Numerical estimates for monochromatic percolation exponents

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Abstract

We present the numerical results of Monte-Carlo simulations, consisting in estimates of monochromatic arm exponents for critical two-dimensional percolation on the square lattice (and in particular, of the backbone exponent). The values indicate a simple link between each of these and the corresponding polychromatic exponent, which leads us to a conjecture about their exact values. In particular, the backbone exponent is measured to be very close to 17/48.

Introduction

This paper can be seen as a complement to our previous article [2], in which we prove the existence of the monochromatic arm exponents $(\alpha'_j)_{j\geqslant 2}$ for critical two-dimensional percolation — in order to keep this note as compact as possible, we refer the reader to that paper for further details and background.

The values of the polychromatic exponents were predicted by Aharony, Aizenman and Duplantier in [1] and later derived rigorously (using SLE techniques) by Smirnov and Werner in [3]. They are equal to

$$\alpha_j = \frac{j^2-1}{12}.$$

In [2] we prove the inequality

$$\alpha'_j > \alpha_j$$

(the inequality $\alpha'_j < \alpha_{j+1}$ is essentially trivial) but as of now, the values of the monochromatic exponents for $j \geqslant 2$ has not been derived rigorously, nor even predicted. Some of them have been estimated through simulation, but so far the results have been inconclusive. The one-arm exponent however is known to be equal to $\alpha'_1 = 5/48$ (thus it is not the "polychromatic one-arm exponent", whatever that should mean).

We simulated percolation in boxes of various sizes in the square lattice, at $p = p_c = 1/2$, and for each sample, counted the number of disjoint arms connecting the neighbors of the origin to the boundary of the box. This leads to estimates for the probability of observing j arms in a box of that size (where for obvious reasons $1 \leqslant j \leqslant 4$), and thus to approximate values of the corresponding exponents; the results of these and a few details about the methods employed are presented in the next section.

As it turns out, for each $j \in \{1,2,3,4\}$, the value we obtain is very close to being of the form $k_j/48$ with $k_j \in \mathbb{Z}_+$; more specifically, we obtain

$$\alpha'_1 \simeq \frac{5}{48}\ ; \quad \alpha'_2 \simeq \frac{17}{48}\ ; \quad \alpha'_3 \simeq \frac{37}{48}\ ; \quad \alpha'_4 \simeq \frac{63}{48}$$

(the fourth one being much less precise, because on boxes of the sizes we considered, the presence of 4 arms is extremely unlikely). The reader can notice that these values are compatible with the following relation:

$$\alpha'_j = \alpha_j + \alpha'_1 = \alpha_j + \frac{5}{48} = \frac{4j^2+1}{48} \tag{1}$$

(at least up to $j = 3$ — but as we noticed above, the measured value of α'_4 is very unreliable).

The first equality in (1) is simple enough to suggest a combinatorial interpretation (and thus a direct proof), possibly involving the coupling of a configuration with j arms of the same color with a pair of configurations, one with j arms of different colors and one with only 1 arm. However, despite our best efforts, we were unable to construct such a coupling; We would gladly hear from any such proof, which is why we made this note public.

Of course, it could as well be a numerical coincidence, or a feature of a continuous scaling limit without any simple counterpart on the discrete level. Besides, it is not impossible that it only holds for small values of j.

The numerical results

The simulations were performed at the Pôle Scientifique de Modélisation Numérique (PSMN) in Lyon. Table 1 contains the raw data: For each value of n and j, the indicated value is the proportion of samples containing j open arms in the box of size $n \times n$ in $\mathbb{Z}^2$, and the number in parenthesis is the standard deviation of the estimator. The number of samples in each case is equal to 200 000 (except for the case $n = 1\,800$ for which it is 180 000, because two of the computers crashed ...). The interested reader can download the source code for our programs at the following web page:

`http://www.umpa.ens-lyon.fr/~vbeffara/simu.php`

Once the raw data are obtained, we estimated the values of the exponents by a least square fit in log-log space with a 3-parameter family of functions, with one harmonic correction term:

$$f_{\alpha,c,\varepsilon}(n) = \frac{c}{n^\alpha} + \frac{\varepsilon}{n^{\alpha+1}}.$$

The results of the fit are shown in Figure 1, and the following values are obtained (since we are looking for exponents of the form $\alpha'_j = k_j/48$, we present estimates for $48\alpha'_j$):

$$\begin{aligned} 48\,\alpha'_1 &\simeq 4.967, \\ 48\,\alpha'_2 &\simeq 17.009, \\ 48\,\alpha'_3 &\simeq 36.992, \end{aligned}$$

with rather narrow confidence intervals.

References

[1] M. Aizenman, B. Duplantier, and A. Aharony, *Path crossing exponents and the external perimeter in 2D percolation*, Phys. Rev. Lett., 83 (1999), pp. 1359–1362.

[2] V. Beffara and P. Nolin, *On monochromatic arm exponents for 2D critical percolation*, submitted, (2009). arXiv:0906.3570.

[3] S. Smirnov and W. Werner, *Critical exponents for two-dimensional percolation*, Mathematical Research Letters, 8 (2001), pp. 729–744.

n	1 arm	2 arms	3 arms	4 arms
10	0.793825 (0.000905)	0.375295 (0.001083)	0.085640 (0.000626)	0.006835 (0.000184)
13	0.775495 (0.000933)	0.341835 (0.001061)	0.070920 (0.000574)	0.005065 (0.000159)
18	0.750905 (0.000967)	0.306425 (0.001031)	0.055705 (0.000513)	0.003315 (0.000129)
24	0.729260 (0.000994)	0.278535 (0.001002)	0.044545 (0.000461)	0.002275 (0.000107)
32	0.707250 (0.001017)	0.251075 (0.000970)	0.036195 (0.000418)	0.001530 (0.000087)
42	0.688170 (0.001036)	0.229360 (0.000940)	0.029835 (0.000380)	0.001090 (0.000074)
56	0.668355 (0.001053)	0.206900 (0.000906)	0.024335 (0.000345)	0.000635 (0.000056)
75	0.648580 (0.001068)	0.185600 (0.000869)	0.018795 (0.000304)	0.000465 (0.000048)
100	0.631185 (0.001079)	0.169845 (0.000840)	0.015520 (0.000276)	0.000435 (0.000047)
130	0.614410 (0.001088)	0.153845 (0.000807)	0.012610 (0.000250)	0.000270 (0.000037)
180	0.594920 (0.001098)	0.136050 (0.000767)	0.009715 (0.000219)	0.000120 (0.000024)
240	0.574295 (0.001106)	0.124620 (0.000739)	0.008050 (0.000200)	0.000105 (0.000023)
320	0.560365 (0.001110)	0.112155 (0.000706)	0.005840 (0.000170)	0.000060 (0.000017)
420	0.544220 (0.001114)	0.100920 (0.000674)	0.005080 (0.000159)	0.000050 (0.000016)
560	0.527030 (0.001116)	0.090820 (0.000643)	0.003975 (0.000141)	0.000040 (0.000014)
750	0.511625 (0.001118)	0.083440 (0.000618)	0.003325 (0.000129)	0.000005 (0.000005)
1000	0.496125 (0.001118)	0.074360 (0.000587)	0.002615 (0.000114)	0.000020 (0.000010)
1300	0.484390 (0.001117)	0.068760 (0.000566)	0.002170 (0.000104)	0.000020 (0.000010)
1800	0.468211 (0.001176)	0.060361 (0.000561)	0.001644 (0.000096)	(0)

Table 1: The results of the Monte-Carlo simulation: the raw data. We are not statisticians, so not all digits are significant ... None of the samples of size 1 800 contained 4 arms — the last column is only here for reference, and further simulation is needed to get significant results.

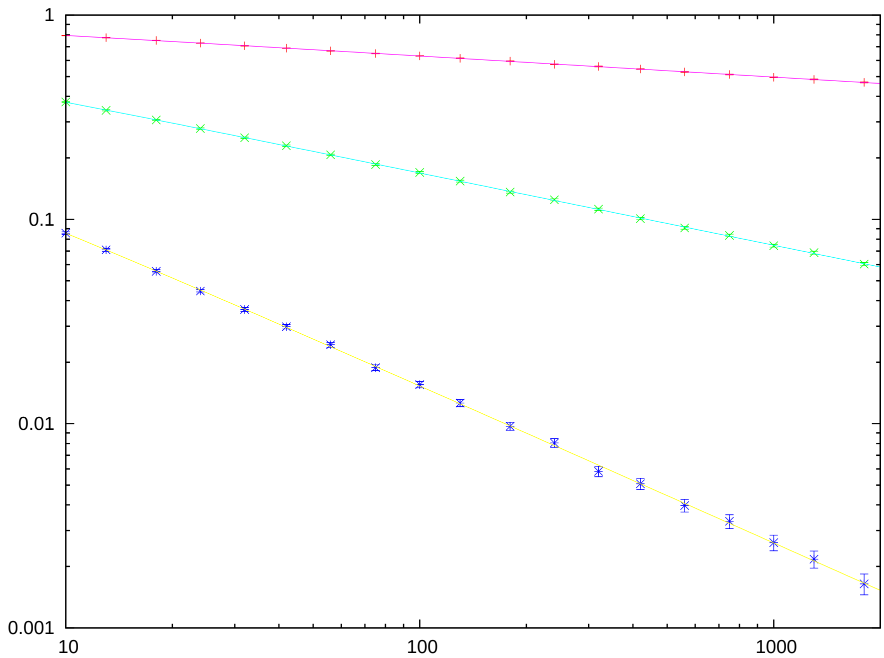

Figure 1: Linear fit of the results of the Monte-Carlo simulations, showing remarkable linearity in the log-log plot. The data for 4 arms are hidden because, given the very small probability of the corresponding event, they are not precise enough to be trusted.