

ESTIMATES ON THE ZEROS OF E_2

ÖZLEM IMAMOĞLU, JONAS JERMANN, ÁRPÁD TÓTH

ABSTRACT. We improve the results from [EBS10] about the Eisenstein series $E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}$. In particular we show that there exists exactly one (simple) zero in each Ford circle and give an approximation to its location.

1. INTRODUCTION

Let $\mathcal{M}_k(\Gamma)$ be the space of holomorphic modular forms of weight k for the full modular group $\Gamma = PSL(2, \mathbb{Z})$. It is well known that $\mathcal{M}_k(\Gamma)$ has dimension $\frac{k}{12} + O(1)$ and a modular form $f \in \mathcal{M}_k$ has $\frac{k}{12} + O(1)$ inequivalent zeros in a fundamental domain $\Gamma \backslash \mathbb{H}$.

For the cuspidal Hecke eigenforms, it is a consequence of the recent proof of the holomorphic Quantum Unique Ergodicity (QUE) by Holowinsky and Soundararajan [HS10] that the zeros are uniformly distributed. More precisely, for a sequence $\{f_k\}$ of cuspidal Hecke eigenforms of weight k we have that as $k \rightarrow \infty$ the zeros of f_k become equidistributed with respect to the normalized hyperbolic measure $\frac{3}{\pi} \frac{dx dy}{y^2}$.

In contrast to this in the case of Eisenstein series, it was conjectured by R.A. Rankin in 1968 and proved by F.K.C. Rankin and Swinnerton-Dyer [RSD70] that all the zeros, in the standard fundamental domain, of the series

$$E_k(\tau) = \frac{1}{2} \sum_{(c,d)=1} (c\tau + d)^{-k}$$

lie on the geodesic arc $\{z \in \mathbb{H} : |z| = 1, 0 \leq \operatorname{Re} z \leq 1/2\}$ and as $k \rightarrow \infty$ they become uniformly distributed on this unit arc. A similar result for the cuspidal Poincaré series was proved by R.A. Rankin [Ran82]. For generalizations of these results to other Fuchsian groups and to weakly holomorphic modular functions see [AKN97], [DJ08], among many others. For some recent work on the zeros of holomorphic Hecke cusp forms that lie on the geodesic segments of the standard fundamental domain see [GS12].

Date: August 6, 2018.

Ö. Imamoglu and J. Jermann are supported by SNF 200020-144285.

Next we turn our attention to the location of the zeros of the non-modular Eisenstein series of weight 2

$$(1) \quad E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \quad q = e^{2\pi i\tau}.$$

As it is well known $E_2(\tau)$ is not modular but it is a quasimodular form and for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ satisfies the transformation property

$$(2) \quad E_2(\gamma\tau)(c\tau + d)^{-2} = E_2(\tau) + \frac{6}{\pi i} \frac{1}{\left(\tau + \frac{d}{c}\right)},$$

The zeros of E_2 has already been investigated in [EBS10]. El Basraoui and Sebbar showed that there are infinitely many non-equivalent zeros of E_2 and two zeros are equivalent if and only if one is a \mathbb{Z} -translate of the other. Of particular interest is the unique zero on the imaginary axis $x = 0$ and the unique zero on $x = \frac{1}{2}$. Those two zeros also occur in [SF07]. They were computed by H. Cohen:

$$(3) \quad \tau_0 = 0.52352170001799926680053440480610976968 \dots i$$

$$(4) \quad \tau_{\frac{1}{2}} = \frac{1}{2} + 0.13091903039676244690411482601971302060 \dots i$$

If

$$\mathcal{F} := \left\{ z \in \mathbb{H} \mid |z| > 1, |\mathrm{Re}(z)| < \frac{1}{2} \right\} \cup \left\{ z \in \mathbb{H} \mid |z| \geq 1, -\frac{1}{2} \leq \mathrm{Re}(z) \leq 0 \right\}$$

is the standard (strict) fundamental domain, El Basraoui and Sebbar also show that there are infinitely many $\mathrm{SL}_2(\mathbb{Z})$ -translates of \mathcal{F} that contain a zero and infinitely many that do not contain a zero.

In this note we improve these results. More precisely for $(a, c) = 1$ recall that the associated Ford circle is the circle on the upper half plane with center $\frac{a}{c} + \frac{1}{2c^2}i$ and radius $\frac{1}{2c^2}$. Then we have

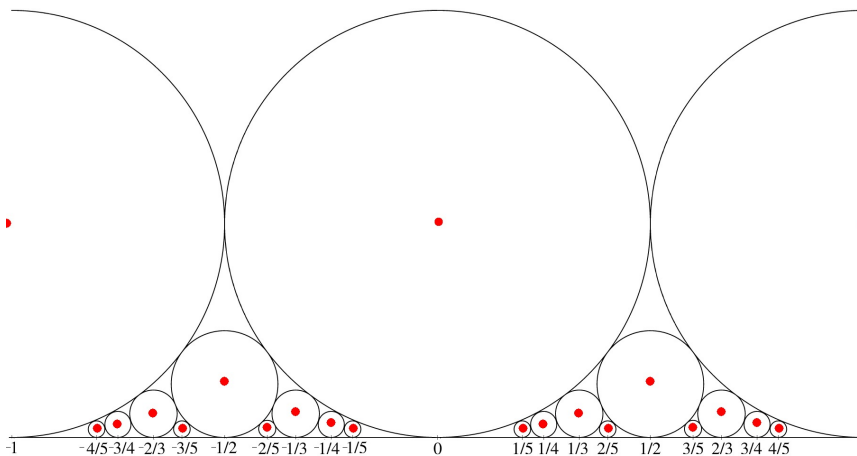
Theorem 1.1. *Inside each Ford circle there is a unique (simple) zero of E_2 and E_2 has no other zeros on the upper half plane.*

Moreover in the Ford circle at the cusp $\frac{a}{c} \in \mathbb{Q}$ the zero satisfies the following approximation:

$$(5) \quad 0.000075 \frac{1}{c^2} < \left| \tau - \frac{a}{c} - \frac{\pi}{6c^2}i \right| < 0.0000777 \frac{1}{c^2}.$$

Since $\pi/6$ is close to $1/2$, the zeros are almost at the center of each Ford circle and in fact when plotted on a not small enough scale as in the following picture, the zeros seem to have remarkable uniformity on the upper half plane.

Theorem 1.1 restricts the possible location of the zeros to a very thin annular region in each Ford circle. For the images of these zeros inside the standard fundamental domain \mathcal{F} we have



Theorem 1.2. *Let $\tau = \gamma z$ be the unique zero of E_2 around $\frac{a}{c}$ with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $c > 0$ and $z \in \mathcal{F}$ then we have*

$$(6) \quad 0.00027 < \left| z - \left(-\frac{d}{c} + \frac{6i}{\pi} \right) \right| < 0.00029.$$

We may rephrase Theorem 1.1 as saying that around each finite cusp there is a unique zero of E_2 . i.e. given $(a, c) = 1$ with $c > 0$, there is a unique matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that $\gamma\mathcal{F}$ contains a zero.

While preparing this manuscript we learned that R. Wood and M. Young [WY] proved a quantitatively equivalent result about the locations of the zeros of E_2 simultaneously and independently of us.

It is possible to give further improvements on these approximations by expanding either in Taylor or Fourier series but we will not pursue this in this paper at any depth. In the next section we give the proofs of Theorem 1.1 and Theorem 1.2 and in the final section we provide some numerical data on the Fourier and Taylor expansions.

2. PROOFS

The proofs of Theorem 1.1 and Theorem 1.2 are based on the mapping properties of the function $f(z)$ defined by

$$(7) \quad f(z) := z - \frac{6i}{\pi E_2(z)}.$$

We will see that f is analytic in the fundamental domain, and even on $\{z : \text{Im}z > 0.53\}$. Let z be in the fundamental domain \mathcal{F} , and let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Then based on the transformation formula of E_2 , one has the fundamental relation

$$(8) \quad E_2(\gamma z) = 0 \iff f(z) = -d/c$$

Therefore we are interested in the solutions of the equation

$$f(z) = -d/c \quad z \in \mathcal{F}.$$

Here $d, c \in \mathbf{Z}$. Let's start with the solution of

$$f(z) \in \mathbb{R} \quad z \in \mathcal{F}.$$

Let $f = u + iv$ and

$$(9) \quad A = \{z \in \overline{\mathcal{F}} : v(z) = 0\}.$$

As a simple application of the implicit function theorem it will be proved below that A is the graph of a function $\phi : [-1/2, 1/2] \rightarrow (0, \infty)$. More precisely if $x + iy \in \mathcal{F}$ and $v(x, y) = 0$ then $y = \phi(x)$.

In view of the relation (8) the possible values $c, d \in \mathbf{Z}$ that arise from a zero of E_2 are those for which $-d/c \in f(A)$. Let $\tau = \gamma z$ be a zero of E_2 with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \mathcal{F}$. Then we will show that $|d/c| \leq 1/2$ and conversely any such $-d/c \in [-1/2, 1/2]$ arises from a zero of $E_2(z)$.

Namely we have

Theorem 2.1. $f(A) = [-1/2, 1/2]$.

The first claim in Theorem 1.1 now follows immediately from

Corollary 2.2. *If $z \in \mathcal{F}$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $E_2(\gamma z) = 0$ then $|d/c| \leq 1/2$.*

In particular Corollary 2.2 uniquely determines the fundamental domain around $\frac{a}{c}$ which contains the zero of E_2 .

The more precise location of zeros of E_2 given in (5) and (6) follow from bounds for E_2 and its derivative.

To prove Theorem 2.1 we will use the implicit function theorem. To ease the notation let $f = u + iv$. We are interested in $v(x, y) = 0$. We need the following estimates.

Lemma 2.3. *Let $z = x + iy \in \mathbb{H}$ and let $q = e^{2\pi iz}$. We have the following estimates.*

$$(10) \quad |E_2(z) - 1| \leq \frac{24|q|}{(1 - |q|)^3},$$

$$(11) \quad 23.48|q| < |E_2(z) - 1| \quad \text{for } y \geq \frac{\sqrt{3}}{2},$$

$$(12) \quad |E_2'(z)| \leq 48\pi \frac{|q|(1 + |q|)}{(1 - |q|)^5}$$

In particular for $y > 0.53$ we have $0 < |E_2(z) - 1| < 1$ and so $E_2(z) \neq 0$ in this region and hence f in (7) is well defined.

Proof. We have

$$|E_2(w) - 1| \leq 24 \sum_{n=1}^{\infty} \frac{n|q|^n}{1 - |q|^n} \leq \frac{24}{1 - |q|} \sum_{n=1}^{\infty} n|q|^n = \frac{24|q|}{(1 - |q|)^3}.$$

The claim for the lower bound for $E_2(z)$ and $E_2'(z)$ follows along the same lines. \square

Lemma 2.4. *If $\text{Im}z \geq 1$ then*

$$(13) \quad |f'(z) - 1| < 0.57$$

Proof. Since

$$|f'(z) - 1| = \frac{6|E_2'(z)|}{\pi|E_2(z)|^2}$$

the result follows from Lemma 2.3 and the bounds (valid for $\text{Im}z \geq 1$)

$$|E_2(z)| \geq |E_2(i\text{Im}z)| \geq |E_2(i)|$$

and

$$|E_2'(z)| \leq |E_2'(i\text{Im}z)| \leq |E_2'(i)|.$$

\square

We will also need the following

Lemma 2.5. *For $x \in [-1/2, 1/2]$ and $y \in [\sqrt{3}/2, 1]$ we have*

$$\text{Im}f(x + iy) < 0.$$

Proof. For any x we have

$$\text{Re}E_2(x + iy) \geq 1 - \frac{24e^{-2\pi y}}{(1 - e^{-2\pi y})^3}$$

and

$$|E_2(x + iy)| \leq 1 + \frac{24e^{-2\pi y}}{(1 - e^{-2\pi y})^3}$$

Therefore for $y \in [\sqrt{3}/2, 1]$ we have

$$\text{Im}f(x + iy) = y - \frac{6\text{Re}E_2(x + iy)}{\pi|E_2(x + iy)|^2} < 1 - \frac{6(1 - t)}{\pi(1 + t)^2}$$

where

$$t = \frac{24e^{-2\pi\sqrt{3}/2}}{(1 - e^{-2\pi\sqrt{3}/2})^3}.$$

The lemma follows from the numerical estimate $t < 0.1054$. \square

Since $\partial_y v = \partial_x u = \text{Re}f'(z)$ we also have

Lemma 2.6. *If $y \geq 1$ then $\partial_y v(x, y) > 0$.*

Proof. By Lemma 2.4,

$$|f'(z) - 1| < 0.57.$$

So $\partial_y v(z) = \operatorname{Re} f'(z) > 0.4$. \square

From Lemma 2.6 it follows that for each $x \in [-1/2, 1/2]$ there can be only one y such that $v(x + iy) = 0$. By Lemma 2.5, it is also clear that such a y exists since $\lim_{y \rightarrow \infty} v(x + iy) = \infty$. Therefore there is a function $\phi : [-1/2, 1/2] \rightarrow (0, \infty)$ such that $x + iy \in \mathcal{F}$ and $v(x, y) = 0$ implies $y = \phi(x)$.

Moreover this ϕ is differentiable by the implicit function theorem, and we have $\phi'(x) = -\partial_x v / \partial_y v$.

Let now the arc A be as in (9). We know from El Basraoui and Sebbar that there are points $z_{1/2}$ and $z_{-1/2}$ on the both of the vertical boundaries of \mathcal{F} , where f takes the values $-1/2$ and $1/2$. They correspond to the zeros of E_2 with real parts $1/2$ and $-1/2$.

By the intermediate value theorem it follows that f restricted to the arc A , (which is the same as u since v vanishes on A) takes all values between $-1/2$ and $1/2$. This part of the claim shows that there is a zero in each Ford circle. To prove that there is only one zero in each Ford circle, we need to show that $f(A) = u(A) = [-1/2, 1/2]$. This we prove by showing that u decreases along A as x increases from $-1/2$ to $1/2$.

Proposition 2.7. *Let $f = u + iv$ and $\phi : [-1/2, 1/2] \rightarrow (1, \infty)$ the function such that*

$$\{z \in \mathcal{F} : v(z) = 0\} = \{(x, \phi(x)) : x \in [-1/2, 1/2]\}$$

For $|x| < 1/2$ we have

$$\frac{d}{dx} u(x, \phi(x)) > 0.$$

Proof. We need to show that

$$\frac{d}{dx} u(x, \phi(x)) = \partial_x u(x, \phi(x)) + \partial_y u(x, \phi(x)) \phi'(x) > 0.$$

By implicit differentiation of $v(x, \phi(x)) = 0$ we have $\phi'(x) = -\partial_x v / \partial_y v$. Therefore

$$\frac{d}{dx} u(x, \phi(x)) = \frac{\partial_x u \partial_y v - \partial_y u \partial_x v}{\partial_y v} = \frac{(\partial_x u)^2 + (\partial_y u)^2}{\partial_y v} = \frac{|f'|^2}{\partial_y v} > 0.$$

\square

This also finishes the proof of Theorem 2.1 and hence the first claim in Theorem 1.1.

The next Proposition on the other hand proves the estimates (5) and (6) about the location of zeros in Theorem 1.1 and Theorem 1.2.

Proposition 2.8. *If $E_2(\gamma z) = 0$, $z = x + iy \in \mathcal{F}$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with $c > 0$ and $\tau = \gamma z$ then $1.909 < y < 1.91$ and*

$$(14) \quad 0.000144 < |E_2(z) - 1| < 0.000149,$$

$$(15) \quad 0.00027 < \left| z - \left(-\frac{d}{c} + \frac{6}{\pi}i \right) \right| < 0.00029,$$

$$(16) \quad 0.0000750 \frac{1}{c^2} < \left| \tau - \left(\frac{a}{c} + \frac{\pi}{6c^2}i \right) \right| < 0.0000777 \frac{1}{c^2}.$$

Proof. We first note that since $y \geq \frac{\sqrt{3}}{2}$ on $\overline{\mathcal{F}}$, using (10) we have

$$(17) \quad |E_2(z) - 1| \leq \frac{24}{(1 - e^{-\pi\sqrt{3}})^3} |q| < 24.32|q| < 0.106.$$

And similarly

$$(18) \quad 23.48|q| < |E_2(z) - 1|$$

We next recall that $E_2(\gamma z) = 0$ gives $f(z) = -d/c$ which in return implies $z - \frac{6i}{\pi E_2(z)} = \frac{-d}{c}$. Hence

$$(19) \quad \left| z - \left(-\frac{d}{c} + \frac{6}{\pi}i \right) \right| = \frac{6}{\pi} \frac{|E_2(z) - 1|}{|E_2(z)|} \leq \frac{6}{\pi} \frac{24|q|}{(1 - |q|)^3 \left(1 - \frac{24|q|}{(1 - |q|)^3} \right)^2}$$

$$(20) \quad < \frac{144}{\pi(1 - 27|q|)^2} |q|$$

$$(21) \quad < \frac{6}{\pi} \frac{0.106}{(1 - 0.106)^2} < 0.26$$

This in turn means that $y > \frac{6}{\pi} - 0.26 > 1.64$ which gives a better estimate for (20). Repeating this procedure twice already gives the nice estimates:

$$(22) \quad y \geq 1.909, \quad |q| \leq 6.18 \times 10^{-6},$$

$$(23) \quad |E_2(z) - 1| < 24 \frac{1}{(1 - |q|)^3} |q| < 0.000149,$$

$$(24) \quad \left| z - \left(-\frac{d}{c} + \frac{6}{\pi}i \right) \right| < 0.00029.$$

The estimates also show that $y < \frac{6}{\pi} + 0.00029 < 1.91$ resp. (using (18)) $|E_2(z) - 1| > 0.000144$, so

$$(25) \quad 0.00027 < \left| z - \left(-\frac{d}{c} + \frac{6}{\pi}i \right) \right|.$$

In particular we also have (14). On the other hand using $\tau = \gamma^{-1}z$ and $E_2(\gamma z) = 0$ we have

$$(26) \quad E(z) = E(\gamma^{-1}\tau) = \frac{6}{\pi i}(-c)(-c\tau + a)$$

and

$$\tau = \frac{a}{c} + \frac{\pi}{6c^2}i + \frac{\pi}{6c^2}i(E_2(z) - 1)$$

So

$$(27) \quad \left| \tau - \frac{a}{c} - \frac{\pi}{6c^2}i \right| < \frac{\pi}{6c^2} |E_2(z) - 1| < 0.0000777 \frac{1}{c^2},$$

where we used (14). Similarly we get:

$$(28) \quad 0.0000750 \frac{1}{c^2} < \frac{\pi}{6c^2} |E_2(z) - 1| < \left| \tau - \frac{a}{c} - \frac{\pi}{6c^2}i \right|.$$

□

3. NUMERICAL CALCULATIONS

In this section we provide additional numerical data on the zeros of $E_2(z)$.

Let $F(x) = f^{-1}(x)$ denote the inverse of $f(z)$ restricted to \mathbb{R} . (Note that F defines an (injective) holomorphic map in a neighbourhood of \mathbb{R} .)

Recall the $\mathrm{SL}_2(\mathbb{Z})$ -translates of the zeros of E_2 inside the strict fundamental domain \mathcal{F} are exactly given by $F(x)$ for $x \in \mathbb{Q} \cap (-\frac{1}{2}, \frac{1}{2}]$.

3.1. Fourier expansion.

Let $H(x) := F(x) - x$. Since $F(x+1) = F(x) + 1$, $H(x)$ is 1-periodic, so it possesses a Fourier expansion:

$$(29) \quad H(x) = \sum_{n \in \mathbb{Z}} c_n \exp(2\pi i n x).$$

To get a formal expression for it we use the standard change of coordinates $q = e^{2\pi i \tau}$. Then $f(\tau) = \tilde{f}(q)$, $H(\tau) = f^{-1}(\tau) - \tau = \frac{1}{2\pi i} (\tilde{H} \circ q)(\tau)$ where

$$(30) \quad \tilde{H}(q) = \left(\log(\tilde{f}^{-1}(\log(q))) - \log(q) \right).$$

Note that although \tilde{f} has a term involving $\log q$ this cancels out and in fact we have that

$$(31) \quad 2\pi i H \left(\tau - \frac{6i}{\pi} \right) = \tilde{H}(q \exp(12)) = \log \left(\frac{\left(q \exp \left(\frac{12}{E_2(q)} - 12 \right) \right)^{-1}}{q \exp(12)} \right) = -12 + \log \left(\frac{\left(q \exp \left(\frac{12}{E_2(q)} - 12 \right) \right)^{-1}}{q} \right) \in \mathbb{Q}[[q]].$$

The existence of the last formal power series and the rationality of its coefficients follow from the facts that $\frac{12}{E_2(q)} - 12 \in \mathbb{Q}[[q]]$ and is divisible by q in this ring.

The coefficients of $\tilde{H}(q \exp(12))$ can be computed explicitly and we have

$$\tilde{H}(qe^{12}) = -12 - 288q + 75168q^2 - 29321856q^3 + 13541649696q^4 + O(q^5)$$

(Note that higher index coefficients no longer all lie in \mathbb{Z} .)

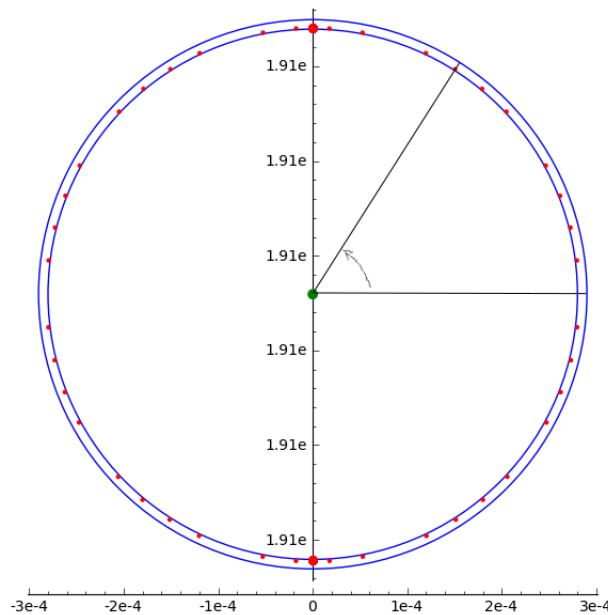
From this the constants c_n can be expressed, for example $c_0 = \frac{6i}{\pi}$ and $c_1 = 144 \frac{e^{-12}}{\pi} i$, and $c_n = r_n \frac{e^{-12n}}{\pi} i$ for some rational number r_n .

Calculations of the Fourier coefficients by numerical integration show perfect agreement with these formal computations:

The numerical results for c_n , $n \geq 0$ were calculated using sage:

$$\begin{aligned} c_0 &= 1.90985931710274402922660516047017234441351574888547738497i \\ c_1 &= 0.00028162994902227980400370919939063856289594529890275357i \\ c_2 &= -0.00000045163288929282012635455207577614911204985274433204i \\ c_3 &= 0.00000000108245596925811696405920054080771200657423178116i \\ c_4 &= -0.00000000000307154282808538137128721799597366123291772057i \\ c_5 &= 0.00000000000000957094344711630129941209014246199967789040i \\ c_6 &= -0.00000000000000003165503372449709626701121359204401518804i \\ c_7 &= 0.0000000000000000010911333723210259127123321555755721374i \\ c_8 &= -0.0000000000000000000038769575689989675972304397185016487i \\ c_9 &= 0.0000000000000000000000140991650336337176376718140570072i \\ c_{10} &= -0.00000000000000000000000522238601930043508518075691844i \end{aligned}$$

Below is a picture of $H(x)$ for $x \in \mathbb{Q} \cap [-\frac{1}{2}, \frac{1}{2}]$, $x = \frac{d}{100}$ with $(d, 100) = 1$ (recall that this is the error term in the approximation $F(x) \approx x$).



The closed curve $H(\mathbb{R}) = H\left(\left[-\frac{1}{2}, \frac{1}{2}\right]\right)$ is already extremely well approximated by the first two terms $c_0 + c_1 \exp(2\pi i x)$. Indeed estimate (6) for $x = \frac{d}{c}$ (in fact for any $x \in \mathbb{R}$) and $H\left(\frac{d}{c}\right) = F\left(\frac{d}{c}\right) - \frac{d}{c}$ shows:

$$(32) \quad 0.00027 < \left| H(x) - \frac{6i}{\pi} \right| < 0.00029.$$

These circular bounds are also drawn in the picture.

A good bound on the remaining Fourier coefficients c_n , $n \geq 2$ would give an estimate on the *angle* of $H(x) - c_0$. Namely that the angle is *close* to $2\pi\left(x + \frac{1}{4}\right)$. Note in particular that $H(0) \in i\mathbb{R}$ is located on the top of $H(\mathbb{R})$, so the angle of $H(0) - c_0$ is exactly $2\pi\frac{1}{4}$. Similarly $H\left(\frac{-1}{2}\right) \in i\mathbb{R}$ is located on the bottom of $H(\mathbb{R})$ and the angle of $H\left(\frac{-1}{2}\right) - c_0$ is exactly $2\pi\frac{-1}{4}$. That's both under the assumption that $c_0 \in i\mathbb{R}$.

3.2. Taylor expansion.

We can also examine the Taylor expansion $F(x) = \sum_{n=0}^{\infty} b_n x^n$ around $x = 0$.

By Lagrange inversion theorem the Taylor coefficients can be calculated in terms of the Taylor coefficients of $f(z)$ at z_0 , where z_0 is the $SL_2(\mathbb{Z})$ -translate in the strict fundamental domain of the zero of E_2 on the imaginary axis.

Note that $f(z)$ is a rational function in $E_2(z)$ and z . It's derivatives can be calculated as rational functions in $E_2(z)$, $E_4(z)$, $E_6(z)$ and z . Therefore the Taylor expansion at z_0 can be expressed as a rational function in $X := E_4(z_0)$, $Y := E_6(z_0)$, $Z := E_2(z_0)$ and z_0 . By the transformation property (2) we have $z_0 = -\frac{12}{2\pi i Z}$, so in fact we get a rational function in X , Y and Z .

For the first coefficient we have $b_0 = \left(\frac{6i}{\pi}\right) \frac{1}{Z}$. For the remaining coefficients ($n > 0$) calculations seem to indicate that b_n is of the following shape:

$$(33) \quad b_n = (-1)^n \left(\frac{6i}{\pi}\right)^{1-n} X^{-2n+1} a_n(X, Y, Z),$$

where $a_n(X, Y, Z) \in \mathbb{Q}[X, Y, Z]$ is a (homogenous) polynomial of degree $10n - 6$, where $\deg(X) := 4$, $\deg(Y) := 6$, $\deg(Z) := 2$. The degree of the (homogenous) rational function $b_n(X, Y, Z)$ is $2(n - 1)$.

Here is a list of the first few polynomials $a_n(X, Y, Z)$ for $n > 0$:

$$\begin{aligned}
a_1 &= Z^2 \\
a_2 &= XZ^5 + X^2Z^3 - 2YZ^4 \\
a_3 &= X^2Z^8 - 2X^3Z^6 - 4XYZ^7 + X^4Z^4 - 4X^2YZ^5 + 8Y^2Z^6 \\
a_4 &= X^3Z^{11} - 5X^4Z^9 - 6X^2YZ^{10} - 9X^5Z^7 + 20X^3YZ^8 \\
&\quad + 20XY^2Z^9 + X^6Z^5 - 6X^4YZ^6 + 24X^2Y^2Z^7 - 40Y^3Z^8 \\
a_5 &= X^4Z^{14} - 8X^5Z^{12} - 8X^3YZ^{13} + \frac{66}{5}X^6Z^{10} \\
&\quad + 56X^4YZ^{11} + 36X^2Y^2Z^{12} - 20X^7Z^8 + 104X^5YZ^9 \\
&\quad - \frac{836}{5}X^3Y^2Z^{10} - 112XY^3Z^{11} + X^8Z^6 - 8X^6YZ^7 \\
&\quad + 48X^4Y^2Z^8 - 160X^2Y^3Z^9 + 224Y^4Z^{10}
\end{aligned}$$

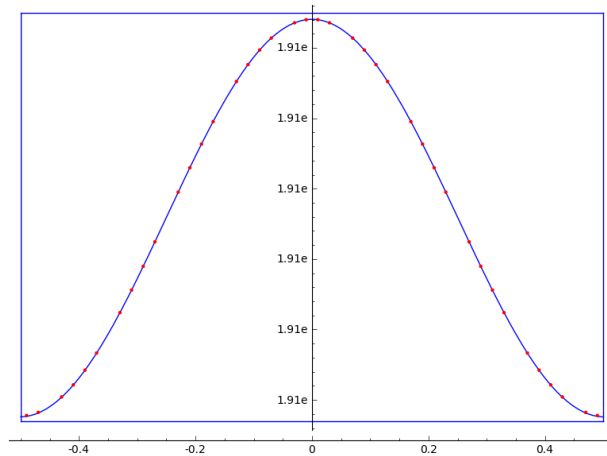
The numerical evaluations of those polynomials ($Z = E_2(z_0)$, etc.) give for Taylor coefficients b_n :

$$\begin{aligned}
b_0 &= 1.9101404964982709820376545357984830913777487030i \\
b_1 &= -0.9982361219015924374815710878280361648431190825 \\
b_2 &= 0.0055236842011260453610739166397990326586651337i \\
b_3 &= -0.01149489150274208316313259093815041563703067326 \\
b_4 &= 0.0178252611095229253133162329291348589135823077i \\
b_5 &= 0.0218243134639575211728774441381952550676521991 \\
b_6 &= 0.0216634844629385759461618124642355350591382590i \\
b_7 &= -0.0173461622715009362175946164162223267332088060 \\
b_8 &= 0.0104172812309514952250501361120571266678673695i \\
b_9 &= 0.0029825116383882005761911965146832408919333302
\end{aligned}$$

For the zeros of E_2 near $1/c$ found in [EBS10] by El Basraoui and Sebbar, the Taylor polynomials give a fast converging asymptotic series in c . In general the Taylor polynomial is inferior to the Fourier approximation and starts to give a reasonably good fit only after $n = 8$.

Below is a picture of the Taylor approximation for $n = 8$ together with plots of $F(x)$ for $x \in \mathbb{Q} \cap [-\frac{1}{2}, \frac{1}{2}]$, $x = \frac{d}{100}$, $(d, 100) = 1$.

The plot also shows the bounds from (6).



REFERENCES

- [AKN97] Tetsuya Asai, Masanobu Kaneko, and Hirohito Ninomiya. Zeros of certain modular functions and an application. *Comment. Math. Univ. St. Paul.*, 46(1):93–101, 1997.
- [DJ08] W. Duke and Paul Jenkins. On the zeros and coefficients of certain weakly holomorphic modular forms. *Pure Appl. Math. Q.*, 4(4, Special Issue: In honor of Jean-Pierre Serre. Part 1):1327–1340, 2008.
- [EBS10] Abdelkrim El Basraoui and Abdellah Sebbar. Zeros of the Eisenstein series E_2 . *Proc. Amer. Math. Soc.*, 138(7):2289–2299, 2010.
- [GS12] Amit Ghosh and Peter Sarnak. Real zeros of holomorphic Hecke cusp forms. *J. Eur. Math. Soc. (JEMS)*, 14(2):465–487, 2012.
- [HS10] Roman Holowinsky and Kannan Soundararajan. Mass equidistribution for Hecke eigenforms. *Ann. of Math. (2)*, 172(2):1517–1528, 2010.
- [Ran82] R. A. Rankin. The zeros of certain Poincaré series. *Compositio Math.*, 46(3):255–272, 1982.
- [RSD70] F. K. C. Rankin and H. P. F. Swinnerton-Dyer. On the zeros of Eisenstein series. *Bull. London Math. Soc.*, 2:169–170, 1970.
- [SF07] Ahmed Sebbar and Thérèse Falliero. Equilibrium point of Green’s function for the annulus and Eisenstein series. *Proc. Amer. Math. Soc.*, 135(2):313–328 (electronic), 2007.
- [WY] R. Wood and M. Young. Zeroes of the weight two Eisenstein series. Preprint, <http://arxiv.org/abs/1312.1930>.

DEPARTMENT OF MATHEMATICS, ETH ZURICH

E-mail address: `ozlem@math.ethz.ch`

DEPARTMENT OF MATHEMATICS, ETH ZURICH

E-mail address: `jjermann@math.ethz.ch`

EOTVOS LORAND UNIVERSITY, BUDAPEST

E-mail address: `toth@cs.elte.hu`