

$$\left.\begin{array}{rcl} X(\tau) & = & \varphi(\tau_j) \;+\; \epsilon \displaystyle\int_{\tau_j}^{\tau} F(X(\sigma), Z(\sigma))d\sigma \\ Z(\tau) & = & Z(\tau_j) \;+\; \displaystyle\int_{\tau_j}^{\tau} \widetilde{G}(X(\sigma), Z(\sigma), \epsilon)d\sigma \end{array}\right. \quad , \ \tau \in [\tau_j, \tau_{j+1})$$

where $\widetilde{G}(x,z,\epsilon) := G(x,z) - \epsilon\, s_x^0(x)\, F(x,z)$.

From this we obtain

$$|\varphi(\tau) - X(\tau)| \;\le\; \epsilon\, L_1 \int_{\tau_j}^{\tau} \Big[|\varphi(\sigma) - X(\sigma)| \,+\, |\psi(\sigma) - Z(\sigma)|\Big] d\sigma$$

$$|\psi(\tau) - Z(\tau)| \;\le\; |\psi(\tau_j) - Z(\tau_j)| + L_2 \int_{\tau_j}^{\tau} \Big[|\varphi(\sigma) - X(\sigma)| \,+\, |\psi(\sigma) - Z(\sigma)|\Big] d\sigma\ .$$

Adding the two inequalities we may apply Gronwall's lemma and thus obtain

$$|\varphi(\tau) - X(\tau)| \,+\, |\psi(\tau) - Z(\tau)| \le |\psi(\tau_j) - Z(\tau_j)|\, e^{(L_2 + \epsilon L_1)\tau^*} \tag{22}$$

for $\tau \in [\tau_j, \tau_{j+1})$.

Introducing this estimate into Eq.(21) we have

$$|\psi(\tau) - s^1(\varphi(\tau))| \le |\psi(\tau_j) - s^1(\varphi(\tau_j))\,|\, e^{(L_2 + \epsilon L_1)\tau^*}\ .$$

Combining this estimate and the estimate (20) we have shown that there is $K^*(\tau^*)$ such that

$$|\psi(\tau) - s^1(\varphi(\tau))| \le K^*\, e^{-\beta^*\tau} |z - s^1(x)| \quad \text{for} \ \ \tau \ge 0\ . \tag{23}$$

$\perp$

iii) *Asymptotic phase:* We again consider the solution $(\varphi(\tau; x, z, \epsilon), \psi(\tau; x, z, \epsilon))$ of Eq.(5) with $|z| \le \delta$, take $\tau > 0$ arbitrary but fixed and introduce $j \in \mathbb{N}_0$ and $\tau_j := j\tau^*$ such that $\tau_j \le \tau < \tau_{j+1}$. From Lemma 3 iii) we know that there is $(\tilde{x}_0, \tilde{z}_0) \in \overline{M}_\epsilon^1$ such that

$$|\varphi(\tau_j) - \widetilde{X}(\tau_j)| \le \epsilon\, \widetilde{C}\, e^{-\beta^*\tau_j}\, |z - s^1(x)| \tag{24}$$

and

$$|\psi(\tau_j) - \widetilde{Z}(\tau_j)| \le e^{-\beta^*\tau_j}\, |z - s^1(x)|$$

where $(\widetilde{X}(\tau), \widetilde{Z}(\tau))$ is the solution of Eq.(5) with initial values $\widetilde{X}(0) = \tilde{x}_0,\ \widetilde{Z}(0) = \tilde{z}_0$ and hence $\widetilde{Z}(\tau) = s^1(\widetilde{X}(\tau))$ for all τ.

We again use the solution $(X(\tau), Z(\tau)) \in \overline{M}^1_\epsilon$ of Eq.(5) introduced in ii). We estimate

$$
\begin{aligned}
|\varphi(\tau) - \widetilde{X}(\tau)| \; &\leq \; |\varphi(\tau) - X(\tau)| \; + \; |X(\tau) - \widetilde{X}(\tau)| \\
|\psi(\tau) - \tilde{Z}(\tau)| \; &\leq \; |\psi(\tau) - Z(\tau)| \; + \; |Z(\tau) - \tilde{Z}(\tau)| \, .
\end{aligned}
\tag{25}
$$

Due to the results of ii) and since

$$|Z(\tau) - \tilde{Z}(\tau)| = |s^1(X(\tau)) - s^1(\widetilde{X}(\tau))| \leq |X(\tau) - \widetilde{X}(\tau)|$$

we only need an estimate for $|X(\tau) - \widetilde{X}(\tau)|$.

From the integral equations of $X(\tau)$ and $\widetilde{X}(\tau)$ we obtain

$$|X(\tau) - \widetilde{X}(\tau)| \; \leq \; |\varphi(\tau_j) - \widetilde{X}(\tau_j)| + 2\,\epsilon\, L_1 \int_{\tau_j}^{\tau} |X(\sigma) - \widetilde{X}(\sigma)| d\sigma \quad \text{for} \;\; \tau \in [\tau_j, \tau_{j+1}) \, .$$

Hence, applying Gronwall's lemma yields

$$|X(\tau) - \widetilde{X}(\tau)| \leq |\varphi(\tau_j) - \widetilde{X}(\tau_j)| \, e^{2\,\epsilon\, L_1\, \tau^*} \, .$$

Now, combining this estimate and the estimates (22), (20) and (24) with Eq.(25) we have shown that there is $\widetilde{K}(\tau^*) > 0$ such that

$$
\left.
\begin{aligned}
|\varphi(\tau) - \widetilde{X}(\tau)| &\leq \widetilde{K}\, e^{-\beta^* \tau} |z - s^1(x)| \\
|\psi(\tau) - \tilde{Z}(\tau)| &\leq \widetilde{K}\, e^{-\beta^* \tau} |z - s^1(x)|
\end{aligned}
\right. \quad , \; \tau \geq 0 \, .
$$

$\perp$

v) *Maximality:* It holds that every solution $(x(\tau), z(\tau))$ of Eq.(5) satisfying $|z(\tau)| \leq \delta$ for all $\tau \in \mathbb{R}$ lies in $\overline{M}^1_\epsilon$, i.e., $z(\tau) = s^1(x(\tau), \epsilon)$ for all τ.

Proof: The set $\{(x(\tau), z(\tau)) \mid \tau \in \mathbb{R}\} \subset \mathbb{R}^m \times D^n_\delta$ is invariant under the flow of Eq.(5). Hence, the set $\{(x(j\,\tau^*), z(j\,\tau^*)) \mid j \in \mathbb{Z}\} \subset \mathbb{R}^m \times D^n_\delta$ is an invariant set of the map P_ϵ. Lemma 3 v) implies that this set lies in $\overline{M}^1_\epsilon$ and therefore $(x(\tau), z(\tau))$ lies in $\overline{M}^1_\epsilon$ for $\tau \in \mathbb{R}$ due to the invariance of $\overline{M}^1_\epsilon$ under Eq.(5). $\perp$

If $(x(\tau),\, z(\tau))$ is a solution of Eq.(5) then $(x(\tau),\, y(\tau))$ with $y(\tau) = s^0(x(\tau)) + z(\tau)$ is a solution of Eq.(4). Hence, defining

$$s(x, \epsilon) := s^0(x) + s^1(x, \epsilon)$$

completes the proof of Theorem 2.

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