Extremal Bounds for Bootstrap Percolation in the Hypercube

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Joint work with Natasha Morrison

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DMO Seminar, McGill University

April 4, 2016
A “Coffee Time” Problem

Consider an $n \times n$ grid in which each square is either infected or healthy. At each step of the process, a healthy square becomes infected if at least two of its neighbouring (vertical or horizontal) squares are infected. The process terminates when no additional squares can be infected.
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**Question:** What is the smallest number of initially infected squares needed to infect the whole $n \times n$ grid?
A Few Hints

Hint 1: An initial infection of size $n$ suffices.

Hint 2: The solution is very simple (some people would say that it can be described in just one word!)
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Bootstrap Percolation

More generally, let $G$ be a graph and fix $r \in \mathbb{N}$. Fix an initial set $A_0 \subseteq V(G)$ of infected vertices. At each step of the process, a healthy vertex $v$ becomes infected if at least $r$ of its neighbours are already infected. This is known as the $r$-neighbour bootstrap process on $G$.

Definition: We say that $A_0$ percolates if every vertex of $G$ is eventually infected.

Extremal Problem: Determine the minimum of $|A_0|$ over all sets $A_0 \subseteq V(G)$ such that $A_0$ percolates. (denoted $m(G, r)$)
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The Hypercube

Recall that the $d$-dimensional hypercube $Q_d$ such that $V(Q_d) = \{0, 1\}^d$ and $uv \in E(Q_d)$ if $u$ and $v$ differ on one coordinate.

Examples:

- $000$
- $100$
- $010$
- $001$
- $011$
- $101$
- $110$
- $111$
- $0000$
- $1000$
- $0100$
- $0010$
- $0110$
- $1010$
- $1100$
- $1110$
- $0001$
- $1001$
- $0101$
- $0011$
- $0111$
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\textbf{Examples:}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{hypercube_graph.png}
\end{figure}
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Examples:
The **2-neighbour process** in the hypercube is well understood:

**Theorem (Balogh and Bollobás 2006)**

The minimum cardinality of a percolating set for 2-neighbour bootstrap percolation in $Q^d$ is $\lceil \frac{d}{2} \rceil + 1$. (That is, $m(Q^d, 2) = \lceil \frac{d}{2} \rceil + 1$.)

What about the $r$-neighbour process for $r \geq 3$?

**Conjecture (Balogh and Bollobás 2006)**

Let $r \geq 3$ be fixed and $d \to \infty$. Then $m(Q^d, r) = 1 + o(1) \cdot r(d - 1)$. Until recently, the best known lower bound was only linear in $d$. 

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There is a natural upper bound construction.

**Definition:**
The $i$th level of the hypercube to be the set of vertices with coordinate sum equal to $i$.

**Upper Bound Construction:**
Take all vertices on level $r-2$ and an approximate Steiner system on level $r$ (which exists by a theorem of Rödl (1985)).

In the first step of the process, every vertex on level $r-1$ becomes infected. The infection now spreads upward and downward through $Q^d$ "level by level.

This proves the upper bound $m(Q^d, r) \leq 1 + o(1) r(d/2 - 1)$.
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Lower Bound

Theorem (Morrison and N. 2015+)

Let $r \geq 3$ be fixed and $d \to \infty$. Then

$$m(Q_d, r) \geq 1 + o(1)$$

$$r(d - 1)$$

The proof has two steps:

Step 1: Relate the problem to a more well-behaved percolation problem on the edges of $Q_d$ (instead of the vertices).

Step 2: Solve the edge problem on $Q_d$ using a linear algebraic technique and induction.

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The Edge Process

Let $G$ be a graph and fix $r \in \mathbb{N}$.

Fix an initial set $E_0 \subseteq E(G)$ of infected edges.

At each step of the process, an edge $e$ becomes infected if at least one of its endpoints is incident to $r$ already infected edges. (In other words, an edge becomes infected if it completes an infected copy of $K_{1,r+1}$)

Definition: The size of the smallest percolating set in the edge process is denoted by $m_e(G,r)$.

Lemma: $m_e(G,r) \geq m_e(G,r) r$ for all $G$ and $r$.

Goal: Prove $m_e(Q_d,r) \geq (1 + o(1))(d r - 1)$.
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**Goal:** Prove $m_e(Q_d, r) \geq (1 + o(1))(r^{-1})^d$. 

Jonathan Noel (Oxford) Bootstrap Percolation in the Hypercube
We obtain a recursive formula for $m_e(Q_d, r)$.

**Theorem (Morrison and N. 2015+)**

For $d \geq r + 1$, 

$$m_e(Q_d, r) = m_e(Q_{d-1}, r) + m_e(Q_{d-1}, r-1).$$

Similar to Pascal's Formula (but with different initial conditions).

**Corollary**:

$$m_e(Q_d, r) = (1 + o(1))(dr - 1)$$

for fixed $r$ and $d \to \infty$.

The upper bound is a simple recursive construction. The lower bound is the interesting part.

Jonathan Noel (Oxford)  Bootstrap Percolation in the Hypercube
Step 2: Solving the Edge Problem on $Q_d$

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**Corollary:** $m_e(Q_d, r) = (1 + o(1))\binom{d}{r-1}$ for fixed $r$ and $d \to \infty$. 
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\[
\begin{array}{c}
\text{Hello} \\
\rightarrow \\
\text{Helloooooooooooooooooooo}
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If every square is eventually infected, then the final perimeter of the infection is $4n$.

So, the perimeter of the initial infection must have been at least $4n$, and so there must have been at least $n$ initially infected squares!
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Suppose we want to prove $m_e(Q_d, r) \geq k$.

Our approach is similar to the grid problem, but with the word "perimeter" replaced with "dimension".

Let $W$ be a vector space. Approach:

1. For each copy of $K_1, r + 1$ the vectors assigned to its edges are linearly dependent with non-zero coefficients,
2. The dimension of the span of $\{ f_e : e \in E(Q_d) \}$ is at least $k$.

Because of the linear dependence, the dimension of the infected edges cannot increase. Therefore, $m_e(Q_d, r) \geq k$.

Similar ideas are used by Kalai (1985) and Balogh, Bollobás, Morris and Riordan (2012) to solve related problems.
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**Approach:** Assign each $e \in E(Q_d)$ to a vector $f_e \in W$ so that:

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2. The dimension of the span of $\{f_e : e \in E(Q_d)\}$ is at least $k$. 

Because of the linear dependence, the dimension of the infected edges cannot increase. Therefore, $m_e(Q_d, r) \geq k$.

Similar ideas are used by Kalai (1985) and Balogh, Bollobás, Morris and Riordan (2012) to solve related problems.
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A More General Edge Process

These linear algebraic tricks can also be used to study more general infection problems. Consider the following:

Let $G$ be a graph containing many copies of a graph $H$.

Start with an initial set of infected edges in $G$.

In each step, a healthy edge becomes infected if it completes an infected copy of $H$.

To show that at least $k$ infected edges are required, one approach is to assign vectors to the edges of $G$ such that

1. the vectors assigned to the edges of any copy of $H$ in $G$ are linearly dependent with non-zero coefficients and
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Jonathan Noel (Oxford) Bootstrap Percolation in the Hypercube
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Recall, Balogh and Bollobás (2006) proved $m(Q_d, 2) = \left\lceil \frac{d}{2} \right\rceil + 1$. For $r = 3$, the previous best known bounds were (for certain $d$)

$$d + 3 \leq m(Q_d, 3) \leq \frac{d}{2} (d + 5) 6.$$  

We obtain a lower bound of $m(Q_d, 3) \geq \left\lceil \frac{d}{2} (d + 3) 6 \right\rceil + 1$. We also obtain a matching upper bound:

**Theorem (Morrison and N. 2015+)**

For $d \geq 3$, $m(Q_d, 3) = \left\lceil \frac{d}{2} (d + 3) 6 \right\rceil + 1$.

**Open Problem:** Determine $m(Q_d, r)$ for $r \geq 4$ and $d \geq r$.  

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Bootstrap Percolation in the Hypercube
An Exact Result for $r = 3$

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Theorem (Morrison and N. 2015+) Let $G$ be the $a_1 \times a_2 \times \cdots \times a_d$ grid. Then

$$m(G, r) \geq r \sum_{S \subseteq [d]} |S| \leq r - 1 \left( \prod_{i \in S} (a_i - 2) \right) \left( \left( r - |S| \right)^2 - r - |S| - 1 \right) \sum_{j=1}^{|S|} \left( d - j - 1 \right) - \frac{1}{j}.$$

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Why Do We Care?

Probabilistic results in bootstrap percolation is of interest to statistical physicists, who introduced it as a model the dynamics of disordered magnetic systems. Extremal results in bootstrap percolation can sometimes be applied in the probabilistic setting. In particular, Balogh, Bollobás and Morris (2010) used the fact that $m(Q_d, 2) = \lceil d^2 \rceil + 1$ to prove a sharp threshold result for 2-neighbour bootstrap percolation in the hypercube.
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Thank you!