AN ANALOGUE OF THE MELLIN TRANSFORM FOR WEAKLY HOLOMORPHIC CUSP FORMS AND A CONVERSE THEOREM

Ö. IMAMOḠLU, Y. MARTIN, AND Á . TÓTH

Abstract. We define an analogue of the classical Mellin transform for vector-valued weakly holomorphic cusp forms for $SL(2,\mathbb{Z})$ and prove a converse theorem for such forms in terms of the new transform. As applications we get converse theorems for scalar valued weakly holomorphic forms for $SL(2,\mathbb{Z})$ as well as $\Gamma_0(p)$ for primes p.

1. Introduction

The classical Mellin transform of a cusp form f for the group $SL_2(\mathbb{Z})$ is the function of s given by

$$\Lambda(f,s) = (-i)\int_0^\infty f(\tau)\left(\frac{\tau}{i}\right)^{s-1} d\tau. \tag{1.1}$$

This integral is well-defined for all $s \in \mathbb{C}$ with $\mathrm{Re}\,(s) \gg 0$ since f has exponential decay at infinity.

In this note we propose a different approach to a meaningful integral transform of cusp forms which is also well defined for all weakly holomorphic cusp forms. Namely, to any weakly holomorphic cusp form f for $\Gamma_0(N)$ of even integral weight we associate

$$D(f,s) = (-i)\int_\rho^{\rho^2} f(\tau)\left(\frac{\tau}{i}\right)^{s-1} d\tau, \tag{1.2}$$

where $\rho = e^{\pi i/3}$. Here in the expression w^s we use the branch of the logarithm for any $w \in \mathbb{C} \setminus (-i\infty, 0]$ by taking the argument of w to lie in $(-\pi/2, 3\pi/2)$.

Even though the integral (1.2) is already present in the literature, most notably in papers about periods of automorphic forms [9, 1], it plays only an auxiliary role in all of them. The goal of this paper is to bring the transform $D(f,s)$ into the forefront.

The Mellin transform (1.1) has a Dirichlet series representation. More precisely if $f(\tau) = \sum_{n=1}^\infty c(n)\exp(2\pi i n\tau)$ is any cusp form of integral weight k for $SL_2(\mathbb{Z})$, one

ETH, Mathematics Dept., CH-8092, Zürich, Switzerland
University of Chile, Santiago, Chile
Eotvos Lorand University, Analysis Department, and Alfréd Rényi Institute of Mathematics, Budapest, Hungary
E-mail addresses: ozlem@math.ethz.ch, ymartin@uchile.cl, toth@cs.elte.hu.
Imamoglu's research is supported by SNF grant 200021-185014.
Á. Tóth is supported by NKFIH (National Research, Development and Innovation Office) grants K 135885 and ELTE TKP 2021-NKTA-62.

has

$$\Lambda(f,s) = (2\pi)^{-s} \sum_{n=1}^{\infty} \frac{c(n)}{n^s} \Gamma(s),$$

where $\Gamma(s)$ is Euler's gamma function. This series converges absolutely and uniformly on any compact subset of the complex half-plane $\mathrm{Re}\,(s) > (k+1)/2$, as $c(n) = O(n^{(k-1)/2})$.

As we shall see in the next section (Lemma 2.3), the integral (1.2) also has a series representation analogous to the above series. Indeed, if N is a positive integer and f is any weakly holomorphic cusp form of weight k for $\Gamma_0(N)$ whose Fourier series at infinity is

$$f(\tau) = \sum_{n=N_0}^{\infty} c(n) \exp(2\pi i n \tau) \quad \text{with} \quad c(0) = 0, \tag{1.3}$$

then

$$D(f,s) = \left(\frac{2\pi}{N}\right)^{-s} \sum_{n=N_0}^{\infty} \frac{c(n)}{n^s} \left(\Gamma\left(s, -\frac{2\pi i n \rho}{N}\right) - \Gamma\left(s, \frac{2\pi i n \overline{\rho}}{N}\right) \right), \tag{1.4}$$

where each term involves a difference of incomplete gamma functions.

One important feature of the integral transform (1.1) is Hecke's converse theorem [7], a fundamental analytic characterization of $\Lambda(f,s)$ for a cusp form f for $SL_2(\mathbb{Z})$, among all Dirichlet series. The main result of this note is to prove an analogous statement for the integral transform $D(f,s)$ in two cases; i) whenever f is a vector-valued, weakly holomorphic cusp form for $SL_2(\mathbb{Z})$, and ii) for any scalar, weakly holomorphic cusp form f for $SL(2(\mathbb{Z})$ as well as $\Gamma_0(p)$, with p a prime number.

1.1. **Results.** In order to state our main results we fix the notation to be used throughout the article and introduce the transform $D(F,s)$ for vector-valued holomorpic functions F on the complex upper half plane $\mathbb{H}$.

If A is any square matrix, ${}^tA$ denotes its transpose. If m is a positive integer, I_m is the identity matrix of size m. We put $e(w) = \exp(2\pi i w)$ for all $w \in \mathbb{C}$.

If N is a positive integer, we put $\zeta_N = e(1/N)$ and $\Gamma_0(N)$ for the Hecke congruence subgroup of level N. As usual, we write $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The symbol $S_k^!(N)$ denotes the $\mathbb{C}$-vector space of weight k, weakly holomorphic cusp forms for $\Gamma_0(N)$, and for any unitary representation $\sigma : SL_2(\mathbb{Z}) \to GL_m(\mathbb{C})$, the symbol $S_{k,\sigma}^!$ refers to the $\mathbb{C}$-vector space of weakly holomorphic cusp forms of weight k and representation σ for $SL_2(\mathbb{Z})$. (For the precise definition of these vector-valued cusp forms see Definition 2.1.)

The main object of our study is the following integral transform.

Definition 1.1. Let m be a positive integer. For any vector-valued, holomorphic function $F : \mathbb{H} \to \mathbb{C}^m$ we define

$$D(F,s) = (-i) \int_{\rho}^{\rho^2} F(\tau) \left(\frac{\tau}{i}\right)^{s-1} d\tau,$$

where $\rho = e^{\pi i/3}$ and $s \in \mathbb{C}$. Here and from now on the branch of the logarithm used in the expression w^s for any $w \in \mathbb{C} \setminus (-i\infty, 0]$ is given by taking the argument of w to lie in $(-\pi/2, 3\pi/2)$.

The map $D(F,s)$ is a vector-valued function of s, which for

$$ {}^tF(\tau) = (f_1(\tau), f_2(\tau), \ldots, f_m(\tau)), \tag{1.5} $$

looks like

$$ {}^tD(F,s) = (-i)\left(\int_\rho^{\rho^2} f_1(\tau)\left(\frac{\tau}{i}\right)^{s-1} d\tau, \ldots, \int_\rho^{\rho^2} f_m(\tau)\left(\frac{\tau}{i}\right)^{s-1} d\tau\right). \tag{1.6} $$

Since every integrand above is a continuous map over $\mathbb{H}$ and the domain of integration is a compact set, one has $D(F,s) \in \mathbb{C}^m$ for all such F and s.

Our first theorem is a converse theorem for vector-valued, weakly holomorphic cusp forms for $SL_2(\mathbb{Z})$ in the case of the integral transform $D(F,s)$.

Theorem 1.2. *Let N, N_0, m, k be integers with $N \geq 1$, $m \geq 1$.*

Let $\sigma : SL_2(\mathbb{Z}) \to GL_m(\mathbb{C})$ be an m-dimensional unitary complex representation such that $\sigma(-I_2) = (-1)^k I_m$ and $\sigma(T) = diag(\zeta_N^{e_1}, \ldots, \zeta_N^{e_m})$.

For each $1 \leq j \leq m$ let $\{c_j(n)\}_{n=N_0}^\infty$ be a sequence of complex numbers such that $c_j(0) = 0$, $c_j(n) = O(e^{C\sqrt{n}})$ for some $C > 0$, and $c_j(n) = 0$ if $n \not\equiv e_j \bmod N$.

Let ${}^tF(\tau) = (f_1(\tau), f_2(\tau), \ldots, f_m(\tau))$, and $D(F,s)$ as in (1.6) with

$$ f_j(\tau) = \sum_{n=N_0}^\infty c_j(n) e(n\tau/N), \ 1 \leq j \leq m. $$

Then

(a) The Fourier series f_j defines a holomorphic function on $\mathbb{H}$, and hence F is a vector-valued holomorphic function from $\mathbb{H}$ to $\mathbb{C}^m$. Moreover, the vector-valued function $D(F,s)$ is an entire function of s.

(b) $F|_k[T] = \sigma(T) F$.

(c) The function F is a cusp form in $S^!_{k,\sigma}$ if, and only if, $D(F,s)$ satisfies the matrix functional equation

$$ D(F, k-s) = i^k \sigma(S) D(F,s). $$

For simplicity we have assumed in this theorem that $\sigma(T)$ is a diagonal matrix. After a change of basis this condition will always hold if $[SL_2(\mathbb{Z}) : \mathrm{Ker}(\sigma)] < \infty$.

It is immediate that Theorem 1.2 with $N = m = 1$, k even and $\sigma =$ the trivial representation, gives the following converse theorem for $D(F,s)$ in the case of weakly holomorphic cusp forms for $SL_2(\mathbb{Z})$.

Theorem 1.3. *Let $\{c(n)\}_{n=N_0}^\infty$ be a sequence of complex numbers such that $c(0) = 0$, and $c(n) = O(e^{C\sqrt{n}})$ for some $C > 0$. Let*

$$ f(\tau) = \sum_{n=N_0}^\infty c(n) e(n\tau) $$

and

$$D(f,s)=(-i)\int_{\rho}^{\rho^2} f(\tau)\left(\frac{\tau}{i}\right)^{s-1} d\tau.$$

Then

(a) *The Fourier series f defines a holomorphic function on $\mathbb{H}$, and the function $D(f,s)$ is an entire function of s.*

(b) *The function f is a cusp form in $S_k^!$ if, and only if $D(f,s)$ satisfies the functional equation*

$$D(f,s)=i^{-k}D(f,k-s).$$

In the case of a weakly holomorphic form $f\in S_k^!(p)$ of prime level p, one can write a vector of Fourier series $(f,g_0,\dots g_{p-1})$ with $g_j:=f|[ST^j]$. For the associated series $D(f,s)$ and $D(g_j,s)$ we then have

Theorem 1.4. *Let p and k be integers with p prime and k even. Assume*

$$f(\tau)=\sum_{n=N_0}^{\infty}c(n)e(n\tau)\in S_k^!(p) \text{ and } f|_k[S]=\sum_{n=N_0}^{\infty}d(n)e\left(n\tau/p\right)$$

for some $N_0\in\mathbb{Z}$ and complex numbers $c(n),d(n)$.

Put $g_0:=f|_k[S]$ and for $1\le j\le p-1$ let $g_j(\tau):=g_0|_k[T^j]=\sum_{n=N_0}^{\infty}\zeta_p^{jn}d(n)e\left(n\tau/p\right)$. Then the functional equations

$$D(f,s)=i^k\,D(g_0,k-s) \text{ and } D(g_j,s)=i^k\,D(g_{j^*},k-s)$$

hold whenever $jj^*\equiv -1 \bmod p$.

Our next result, which is also a consequence of Theorem 1.2, is a converse theorem for weakly holomorphic cusp forms of prime level. Composite levels can also be treated but is more complicated due to the larger number of cusps.

Theorem 1.5. *Let p, N_0 and k be integers with p prime and k even.*

Let $\{c(n)\}_{n=N_0}^{\infty}$ and $\{d(n)\}_{n=N_0}^{\infty}$ be two sequences of complex numbers such that $c(0)=d(0)=0$ and $c(n)=O\left(e^{C\sqrt{n}}\right)$, $d(n)=O\left(e^{C\sqrt{n}}\right)$ for some $C>0$.

Then

(a) *The Fourier series*

$$f(\tau)=\sum_{n=N_0}^{\infty}c(n)e(n\tau) \text{ and } g_j(\tau)=\sum_{n=N_0}^{\infty}\zeta_p^{jn}d(n)e\left(n\tau/p\right) \tag{1.7}$$

for $0\le j\le p-1$, define holomorphic functions of τ on $\mathbb{H}$. Moreover the integral transforms $D(f,s),D(g_j,s)$ given in (1.2) are entire functions of s.

(b) *If the functional equations*

$$D(f,s)=i^k\,D(g_0,k-s) \text{ and } D(g_j,s)=i^k\,D(g_{j^*},k-s)$$

hold whenever $jj^*\equiv -1 \bmod p$, for $j=1,\dots,p-1$, then the series f is a cusp form in $S_k^!(p)$.

1.2. **Comparison to other approaches.** It is important to note that a generalization of the Mellin transform (1.1) to weakly holomorphic cusp forms is not at all obvious as f may have a pole at infinity. We circumvent this problem by integrating on a cycle which avoids the cusp.

On the other hand in [2] Bringmann, Fricke and Kent introduced certain regularization of (1.1) via analytic continuation which allowed them to pick f from the space of weakly holomorphic cusp forms for $SL_2(\mathbb{Z})$. The regularization in [2] gives rise to the series

$$(2\pi)^{-s}\sum_{n=N_0}^{\infty}\frac{c(n)}{n^s}\Gamma(s,2\pi int_0)+i^k(2\pi)^{-(k-s)}\sum_{n=N_0}^{\infty}\frac{c(n)}{n^{k-s}}\Gamma\left(k-s,\frac{2\pi in}{t_0}\right) \tag{1.8}$$

for any $t_0>0$ (see Theorem 2.2 in [2]).

Note that once the modularity is used then $\Lambda(f,s)$, for a holomorphic cusp form $f\in S_k$, can be written in terms of incomplete gamma functions. This representation makes its functional equation also apparent. More precisely

$$\Lambda(f,s)=(2\pi)^{-s}\sum_{n=1}^{\infty}\frac{c(n)}{n^s}\Gamma(s,2\pi n)+(2\pi)^{-(k-s)}\sum_{n=1}^{\infty}\frac{c(n)}{n^{k-s}}\Gamma(k-s,2\pi n).$$

The series (1.8) looks similar to the series representation (1.4) for $D(f,s)$ but they are not the same. As it is also pointed out in [4], the built in functional equation in (1.8) prevents a meaningful formulation of a converse theorem for such a series. This obstacle is circumvented in [4] by the introduction of a regularization using general test functions and their Laplace transforms, in a way that a converse theorem is established in Theorem 5.1 of [4].

Equation (1.4) allow us to make a comparison between the integral transform $D(f,s)$ and the collection of L-series introduced by Diamantis et al [4] for their converse theorem. In the introduction of [4] the authors state a converse theorem for integer weight, weakly holomorphic modular forms of level 1 using what they call L-series for general harmonic Maass forms. Namely, they associate to any sequence of complex numbers $\{c(n)\}_{n=N_0}^{\infty}$ as in Theorem 1.3 the Fourier series $f(\tau)=\sum_{n=N_0}^{\infty}c(n)e(n\tau)$ and the functions

$$L_f(\varphi):=\sum_{n=N_0}^{\infty}c(n)\mathcal{L}_{\varphi}(2\pi n),$$

where $\mathcal{L}_{\varphi}(s)$ denotes the Laplace transform of φ, and φ runs over all compactly supported, smooth functions $\varphi:\mathbb{R}_{>0}\to\mathbb{C}$.

Using (1.4) we can see that $D(f,s)$ resembles somehow these series, as it can be written as the difference

$$D\,(f,s)=\sum_{n=N_0}^{\infty}c(n)\mathcal{L}_{\varphi_s^+}\left(2\pi n\left(\frac{\rho}{i}\right)\right)-\sum_{n=N_0}^{\infty}c(n)\mathcal{L}_{\varphi_s^-}\left(2\pi n\left(\frac{\rho}{i}\right)^{-1}\right), \tag{1.9}$$

where

$$\varphi_s^{\pm}(t)=\left(\frac{\rho}{i}\right)^{\pm s}\phi_s(t) \text{ and } \phi_s:\mathbb{R}_{>0}\to\mathbb{C} \text{ is the map } \phi_s(t)=\begin{cases}0 & \text{if } t<1\\ t^{s-1} & \text{if } t\geq 1\end{cases}$$

for every $s \in \mathbb{C}$. Here the functions $\varphi_s^{\pm}(t)$ are not of compact support and the Laplace transforms are evaluated not at the real numbers $2\pi n$ but at the points $2\pi n \left(\frac{\rho}{i}\right)^{\pm 1}$.

We note that the converse theorem from [4] requires the verification of infinitely many functional identities, one for each $L_f(\varphi)$ where φ runs over all compactly supported smooth test functions and has a generalization to a certain class of harmonic Maass forms for $\Gamma_0(N)$, where N is any positive integer. In contrast, our Theorem 1.5 asks for the verification of only finitely many functional equations in s but is valid only for $\Gamma_0(p)$ with p prime. On the other hand it might be worth to observe that once $D(f,s)$ is written as in (1.9) in terms of the Laplace transforms of test functions, the finitely many functional equations for $D(f,s)$ in s can be viewed as infinitely many functional identities, one for each of the infinitely many (non-compactly supported) test functions $\{\varphi_s\}_s$ given above. The collection of these identities is simpler in the sense that they form a 1-dimensional complex analytic family among distributions.

All the series in the works mentioned above specialize to the classical L-function for cusp forms, but their integral representation need to be regularized for non cuspidal forms. In contrast to this situation, our integral transform $D(f,s)$ does not require regularization for weakly holomorphic forms but it does not specialize to the classical L-function in the case of holomorphic cusp forms. Nevertheless, $D(f,s)$ carries all the information that the classical L-function does and this is our motivation for studying this simpler transform (1.2).

In a subsequent article we will discuss the special values of $D(f,s)$ and their relations to the special values of $\Lambda(f,s)$.

1.3. **Organization of the paper.** In the following section we recall basic definitions and establish a series representation for the components of $D(F,s)$. In Section 3 we prove Theorem 1.2, and in Section 4 we prove Theorems 1.4 and 1.5.

2. Basic definitions

Let k, $N \geq 1$ and $m \geq 1$ be integers. Let $\sigma : SL_2(\mathbb{Z}) \to GL_m(\mathbb{C})$ be an m-dimensional unitary representation such that $\sigma(-I_2) = (-1)^k I_m$ and $\sigma(T)$ has finite order in $GL_m(\mathbb{C})$ dividing N. As in [8] we introduce the following

Definition 2.1. A vector-valued function $F : \mathbb{H} \to \mathbb{C}^m$, say

$$ {}^tF(\tau) = (f_1(\tau), f_2(\tau), \ldots, f_m(\tau)), $$

is called a weakly holomorphic modular form of weight k and representation σ for the group $SL_2(\mathbb{Z})$ if:

(i) The map $f_j : \mathbb{H} \to \mathbb{C}$ is holomorphic on $\mathbb{H}$ for every $1 \leq j \leq m$.
(ii) The functional equation

$$ F|_k \left[M\right](\tau) := (c\tau + d)^{-k} F\left(\frac{a\tau + b}{c\tau + d}\right) = \sigma(M)F(\tau) $$

holds for every $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$.

(iii) There is an integer N_0, depending on F, so that every f_j has a Fourier series representation of type

$$f_j(\tau) = \sum_{n=N_0}^{\infty} c_j(n) e\left(\frac{n}{N}\tau\right). \tag{2.1}$$

Any such F is said to be cuspidal if $c_j(0) = 0$ for all $1 \leq j \leq m$.

We denote by $M^!_{k,\sigma}$ (resp. $S^!_{k,\sigma}$) the $\mathbb{C}$-vector space of vector-valued weakly holomorphic modular forms (resp. cusp forms) of weight k and representation σ for $SL_2(\mathbb{Z})$.

In the scalar case it is well known that the Fourier coefficients $c_1(n)$ of $F(\tau) = f_1(\tau)$ are bounded by $e^{C\sqrt{n}}$. See for example [3]. It is easy to adapt the argument for the vector-valued case.

Lemma 2.2. *Let F be a vector-valued weakly holomorphic modular form as above. Then there exist $A, B \in \mathbb{R}$ such that for each j and all $n > 0$ we have*

$$|c_j(n)| \leq Ae^{B\sqrt{n}}.$$

Proof. Since $\sigma(S)^4 = I_m$, $\sigma(S)$ is diagonalizable. Hence there are holomorphic functions $g_j : \mathbb{H} \to \mathbb{C}$ with $1 \leq j \leq m$ such that every f_j is in the $\mathbb{C}$-linear span of $\{g_1, g_2, \ldots, g_m\}$, every g_j is in the $\mathbb{C}$-linear span of $\{f_1, f_2, \ldots, f_m\}$ and

$$g_j|_k\,[S] = \epsilon_j g_j \text{ for some } \epsilon_j \in \{\pm 1, \pm i\}. \tag{2.2}$$

Equation (2.1) and the fact that g_j is in the $\mathbb{C}$-linear span of $\{f_1, f_2, \ldots, f_m\}$ give $g_j(\tau) = O\left(e^{C\,\mathrm{Im}\,(\tau)}\right)$ for some real number C, and (2.2) yields

$$|g_j(\tau)| = O\left(e^{C\,\mathrm{Im}\,(-1/\tau)}\right) = O\left(e^{C/\,\mathrm{Im}\,(\tau)}\right).$$

If we use this estimate in the integral representation of the n-th Fourier coefficient $d_j(n)$ of g_j, and compute such integral with $\tau = x + i/\sqrt{n}$, $x \in \mathbb{R}$, we end up with the bound $|d_j(n)| = O\left(e^{C'\sqrt{n}}\right)$. The lemma follows from this fact and the linear dependence of each f_j on $\{g_1, g_2, \ldots, g_m\}$. $\square$

For a fixed s let $\Gamma(s, z)$ be the incomplete gamma function on $\mathbb{C}$ with the non-positive imaginary axis removed so that we have

$$\frac{d}{dz}\Gamma\left(s, z\right) = e^{-z} z^{s-1}. \tag{2.3}$$

Note that $\mathbb{C} \setminus (-i\infty, 0]$ is simply connected, so such primitives exist, and our choice is normalized by requiring that

$$\Gamma(s, w) = \int_w^{\infty} e^{-z} z^{s-1} dz \tag{2.4}$$

for $w \in (0, \infty)$. Note also that for any parameterized curve η in $\mathbb{C} \setminus (-i\infty, 0]$ from a to b one has

$$\int_\eta e^{-z} z^{s-1} dz = \Gamma(s, b) - \Gamma(s, a).$$

Lemma 2.3. *Let N, m and F be as in Definition 1.1. If F is the vector-valued function (1.5) and the j-th component f_j of F has a Fourier series representation (2.1) with coefficients $\{c_j(n)\}_{n=N_0}^{\infty}$ such that $c_j(n) = O(e^{C\sqrt{n}})$ for some real number C, then the j-th entry of the column vector $D(F,s)$ is*

(2.5)
$$(-i)\int_{\rho}^{\rho^2} f_j(\tau)\left(\frac{\tau}{i}\right)^{s-1} d\tau = \left(\frac{2\pi}{N}\right)^{-s} \sum_{n=N_0}^{\infty} \frac{c_j(n)}{n^s}\left(\Gamma\left(s, -\frac{2\pi in\rho}{N}\right) - \Gamma\left(s, \frac{2\pi in\overline{\rho}}{N}\right)\right).$$

Proof. The growth of the sequence $\{c_j(n)\}_{n=N_0}^{\infty}$ implies that $f_j(\tau)$ is a holomorphic function on the complex upper half-plane, hence every integral in the vector $D(F,s)$ is independent of the path of integration in $\mathbb{H}$. Moreover,

$$(-i)\int_{\rho}^{\rho^2} f_j(\tau)\left(\frac{\tau}{i}\right)^{s-1} d\tau = (-i)\sum_{n=N_0}^{\infty} c_j(n)\int_{\rho}^{\rho^2} e^{\frac{2\pi in\tau}{N}}\left(\frac{\tau}{i}\right)^{s-1} d\tau.$$

For the computation of the inner integral in this expression, note that by (2.3)

$$\frac{d}{d\tau}\Gamma\left(s, -\frac{2\pi in\tau}{N}\right) = e^{\frac{2\pi in\tau}{N}}\left(-\frac{2\pi in\tau}{N}\right)^{s-1}\left(\frac{2\pi in}{N}\right).$$

Now we choose the portion of the unit circle in $\mathbb{H}$ limited by ρ and $-\overline{\rho} = \rho^2$ as a path of integration, and observe the following; as τ runs over such a curve, $-i\tau$ runs over the portion of the unit circle which goes from $-i\rho$ to $i\,\overline{\rho}$ at the right hand side of the imaginary axis. Thus

$$\left(-\frac{2\pi in\tau}{N}\right)^{s-1} = \left(\frac{2\pi n}{N}\right)^{s-1}(-i\tau)^{s-1}$$

for all $n \in \mathbb{Z}$, due to our choice of branch for the logarithm. Therefore

$$\frac{d}{d\tau}\Gamma\left(s, -\frac{2\pi in\tau}{N}\right) = \left(\frac{2\pi n}{N}\right)^{s} ie^{\frac{2\pi in\tau}{N}}(-i\tau)^{s-1}$$

for all τ on the path of integration mentioned above, and so

$$(-i)\int_{\rho}^{\rho^2} f_j(\tau)\left(\frac{\tau}{i}\right)^{s-1} d\tau = -\sum_{n=N_0}^{\infty} c_j(n)\frac{1}{(2\pi n/N)^s}\Gamma\left(s, -\frac{2\pi in\tau}{N}\right)\Big|_{\rho}^{\rho^2}.$$

This identity yields the series representation (2.5). □

3. Proof of Theorem 1.2

Part (a) of the theorem follows from the fact that the individual terms in both series, (2.1) and (2.5), are holomorphic functions with exponential decay (so, the convergence of each series is absolute and uniform on compact subsets of the corresponding domain). This is evident for (2.1), and a consequence of the following for (2.5); the map $s \mapsto \Gamma(s,w)$ is entire for any $w \in \mathbb{C}$, and $\Gamma(s,w) = w^{s-1}e^{-w}(1 + R(s,w))$ where the function $R(s,w)$ satisfies $R(s,w) = O(w^{-1})$ as $w \to \infty$ and $|\arg w| < \pi/2$, for any fixed s (see [10] p. 341).

This condition and the hypothesis satisfied by the sequences $\{c_j(n)\}_{n=N_0}^{\infty}$ yield the desired exponential decay.

For the proof of (b) we notice that

$$ {}^tF|_k\,[T]\,(\tau) = {}^tF(\tau+1) = \left(\ldots, \sum_{n=N_0}^{\infty} \zeta_N^n\, c_j(n) e\left(\frac{n}{N}\tau\right), \ldots\right). $$

By hypothesis we have $\zeta_N^n = \zeta_N^{e_j}$ whenever $c_j(n) \neq 0$, hence

$$ {}^tF|_k\,[T]\,(\tau) = \left(\ldots, \zeta_N^{e_j} f_j(\tau), \ldots\right) = {}^tF(\tau)\,\sigma\,(T)\,. $$

For the proof of the equivalence in part (c) we assume first that F satisfies

$$ \tau^{-k} F\left(\frac{-1}{\tau}\right) = \sigma\,(S)\,F(\tau) \tag{3.1} $$

for all $\tau \in \mathbb{H}$. Then we have

$$ \begin{aligned} D(F,s) &= (-i)\int_{\rho}^{\rho^2} F\,(\tau)\left(\frac{\tau}{i}\right)^{s-1} d\tau \\ &= (-i)\int_{\rho}^{\rho^2} \sigma\,(S)^{-1}\,\tau^{-k} F\left(\frac{-1}{\tau}\right)\left(\frac{\tau}{i}\right)^{s-1} d\tau \\ &= -(-i)\sigma(S)^{-1}\int_{\rho}^{\rho^2}\left(\frac{-1}{\tau}\right)^{-k} F(\tau)\left(\frac{-1}{i\tau}\right)^{s-1} d\left(\frac{-1}{\tau}\right) \\ &= (-i)i^{-k}\sigma\,(S)^{-1}\int_{\rho}^{\rho^2} F\,(\tau)\left(\frac{\tau}{i}\right)^{k-s-1} d\tau \\ &= i^{-k}\,\sigma\,(S)^{-1}\,D(F,k-s). \end{aligned} $$

Here we have used that the integral in Definition 1.1 is valid for $s \in \mathbb{C}$.

For the implication in the opposite direction we first note that the components of the vector of Fourier series F are periodic holomorphic functions from $\mathbb{H}$ to $\mathbb{C}$ by part (a), and that F satisfies the functional equation in (b). Hence, it suffices to prove that F also satisfies (3.1) for all τ on some curve in $\mathbb{H}$. To this end we define $\widetilde{F}(\theta) = iF(ie^{-i\theta})$, and get

$$ D(F,s) = \int_{-\pi/6}^{\pi/6} \widetilde{F}(\theta) e^{-i\theta s} d\theta $$

from Definition 1.1 and the substitution $\tau = ie^{-i\theta}$. Clearly, the last integral is the Fourier transform of $\widetilde{F}\chi_{[-\pi/6,\pi/6]}$, where $\chi_{[-\pi/6,\pi/6]}$ is the characteristic function of the interval $[-\pi/6, \pi/6]$.

As F is holomorphic on $\mathbb{H}$, the function $\theta \mapsto \widetilde{F}\chi_{[-\pi/6,\pi/6]}(\theta)$ is integrable and piecewise smooth on $\mathbb{R}$. Then, a standard result in Fourier Analysis (see for example Theorem 7.6 in [6] p. 220) yields

$$ \widetilde{F}(\theta) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} D(F,s) e^{i\theta s} ds \quad \text{for all } \frac{-\pi}{6} < \theta < \frac{\pi}{6}. \tag{3.2} $$

Given (3.2), the hypothesis $D(F,s) = i^{-k}\sigma(S)^{-1}D(F,k-s)$ leads to

$$\begin{aligned}
\widetilde{F}(\theta) =& i^{-k}e^{i\theta k}\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\sigma(S)^{-1}D(F,k-s)e^{-i\theta(k-s)}ds \\
&= \left(ie^{-i\theta}\right)^{-k}\lim_{T\to\infty}\frac{1}{2\pi}\int_{k-T}^{k+T}\sigma(S)^{-1}D(F,s)e^{-i\theta s}ds \\
&= \left(ie^{-i\theta}\right)^{-k}\lim_{T\to\infty}\frac{1}{2\pi}\int_{k-T}^{T-k}\sigma(S)^{-1}D(F,s)e^{-i\theta s}ds \\
&= \left(ie^{-i\theta}\right)^{-k}\sigma(S)^{-1}\widetilde{F}(-\theta),
\end{aligned}$$

for all $-\pi/6 < \theta < \pi/6$. In the last steps we have used equation (3.2), the decomposition

$$\int_{k-T}^{k+T} = \int_{k-T}^{T-k} + \int_{T-k}^{T+k}, \quad \text{and} \quad \lim_{T\to\infty}\int_{T-k}^{T+k} D(F,s)e^{-i\theta s}ds = 0.$$

In turn, this last limit follows from the Riemann-Lebesgue Lemma (see [6] p. 117) since $D(F,s)$ is the Fourier transform of a function in $L^1(\mathbb{R})$, and so $D(F,s)\to 0$ as $s\to\pm\infty$.

Finally we observe that $\widetilde{F}(\theta) = \left(ie^{-i\theta}\right)^{-k}\sigma(S)^{-1}\widetilde{F}(-\theta)$ for all $-\pi/6 < \theta < \pi/6$ is equivalent to $F(\tau) = \tau^{-k}\sigma(S)^{-1}F(-1/\tau)$ for all τ in the portion of the unit circle in $\mathbb{H}$ limited by ρ and ρ^2. This identity finishes the proof of part (c). □

For later purposes we note that the proof of part (c) in Theorem 1.2 with $m = 1$ can be easily adapted to yield the following

Proposition 3.1. *Let p, N_0 and k be integers with p prime and k even.*

Let $\{c(n)\}_{n=N_0}^{\infty}$ and $\{d(n)\}_{n=N_0}^{\infty}$ be two sequences of complex numbers such that $c(0) = d(0) = 0$ and $c(n) = O\left(e^{C\sqrt{n}}\right)$, $d(n) = O\left(e^{C\sqrt{n}}\right)$ for some $C > 0$.

Then the Fourier series

$$f(\tau) = \sum_{n=N_0}^{\infty} c(n)e(n\tau) \text{ and } g(\tau) = \sum_{n=N_0}^{\infty} d(n)e\left(n\tau/p\right)$$

define holomorphic functions of τ on $\mathbb{H}$. Moreover the integral transforms $D(f,s)$ and $D(g,s)$ given in (1.2) are entire functions of s. Furthermore,

$$f|_k[S](\tau) = g(\tau) \text{ if, and only if } D(f,s) = i^k\, D(g,k-s).$$

4. Proofs of Theorem 1.4 and Theorem 1.5

Let k and $N \geq 1$ be integers. For completeness we first recall the definition of (scalar) weakly holomorphic modular forms over $\Gamma_0(N)$.

Definition 4.1. A function $f : \mathbb{H} \to \mathbb{C}$ is a weakly holomorphic modular form of weight k for the group $\Gamma_0(N)$ if:

(i) The map f is holomorphic on $\mathbb{H}$.

(ii) The functional equation

$$f|_k[M](\tau) = f(\tau)$$

holds for every $M \in \Gamma_0(N)$.

(iii) There is an integer N_0, depending on f, so that every $f|_k[A]$ with $A \in SL_2(\mathbb{Z})$ has a Fourier series representation of type

$$f|_k[A](\tau) = \sum_{n=N_0}^{\infty} c_A(n) e\left(\frac{n}{N}\tau\right).$$

Any such f is said to be cuspidal if $c_A(0) = 0$ for all $A \in SL_2(\mathbb{Z})$.

We denote by $M_k^!(N)$ (resp. $S_k^!(N)$) the $\mathbb{C}$-vector space of all weakly holomorphic modular forms (resp. cusp forms) of weight k for $\Gamma_0(N)$. Notice that $M_k^!(N) = \{0\}$ whenever k is odd.

We start with the following

Lemma 4.2. *Let p, N_0 and k be integers with p prime and k even.*

Let $\{c(n)\}_{n=N_0}^{\infty}$ be a sequence in $\mathbb{C}$ with $c(0) = 0$ and $c(n) = O\left(e^{C\sqrt{n}}\right)$ for some $C > 0$. Set

$$f(\tau) = \sum_{n=N_0}^{\infty} c(n) e(n\tau) \ \text{ and } F = {}^t\left(f, f|_k[S], f|_k[ST], \dots, f|_k[ST^{p-1}]\right).$$

Assume that $f|_k[S](\tau) = \sum_{n=N_0}^{\infty} d(n) e(n\tau/p)$ for some $d(n)$ in $\mathbb{C}$ such that $d(0) = 0$ and $d(n) = O\left(e^{C\sqrt{n}}\right)$.

Then f is in $S_k^!(p)$ if, and only if $F \in S_{k,\phi}^!$, where ϕ is the linear representation associated to the right action of $SL_2(\mathbb{Z})$ on $\Gamma_0(p)\backslash SL_2(\mathbb{Z})$.

Proof. The growth conditions of the sequences $\{c(n)\}$ and $\{d(n)\}$ imply that f, $f|_k[S]$ and all the other components in F define holomorphic functions on $\mathbb{H}$. Moreover, the constant term in the Fourier series representation of each of these components is zero.

Next we note that the representation ϕ arises from the permutations of X_p, the ordered set of right $\Gamma_0(p)$-cosets in $SL_2(\mathbb{Z})$,

$$X_p = \{\Gamma_0(p)\} \cup \{\Gamma_0(p)ST^j \mid j = 0, ..., p-1\}. \tag{4.1}$$

Clearly, if $f \in S_k^!(p)$ then F transforms according to ϕ.

For the converse, assume $F = {}^t(f, f|_k[S], f|_k[ST], \dots, f|_k[ST^{p-1}]) \in S_{k,\phi}^!$. As for any $\gamma \in \Gamma_0(p)$ one has $\Gamma_0(p)\gamma = \Gamma_0(p)$, the first row of the matrix $\phi(\gamma)$ is $(1, 0, \cdots, 0)$.

Hence, for the first component f of F we have $f|_k[\gamma] = f$. □

Proof of Theorem 1.4. By hypothesis $f \in S_k^!(p)$ and $f|_k[S] = g_0$. Hence Proposition 3.1 implies $D(f, s) = i^k D(g_0, k-s)$.

On the other hand, a simple calculation shows that for all $1 \le j \le p-1$,

$$\Gamma_0(p)ST^jS = \Gamma_0(p)ST^{j^*}$$

where $jj^* \equiv -1 \bmod p$. □

Proof of Theorem 1.5. Let f and g_j be the series given in (1.7). By the estimates $c(n) = O\left(e^{C\sqrt{n}}\right)$ and $d(n) = O\left(e^{C\sqrt{n}}\right)$ we know that f and every g_j define holomorphic functions on $\mathbb{H}$. Consequently, the integral transforms $D(f,s)$ and $D(g_j,s)$ are entire functions of s. This proves the first claim.

For the proof of (b) let F be the column vector

$$^{t}F(\tau) = (f(\tau), g_0(\tau), g_1(\tau), \ldots, g_{p-1}(\tau)).$$

From Proposition 3.1 we get that the functional equation $D(f,s) = i^k D(g_0, k-s)$ implies $f|_k[S] = g_0$. Then the definition of g_j in (1.7) gives that $g_j = f|_k[ST^j]$ for every $0 \le j \le p-1$. Thus

$$F = {}^{t}\left(f, f|_k[S], f|_k[ST], \ldots, f|_k[ST^{p-1}]\right).$$

Let $\phi : SL_2(\mathbb{Z}) \to GL_{p+1}(\mathbb{C})$ be the permutation representation induced from the $SL_2(\mathbb{Z})$-action on $\Gamma_0(p)\backslash SL_2(\mathbb{Z})$ by right multiplication as in Lemma 4.2. By that lemma we know that $f \in S_k^!(p)$ if $F \in S_{k,\phi}^!$. Hence, it is enough to prove $F \in S_{k,\phi}^!$.

Since $\phi(T)$ is not diagonal, we cannot use Theorem 1.2 directly to show that $F \in S_{k,\phi}^!$. On the other hand from the definitions of f, g_j in (1.7) we have

$$f(\tau+1) = f(\tau) \text{ and } g_j(\tau+1) = g_{j+1}(\tau)$$

for all $j = 1, 2, \ldots, p$, where the sub-indices are read modulo p. Therefore

$$F(\tau+1) = \phi(T)\, F(\tau)$$

with $\phi(T)^p = I_{p+1}$. Consequently there exists $R \in GL_{p+1}(\mathbb{C}))$ such that $R\phi(T)\, R^{-1}$ is a diagonal matrix. Now we consider the conjugate representation $\sigma := R\phi R^{-1}$ of $SL_2(\mathbb{Z})$, and observe that $F \in S_{k,\phi}^!$ if and only if $RF \in S_{k,\sigma}^!$.

Clearly the vector-valued function $RF : \mathbb{H} \to \mathbb{C}^{p+1}$ is holomorphic and its $p+1$ components are given by Fourier series whose coefficients satisfy the hypothesis of Theorem 1.2 with $N = p$ and $m = p+1$. As $\sigma(T)$ is diagonal, this result yields that $RF \in S_{k,\sigma}^!$ if, and only if

$$D(RF, s) = i^{-k}\sigma(S)^{-1} D(RF, k-s). \tag{4.2}$$

But since $D(RF,s) = RD(F,s)$ and $\sigma = R\phi R^{-1}$, this functional equation is equivalent to

$$D(F, s) = i^{-k}\phi(S)^{-1} D(F, k-s). \tag{4.3}$$

In order to prove the matrix functional equation (4.3), we note that the action of S on the set X_p in (4.1) is as follows; it interchanges $\Gamma_0(p)$ with $\Gamma_0(p)S$, and interchanges $\Gamma_0(p)S^j$ with $\Gamma_0(p)S^{j^*}$ whenever $jj* \equiv -1 \bmod p$. Therefore (4.3) is exactly the matrix form of the set of scalar functional equations that are assumed in Theorem 1.5 part (b). This shows that (4.3) holds, and therefore $F \in S_{k,\phi}^!$ as desired. □

References

[1] Bringmann, K., P. Guerzhoy, Z. Kent and K. Ono. "Eichler-Shimura theory for mock modular forms." *Math. Annalen* 355 no. 3 (2013): 1085–1121.

[2] Bringmann, K., K. Fricke and Z. Kent. "Special L-values and periods of weakly holomorphic modular." *Proc. Amer. Math. Soc.* 142 (2014): 3425–3439.

[3] Bruinier, J. H. and J. Funke. "On two geometric theta lifts." *Duke Mathematical Journal* 125 no.1 (2004): 45–90.
[4] Diamantis, N., M. Lee, W. Raji and L. Rolen. "L-series of harmonic Maass forms and a summation formula for harmonic lifts." arXiv:2107.12366
[5] Diamantis, N. and L. Rolen. "L-values of harmonic Maass forms." arXiv:2201.10193
[6] Folland, G. *Fourier analysis and its applications*, Pacific Grove, CA: Wadsworth & Brooks/Cole, 1992.
[7] Hecke, E. "Über die Bestimmung Dirichletscher Reichen durch ihre Funktionalgleichung." *Math. Ann.* 112 (1936): 664–699 (Werke 33).
[8] Knopp, M. and G. Mason. "Vector-valued modular forms and Poincaré series." *Illinois Journal of Mathematics* 48 no. 4 (2004): 1345–1366.
[9] Kohnen, W. and D. Zagier, "Modular forms with rational periods." *Modular forms (Durham 1983), ed. R. A. Rankin*, John Wiley, 1984.
[10] Magnus, W., F. Oberhettinger and R. P. Soni. *Formulas and Theorems for the Special Functions of Mathematical Physics.* Die Grundlehren der mathematischen Wissenschaften (Vol. 52). 2013. Springer Science & Business Media.