Calculating Ramsey numbers by partitioning coloured graphs

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Ramsey Theory

The Ramsey Number $R(G, H)$ is the smallest $n$ for which any 2-edge-colouring of $K_n$ contains either a red $G$ or a blue $H$.

**Theorem (Ramsey, 1930)**

$R(K_n, K_n)$ is finite for every $n$.

The following bounds hold

\[ \sqrt{2^n} \leq R(K_n, K_n) \leq 4^n. \]
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### Theorem (Erdős, 1947)

$$R(P_n, K_m) = (n - 1)(m - 1) + 1.$$
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**Proof.**
Ramsey Theory

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Theorem (Pósa, 1963)

*The vertices of every graph G can be covered by \( \alpha(G) \) disjoint cycles.*
Ramsey Theory

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\[ R(P_n, K_m) = (n - 1)(m - 1) + 1. \]

Proof.

Theorem (Pósa, 1963)

The vertices of every graph \( G \) can be covered by \( \alpha(G) \) disjoint cycles.

Theorem (Gallai-Milgram, 1960)

The vertices of every directed graph \( D \) can be covered by \( \alpha(D) \) disjoint directed paths.
Ramsey Theory

Theorem (Gerencsér and Gyárfás, 1966)

For $n \geq m$, 

$$R(P_n, P_m) = n + \left\lfloor \frac{m}{2} \right\rfloor - 1.$$
Theorem (Gerencsér and Gyárfás, 1966)

For $n \geq m,$

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Theorem (Gerencsér and Gyárfás, 1966)

Every 2-edge-coloured complete graph can be covered by 2 disjoint monochromatic paths with different colours.
Theorem 1. For \( k \geq l \) we have
\[
g(k, l) = k + \left[ \frac{l+1}{2} \right]
\]

Considering the other special case of this type of problems, let \( f_2(n) \) denote the smallest integer with the property, that colouring the edges of a complete \( n \)-tuple graph with \( r \) colours arbitrarily, there exists always a one-coloured connected subgraph with at least \( f_2(n) \) vertices.

It is easy to see the following remark of P. Erdős: if a graph is not connected then its complement is connected, i.e., \( f_2(n) = n \). We shall prove

Theorem 2.
\[
f_2(n) = \left[ \frac{n+1}{2} \right]
\]

Now we turn to the proof of Theorem 1. First we prove \( g(k, l) \geq k + \left[ \frac{l+1}{2} \right] \),

by induction on \( k \). For \( k = 1 \) the Theorem evidently holds and let us suppose that for all \( k \) less than this statement is true. Let us construct a graph \( G \) with \( k + \left[ \frac{l+1}{2} \right] \) vertices. If \( l \leq k \), then for any subgraph of \( G \) with \( k + \left[ \frac{l+1}{2} \right] \) points holds that either itself contains a path of length \( k - 1 \), or its complement a path of length \( l \). For \( l > k \) we consider a subgraph with \( k + \left[ \frac{l+1}{2} \right] \) points.

This or its complement contains a path of length \( k - 1 \). Thus in every case can be supposed that the length of the longest path of \( G \) is \( k - 1 \). Let \( U_1, U_2, \ldots, U_k \) be the consecutive vertices of such a path and \( U = \{ U_1, \ldots, U_k \} \). We denote the remaining vertices by \( V_1, \ldots, V_{l-k} \) and the set of them by \( V = \{ V_1, \ldots, V_{l-k} \} / \{ U_1, \ldots, U_k \} \).

It clearly holds that

(i) for all \( i \in V \) either \( V_i U_j \in \bar{G} \) or \( V_i U_{j+1} \in \bar{G} \)

(ii) for all \( i \in V \) \( V_i U_1 \in \bar{G} \) and \( V_i U_k \in G \)

(iii) for \( V_{l-k}, V_{l-k+2}, V_{l-k} \in V \) and \( U_{l+1-k} \in U \)

at least one of the latest points is connected in \( G \) with at least two of \( V_{l-k}, V_{l-k+2}, V_{l-k} \).

Consider a maximal path of \( G \) not containing \( U_1, U_k \) with the property that any edge of it connects a point of \( U \) with a point of \( V \), and its endpoints are in \( V \); let us denote the endpoints by \( A \) and \( B \), and the path by \( S \). If \( S \) contains all points of \( V \), then by adding the edges \( U_2 A, U_k B \) we have a path of length \( 2 \left[ \frac{l+1}{2} \right] = l \) in \( \bar{G} \). So we may suppose that the set of points \( V \) not contained by \( S \) is not empty. Let this set be called \( W \). Consider a maximal path \( q \) of \( G \) not containing \( U_1, U_k \) and having no common points with \( S \), such that any edge of it connects a point of \( U \) with a point of \( W \) and the endpoints of it, called by \( C \) and \( D \), are in \( W \). We show that all points of \( V \) are contained either in \( S \) or in \( q \). Suppose that \( X \in V \) but \( X \notin S \) \& \( X \notin q \). It is clear, that the number of vertices of \( S \) and \( q \) in \( U \) is at most \( \left[ \frac{l+1}{2} \right] - 3 \leq \left[ \frac{k-2}{2} \right] = \left[ \frac{k-2}{2} \right] \). Since \( l \leq k \). So there exist two points \( U_1, U_1+1 \in \{ U_1, \ldots, U_k \} \) which do not belong either to \( S \) or to \( q \). Applying (iii) for \( A, C, X \in V \) and \( U_1, U_1+1 \in U \) we have a contradiction to the maximal properties of \( S \) and \( q \).

So the sum of the length of \( S \) and \( q \) is \( 2 \left[ \frac{l+1}{2} \right] = 4 \). We add them the edges

\[ U_1 A, U_1 B, U_1 C, U_1 D \]

and so we have a circuit of length \( 2 \left[ \frac{l+1}{2} \right] + 1 \) in \( \bar{G} \). For odd \( l \) this contains a desired path with length \( l \). For even \( l \) an easy reasoning shows that there are \( U_1, U_1+1 \in U \) which do not belong to this circuit. Hence one of them is connected with a vertex of the circuit (see (ii)) and so we have again a path with length \( l \) in \( G \). That completes the proof.

Now we give examples for graphs \( G \) with \( k + \left[ \frac{l+1}{2} \right] \) - 1 points that have no path of length \( k \), and do not have a path of length \( l \).

a) Let \( G \) consist of the disjoint graphs \( H_1, H_2, H_3 \) with \( k \) and \( \left[ \frac{l+1}{2} \right] \) - 1 points respectively, where the graph \( H_2 \) is complete.

b) For even \( l \) we can leave one of the edges of \( H_2 \). These graphs possess obviously the desired property.

Now we turn to the proof of Theorem 2. We consider a classification of the edges of a complete graph \( G \) into three classes, i.e., let the edges of \( G \) be coloured with red, yellow and blue colours. So we get the graphs \( G_r, G_y, G_b \) formed by the red, yellow and blue edges respectively. We say that a subgraph is for example red-connected if it is a connected subgraph of \( G_r \). Let us take a maximal red-connected subgraph \( R \). It may be supposed that \( R \) is not empty and \( \pi(R) \leq \pi(G) = n \). Let \( B \) be a point of \( G \) such that \( B \notin R \). Since \( R \) is a maximal connected subgraph of \( G \), \( BR \) is not red for \( R \subseteq B \). So one may suppose that there are at least \( \frac{1}{2} \pi(G) \) points of \( R \) which are connected with \( B \) by blue edges.

Let \( V \) denote the set of these points of \( R \) and \( W \) be the maximal blue-connected subgraph that contains \( B \). If \( Y \) is a point such that \( Y \notin R \) and \( Y \notin W \), then \( YY \) is yellow for \( V \). Let \( Q \) denote the maximal yellow-connected subgraph that contains \( V \). If there is no such \( Y \), \( Q \) denotes the empty set. Then \( W, Q \) contain together all points of \( G \). Namely any points \( S \notin R \) is connected with a point of \( B \).
Theorem 1. For $k \geq l$ we have

\[ g(k, l) = k + \left\lfloor \frac{l+1}{2} \right\rfloor \]

Considering the other special case of this type of problems, let $f(n)$ denote the greatest integer with the property, that colouring the edges of a complete $n$-tuple $g$ with $r$ colours arbitrarily, there exists always a one-coloured connected subgraph with at least $f(n)$ vertices.

It is easy to see the following remark of P. Erdős: if a graph is not connected then its complement is connected, i.e. $f(n) = n$. We shall prove

Theorem 2.

\[ f(n) = \left\lfloor \frac{n+1}{2} \right\rfloor \]

Now we turn to the proof of Theorem 1. First we prove $g(k, l) \geq k + \left\lfloor \frac{l+1}{2} \right\rfloor$ by induction on $k$. For $k = 1$ the Theorem evidently holds and let us suppose that for all $k$ less than this statement is true. Let us consider a graph $G$ with $k + \left\lfloor \frac{l+1}{2} \right\rfloor$ vertices. If $l < k$, then for any subgraph of $G$ with $k - 1 + \left\lfloor \frac{l+1}{2} \right\rfloor$ points holds that either itself contains a path of length $k - 1$, or its complement a path of length $l$. For $l = k$ we consider a subgraph with $k - 1 + \left\lceil \frac{l}{2} \right\rceil$ points.

This or its complement contains a path of length $k - 1$. Thus in every case can be supposed, that the length of the longest path of $G$ is $k - 1$. Let $U_1, U_2, \ldots, U_k$ be the consecutive vertices of such a path and $U = \{U_1, \ldots, U_k\}$. We denote the remaining vertices by $V_1, \ldots, V_\left\lceil \frac{l}{2} \right\rceil$ and the set of them by $V = V \setminus \{U_1, \ldots, V_\left\lceil \frac{l}{2} \right\rceil\}$.

It clearly holds that

(i) for all $V_i \in V$ either $V_iU_j \in G$ or $V_iU_{j+1} \in \overline{G}$
(ii) for all $V_i \in V$ $V_iU_j \in G$ and $V_iU_k \in \overline{G}$
(iii) for $V_\alpha, V_\beta, V_\gamma \in V$ and $U_\alpha, U_\beta, U_\gamma \in U$

at least one of the last points is connected in $\overline{G}$ with at least two of $V_\alpha, V_\beta, V_\gamma$.

Consider a maximal path of $G$ not containing $U_1, U_k$ with the property that any edge of it connects a point of $U$ with a point of $V$, and its endpoints are in $V$; let us denote the endpoints by $A$ and $B$, and the path by $S$. If $S$ contains all points of $V$, then by adding the edges $U_2A, BU_k$ we have a path of length $2\left\lfloor \frac{l+1}{2} \right\rfloor + 1$ in $\overline{G}$. So we may suppose that the set of points $V$ not contained by $S$ is not empty. Let this set be called $W$. Consider a maximal path $q$ of $G$ not containing $U_1, U_k$ and having no common points with $S$, such that any edge of it connects a point of $U$ with a point of $W$ and the endpoints of it, called by $C$ and $D$, are in $W$. We show that all points of $V$ are contained either in $S$ or in $q$. Suppose that $X \in V$ but $X \notin S, X \notin q$. It is clear, that the number of vertices of $S$ and $q$ in $U$ is at most \[ \frac{\left\lfloor \frac{l+1}{2} \right\rfloor + 3 - \left\lfloor \frac{k-2}{2} \right\rfloor}{2} = \frac{k-2-1}{2} \]

since $l \geq k$. So there exist two points $U_i, U_{i+1} \in \{U_1, \ldots, U_k\}$ which do not belong either to $S$ or to $q$. Applying (iii) for $A, C, X \in V$ and $U_i, U_{i+1} \in U$ we have a contradiction to the maximal properties of $S$ and $q$.

So the sum of the length of $S$ and $q$ is $2\left\lfloor \frac{l+1}{2} \right\rfloor - 4$. We add them the edges $U_iA, BU_k, U_1C, DU_1$ and so we have a circuit of length $2\left\lfloor \frac{l+1}{2} \right\rfloor$ in $\overline{G}$. For odd $l$ this contains a desired path with length $l$. For even $l$ an easy reasoning shows that there are $U_i, U_{i+1} \in U$ which do not belong to this circuit. Hence one of them is connected with a vertex of the circuit (see (ii)) and so we have again with path $l$ in $\overline{G}$. That completes the proof.

Now we give examples for graphs $G$ with $k + \left\lfloor \frac{l+1}{2} \right\rfloor - 1$ points that have no path of length $k$, and for them at the same time $\overline{G}$ have no path of length $l$.

a). Let $G$ consist of the disjoint graphs $H_1, H_2$ with $k$ and $\left\lfloor \frac{l+1}{2} \right\rfloor - 1$ points respectively, where the graph $H_2$ is complete.

b). For even $l$ we can leave one of the edges of $H_1$. These graphs possess obviously the desired property.

Now we turn to the proof of Theorem 2. We consider a classification of the edges of a complete graph $G$ into three classes, i.e. let the edges of $G$ be coloured with red, yellow and blue colours. So we get the graphs $G_1, G_2, G_3$ formed by the red, yellow and blue edges respectively. We say that a subgraph is for example red-connected if it is a connected subgraph of $G_1$. Let us take a maximal red-connected subgraph $R$. It may be supposed that $R$ is not empty and $\pi(R) < \pi(G) = n$. Let $G$ be a point of $G$ such that $B \notin R$. Since $R$ is a maximal connected subgraph of $G$, $BR_i$ is not red for $R_i \in R$. So one may suppose that there are at least $\left\lfloor \frac{l}{2} \right\rceil \pi(R)$ points of $R$ which are connected with $B$ by blue edges.

Let $V$ denote the set of these points of $R$ and $W$ be the maximal blue-connected subgraph that contains $B$. If $Y$ is a point such that $Y \notin R$ and $Y \notin W$ then $YV$ is yellow for $V \notin V$. Let $Q$ denote the maximal yellow-connected subgraph that contains $Y$. If there is no such $Y$, $Q$ denotes the empty set. $R, W, Q$ contain together all points of $G$. Namely any points $S \notin R$ is connected with a

\[ \text{\textsuperscript{1}}\text{The weaker result } g(k, l) = k + l \text{ can be easily proved. Let us consider any vertex } P \text{ and a pair of paths of } G \text{ and } \overline{G} \text{ without common vertices except } P \text{. It can be proved that a pair of paths with maximal sum of lengths contains all points. (Maximality with respect to all } P \text{ and all pairs.) From that the statement follows.} \]
Ramsey Theory

Calculating Ramsey numbers by partitioning coloured graphs

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Theorem 1. For $k \geq l$ we have

$$g(k, l) = k + \left\lfloor \frac{l+1}{2} \right\rfloor.$$  

Considering the other special case of this type of problems, let $f_4(n)$ denote the greatest integer with the property, that colouring the edges of a complete $n$-tuple $g$ with $r$ colours arbitrarily, there exists always a one-coloured connected subgraph with at least $f_4(n)$ vertices.

It is easy to see the following remark of P. Erdős: if a graph is not connected then its complement is connected, i.e., $f_4(n) = n$. We shall prove

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$$f_4(n) = \left\lfloor \frac{n+1}{2} \right\rfloor.$$  

Now we turn to the proof of Theorem 1. First we prove $g(k, l) \geq k + \left\lfloor \frac{l+1}{2} \right\rfloor$ by induction on $k$. For $k = 1$ the Theorem evidently holds and let us suppose that is true for $k = k - 1$. We shall prove $g(k, l) \geq k + \left\lfloor \frac{l+1}{2} \right\rfloor$ for $k$.

The weaker result $g(k, l) \leq k + l$ can be easily proved. Let us consider any vertex $P$ and a pair of paths of $G$ and its complement $\bar{G}$ without common vertices except $P$. It can be proved that a pair of paths with maximal sum of lengths contains all points. (Maximality with respect to all $P$ and all pairs.) From that the statement follows.

Theorem 2 would be completely proved if we had $V = \{V_0, \ldots, V_{\left\lfloor \frac{1}{2} \right\rfloor}\}$ and the set of them by $V = \{V_0, \ldots, V_{\left\lfloor \frac{1}{2} \right\rfloor}\}$.

It clearly holds that

(i) for all $V_i \in V$ either $V_i U_j \in G$ or $V_i U_j \notin G$

(ii) for all $V_i \in V$, $U_j \in G$ and $V_i U_j \in G$

(iii) for $V_i, V_j, V_k \in V$ and $U_i, U_j \in U$

at least one of the latest points is connected in $G$ with at least two of $V_0, V_1, V_2$.

Consider a maximal path of $G$ not containing $U_i, U_j$ with the property that any edge of it connects a point of $U$ with a point of $V$, and its endpoints are in $V$; let us denote the endpoints by $A$ and $B$, and the path by $S$. If $S$ contains all points of $V$, then by adding the edges $U_i A, B U_k$ we have a path of length $2 \left\lfloor \frac{l+1}{2} \right\rfloor$ in $\bar{G}$. So we may suppose that the set of points $V$ not contained by $S$ is not empty. Let this set be called $W$. Consider a maximal path $q$ of $G$ not containing $U_i, U_j$ and having no common points with $S$, such that any edge of it connects a point of $U$ with a point of $W$ and the endpoints of it, called by $C$ and $D$, are in $W$. We show that all points of $V$ are contained either in $S$ or in $q$. Suppose that $x \in V$ but $x \notin S, x \notin q$. It is clear, that the number of vertices of $S$ and $q$ in $U$ is at most $\left\lfloor \frac{l+1}{2} \right\rfloor + 3 < \frac{(k-3)}{2} = \frac{k-2}{2}$ since $l \geq k$. So there exist two points $U_i, U_{i+1} \in U_k, \ldots, U_{i+j} \in U_k$ which do not belong to $S$ or to $q$. Applying (iii) for $A, C, x \in V$ and $U_i, U_{i+1} \in U$ we have a contradiction to the maximal properties of $S$ and $q$.

So the sum of the length of $S$ and $q$ is $2 \left\lfloor \frac{l+1}{2} \right\rfloor$. We add them the edges $U_i A, B U_i, U_i C, D U_i$ and so we have a circuit of length $2 \left\lfloor \frac{l+1}{2} \right\rfloor$ in $\bar{G}$. For odd $l$ this contains a desired path with length $l$. For even $l$ an easy reasoning shows that there are $U_i, U_{i+1} \in U_k$ which do not belong to this circuit. Hence one of them is connected with a vertex of the circuit (see $(ii)$) and so we have again a path with length $l$ in $\bar{G}$. That completes the proof.

Now we give examples for graphs $G$ with $k + \left\lfloor \frac{l+1}{2} \right\rfloor$ points that have...
Theorem (Gyárfás and Lehel; Faudree and Schelp, 1973)

\[ R_{K_{n,n}}(P_n, P_m) \approx n + m \]
**Theorem (Gyárfás and Lehel, 1973)**

\[
R_{K_{n,n}}(P_n, P_m) \approx n + m
\]

Let \( G \) be a 2-edge-coloured balanced complete bipartite graph. Then one of the following holds.

- \( G \) looks like this:

\[
\begin{align*}
X_1 & \quad X_2 \\
Y_1 & \quad Y_2
\end{align*}
\]

- Then there are two disjoint monochromatic paths covering all, except possibly one vertex in \( G \).
Partitioning coloured graphs

Theorem (Gerencsér and Gyárfás, 1966)

Every 2-edge-coloured complete graph can be covered by 2 disjoint monochromatic paths with different colours.

Conjecture (Gyárfás, 1989)

Every $r$-edge-coloured complete graph can be covered by $r$ disjoint monochromatic paths.

This theorem and conjecture gave rise to a number of results.
### Conjecture (Gyárfás, 1989)

*Every r-edge-coloured complete graph can be covered by r disjoint monochromatic paths.*

This conjecture led to...

- Every $r$-edge-coloured infinite complete graph can be covered by $r$ infinite monochromatic paths. [Rado, 1987]
- Every $r$-edge-coloured $K_n$ can be covered by $O(r^2 \log r)$ disjoint monochromatic cycles. [Erdős, Gyárfás and Pyber, 1991]
- Every $r$-edge-coloured $K_n$ can be covered by $O(r \log r)$ disjoint monochromatic cycles. [Gyárfás, Ruszinkó, Sárközy and Szemerédi, 2006]
- Every 2-edge-coloured $K_n$ can be covered 2 disjoint monochromatic cycles. [Łuczak, Rödl and Szemerédi, 1998; Allen, 2008; Bessy and Thomassé, 2010]
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Partitioning coloured graphs

Conjecture (Gyárfás, 89)

Every $r$-edge-coloured complete graph can be covered by $r$ disjoint monochromatic paths.

This conjecture led to...

- Every 3-edge-coloured $K_n$ has 3 monochromatic cycles covering $n - o(n)$ vertices. [Gyárfás, Ruszinkó, Sárközy and Szemerédi, 2011]
- Not every 3-edge-coloured $K_n$ can be covered by 3 disjoint monochromatic cycles. [P., 2013]
- Every 3-edge-coloured $K_n$ can be covered by 3 disjoint monochromatic paths. [P., 2013]
- Suppose that we have a sequence $G = \{G_0, G_1, G_2, \ldots \}$ of graphs with maximum degree $\leq \Delta$. Every 2-edge-coloured complete graph can be covered by at most $2^{C\Delta \log \Delta}$ monochromatic copies of graphs from $G$. [Grinshpun and Sárközy, 2013]
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- Every 3-edge-coloured \( K_n \) has 3 monochromatic cycles covering \( n - o(n) \) vertices. [Gyárfás, Ruszinkó, Sárközy and Szemerédi, 2011]
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- Every 3-edge-coloured \( K_n \) can be covered by 3 disjoint monochromatic paths. [P., 2013]
- Suppose that we have a sequence \( G = \{ G_0, G_1, G_2, \ldots \} \) of graphs with maximum degree \( \leq \Delta \). Every 2-edge-coloured complete graph can be covered by at most \( 2^{c\Delta \log \Delta} \) monochromatic copies of graphs from \( G \). [Grinshpun and Sárközy, 2013]
Conjecture (Gyárfás, 89)

Every $r$-edge-coloured complete graph can be covered by $r$ disjoint monochromatic paths.

This conjecture led to...

- Every 3-edge-coloured $K_n$ has 3 monochromatic cycles covering $n - o(n)$ vertices. [Gyárfás, Ruszinkó, Sárközy and Szemerédi, 2011]
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- Suppose that we have a sequence $G = \{G_0, G_1, G_2, \ldots \}$ of graphs with maximum degree $\leq \Delta$. Every 2-edge-coloured complete graph can be covered by at most $2^{c\Delta \log \Delta}$ monochromatic copies of graphs from $G$. [Grinshpun and Sárközy, 2013]
Results

Theorem (P., 2014+)

Suppose that the edges of $K_n$ are 2-coloured. Then $K_n$ can be covered by $k$ disjoint red paths and a disjoint blue balanced complete $(k + 1)$-partite graph.
Theorem (P., 2014+)

Suppose that the edges of $K_n$ are $2$-coloured. Then $K_n$ can be covered by $k$ disjoint red paths and a disjoint blue balanced complete $(k + 1)$-partite graph.

Theorem (P., 2014+)

Suppose that the edges of $K_n$ are $2$-coloured such that the red subgraph is connected. Then $K_n$ can be covered by $k$ disjoint red paths and a disjoint blue balanced complete $(k + 2)$-partite graph.
Applications

Theorem (P., 2014+)

Suppose that the edges of $K_n$ are 2-coloured. Then $K_n$ can be covered by $k$ disjoint red paths and a disjoint blue balanced complete $(k + 1)$-partite graph.

- Generalises original Gerencsér-Gyárfás path partitioning theorem.
Theorem (P., 2014+)

Suppose that the edges of $K_n$ are 2-coloured. Then $K_n$ can be covered by $k$ disjoint red paths and a disjoint blue balanced complete $(k + 1)$-partite graph.

- Generalises original Gerencsér-Gyárfás path partitioning theorem.
- Can be used to prove the $r = 3$ case of Gyárfás Conjecture.
Applications

Theorem (P., 2014+)

Suppose that the edges of $K_n$ are 2-coloured. Then $K_n$ can be covered by $k$ disjoint red paths and a disjoint blue balanced complete $(k + 1)$-partite graph.

- $R(P_n, K_i^t) = (t - 1)(n - 1) + t(i - 1) + 1$ for $i \equiv 1 \pmod{n - 1}$. 

This generalises:

$R(P_n, K_m) = (n - 1)(m - 1) + 1 \quad \text{[Erdős]}

$R(P_n, K_i, i) = n + i - 1$ for $m \equiv 1 \pmod{n - 1} \quad \text{[Häggkvist]}

$R(P_n, P_{kn}) = (n - 1)k + \left\lfloor \frac{n}{k} + 1 \right\rfloor \quad \text{[Conjectured by Allen, Brightwell and Skokan]}

Might be useful for finding $R(P_n, H)$ for other graphs $H$.
Applications

Theorem (P., 2014+)

Suppose that the edges of $K_n$ are 2-coloured. Then $K_n$ can be covered by $k$ disjoint red paths and a disjoint blue balanced complete $(k + 1)$-partite graph.

- $R(P_n, K^t_i) = (t - 1)(n - 1) + t(i - 1) + 1$
  for $i \equiv 1 \pmod{n - 1}$. This generalises:
  - $R(P_n, K_m) = (n - 1)(m - 1) + 1$ [Erdős].
  - $R(P_n, K_{i,i}) = n + i - 1$ for $m \equiv 1 \pmod{n - 1}$ [Häggkvist].
Applications

Theorem (P., 2014+)

Suppose that the edges of $K_n$ are 2-coloured. Then $K_n$ can be covered by $k$ disjoint red paths and a disjoint blue balanced complete $(k + 1)$-partite graph.

- $R(P_n, K^t_i) = (t - 1)(n - 1) + t(i - 1) + 1$ for $i \equiv 1 \pmod{n - 1}$. This generalises:
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  - $R(P_n, K_{i, i}) = n + i - 1$ for $m \equiv 1 \pmod{n - 1}$ [Häggkvist].
- $R(P_n, P^k_n) = (n - 1)k + \left\lfloor \frac{n}{k+1} \right\rfloor$ (Conjectured by Allen, Brightwell and Skokan).
Applications

Theorem (P., 2014+)

_Suppose that the edges of \( K_n \) are 2-coloured. Then \( K_n \) can be covered by \( k \) disjoint red paths and a disjoint blue balanced complete \((k + 1)\)-partite graph._

\[
R(P_n, K^t_i) = (t - 1)(n - 1) + t(i - 1) + 1 \\
\text{for } i \equiv 1 \pmod{n - 1}. \text{ This generalises:}
\]

\[
\begin{align*}
& R(P_n, K_m) = (n - 1)(m - 1) + 1 \quad \text{[Erdös].} \\
& R(P_n, K_{i,i}) = n + i - 1 \quad \text{for } m \equiv 1 \pmod{n - 1} \quad \text{[Häggkvist].}
\end{align*}
\]

\[
R(P_n, P^k_n) = (n - 1)k + \left\lfloor \frac{n}{k+1} \right\rfloor \quad \text{(Conjectured by Allen, Brightwell and Skokan).}
\]

\[
\text{ Might be useful for finding } R(P_n, H) \text{ for other graphs } H.\ldots
\]
Proof

Theorem

Every 2-edge-coloured complete graph can be covered by a red path and a disjoint blue balanced complete bipartite graph.

Proof.
Open problems

Conjecture

Every 2-edge-coloured complete tripartite graph can be covered by two disjoint monochromatic paths.

Conjecture (Gyárfás and Sarközy)

Every complete $r$-uniform hypergraph $H$ can be covered by $\alpha(H)$ disjoint loose cycles.

Problem

Every $r$-edge-coloured complete graph can be covered by $1000r$ monochromatic paths.
Problem

Prove natural statements of the form “Every 2-edge-coloured complete graph can be covered by a red graph $G$ and a disjoint blue graph $H$ with $G$ and $H$ having particular structures”.

Known results of this type:

- $G$ and $H$ paths [Gerencsér and Gyárfás].
- $G$ and $H$ cycles [Łuczak, Rödl, and Szemerédi; Allen; Bessy and Thomassé].
- $G$ a matching, $H$ a complete graph [folklore].
- $G$ a forest of $k$ paths, $H$ a balanced complete $(k+1)$-partite graph. [P.]
- $G$ a cycle, $H$ a graph with $\Delta(H) \geq \frac{1}{2}(|H| - 1)$. [P.]