

Existence, minimality and approximation of solutions to BSDEs with convex drivers

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Abstract

We study the existence of solutions to backward stochastic differential equations with drivers $f(t, W, y, z)$ that are convex in z . We assume f to be Lipschitz in y and W but do not make growth assumptions with respect to z . We first show the existence of a unique solution (Y, Z) with bounded Z if the terminal condition is Lipschitz in W and that it can be approximated by the solutions to properly discretized equations. If the terminal condition is bounded and uniformly continuous in W we show the existence of a minimal continuous supersolution by uniformly approximating the terminal condition with Lipschitz terminal conditions. Finally, we prove existence of a minimal RCLL supersolution for bounded lower semicontinuous terminal conditions by approximating the terminal condition pointwise from below with Lipschitz terminal conditions.

Keywords Backward stochastic differential equations, backward stochastic difference equations, convex drivers, discrete-time approximations, supersolutions.

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1 Introduction

We consider BSDEs (backward stochastic differential equations) of the form

$$Y_t = \xi + \int_t^T f(s, W, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \quad (1.1)$$

with drivers f that are convex in Z_s . We assume f to be Lipschitz-continuous in W and Y_s but only locally Lipschitz-continuous in Z_s . In particular, f can grow arbitrarily fast in Z_s . $(W_t)_{t \in [0, T]}$ is a d -dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $Z_s dW_s$ is understood as $\sum_{k=1}^d Z_s^k dW_s^k$. The terminal condition ξ is a random variable measurable with respect to the sigma-algebra generated by $(W_t)_{t \in [0, T]}$.

BSDEs with drivers linear in (y, z) were introduced by Bismut [4]. Pardoux and Peng [17] showed that BSDEs with drivers that are Lipschitz in (y, z) have a unique solution if the terminal condition is square-integrable. Kobylanski [16] proved existence and uniqueness of solutions to BSDEs with bounded terminal conditions and drivers that grow at most quadratically in z . Extensions to unbounded terminal conditions were given by Briand and Hu [6, 7] as well as Delbaen et al. [11].

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BSDEs with drivers that are convex and of unrestricted growth in z have already been studied in Delbaen et al. [10]. In that paper, the Brownian motion is one-dimensional, the terminal condition is bounded and the driver is of the form $f(z)$ for a deterministic convex function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $f(0) = 0$ and $\lim_{z \rightarrow \pm\infty} f(z)/|z|^2 = \infty$. It is shown in Delbaen et al. [10] that, depending on the terminal condition, BSDEs of this form have either no or infinitely many bounded solutions. Moreover, it is proved that a bounded solution exists if the terminal condition is of the form $\varphi(X_T)$, where $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a deterministic bounded continuous function and X a forward process driven by the underlying Brownian motion. In this special case, BSDEs can be formulated as parabolic PDEs. Related PDE results have been obtained by Ben-Artzi et al. [3] and Gilding et al. [14].

The purpose of this paper is to show the existence and uniqueness of a solution if the driver f depends on (t, W, y, z) and the terminal condition ξ is a possibly unbounded function of the whole underlying Brownian motion W_t , $0 \leq t \leq T$. However, in view of the results of Delbaen et al. [10] it cannot be hoped that solutions exist for arbitrary terminal conditions or that uniqueness holds without restrictions on the Z -process. Therefore, we first study terminal conditions that are Lipschitz in the underlying Brownian motion and then approximate more general terminal conditions with Lipschitz ones. In Theorem 2.4 we show that (1.1) has a unique solution (Y, Z) with bounded Z if the terminal condition is of the form $\varphi(W)$, where φ is a Lipschitz-continuous function on the space of continuous functions. Our method of proof is to approximate (1.1) by discrete-time equations and show that their solutions converge to a solution of the continuous-time BSDE. In Theorem 2.5 we prove that for bounded terminal conditions that can uniformly be approximated by Lipschitz terminal conditions the BSDE (1.1) has a bounded continuous supersolution in the sense of Peng [18] such that Z is a BMO process. This covers the case of bounded terminal conditions that are uniformly continuous in the underlying Brownian motion. Theorem 2.7 treats bounded terminal conditions that are pointwise limits of an increasing sequence of Lipschitz terminal conditions. In this case we show that the BSDE (1.1) has a bounded RCLL supersolution such that Z is BMO. This gives the existence of a RCLL supersolution for bounded terminal conditions that are lower semicontinuous in the underlying Brownian motion. If the driver is monotone in y , we are also able to show that the BSDE (1.1) satisfies a one-sided comparison principle, from which we deduce that the supersolutions constructed in Theorems 2.5 and 2.7 are minimal.

The structure of the paper is as follows: In Section 2 we introduce the notation and state our main results. In Section 3 we prove results on BSDEs (backward stochastic difference equations) that are needed in the proof of Theorem 2.4 given in Section 4. In Section 5 we use convex duality to show comparison results. In Section 6 we give the proofs of Theorem 2.5 and 2.7. In the Appendix we show that a convergence result of Briand et al. [5] which we need in the proof of Theorem 2.4 still holds in our setting.

2 Notation and statement of results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space carrying a d -dimensional Brownian motion $(W_t)_{0 \leq t \leq T}$. By (\mathcal{F}_t) we denote the augmented filtration generated by (W_t) . As usual, we identify random variables that agree almost surely and understand equalities as well as inequalities between them in the almost sure sense. Fix $T \in (0, \infty)$ and denote by $C^d[0, T]$ the space of all continuous functions $w : [0, T] \rightarrow \mathbb{R}^d$. Let (\mathcal{E}_t) be the filtration on $C^d[0, T]$ generated by the coordinate process and \mathcal{P} the predictable sigma-algebra on $[0, T] \times C^d[0, T]$. We call a function

$$f : [0, T] \times C^d[0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$$

a driver if it is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable. We always assume f in equation (1.1) to be a driver and the terminal condition ξ an \mathcal{F}_T -measurable random variable. We call a stochastic process RCLL

if almost all of its paths are right-continuous and have left limits. We call a stochastic process (A_t) increasing if $A_s \leq A_t$ for $s \leq t$. Similarly, we say a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is increasing (decreasing) if $f(x) \leq (\geq) f(y)$ for $x \leq y$.

Definition 2.1 A solution of the BSDE (1.1) consists of a pair $(Y_t, Z_t)_{0 \leq t \leq T}$ of predictable processes with values in $\mathbb{R} \times \mathbb{R}^d$ such that

$$\int_0^T |f(s, W, Y_s, Z_s)| ds < \infty, \quad \int_0^T |Z_s|^2 ds < \infty \quad (2.1)$$

and

$$Y_t = \xi + \int_t^T f(s, W, Y_s, Z_s) ds - \int_t^T Z_s dW_s \quad \text{for all } t \in [0, T].$$

A supersolution of the BSDE (1.1) consists of a triple $(Y_t, Z_t, A_t)_{0 \leq t \leq T}$ of predictable processes with values in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ such that (Y_t) is RCLL, (A_t) starts at 0 and is increasing RCLL, (2.1) holds and

$$Y_t = \xi + \int_t^T f(s, W, Y_s, Z_s) ds - \int_t^T Z_s dW_s + A_T - A_t \quad \text{for all } t \in [0, T].$$

Definition 2.2 We say a supersolution (Y_t, Z_t, A_t) of the BSDE (1.1) satisfies bounded comparison from above if for every supersolution (Y'_t, Z'_t, A'_t) of (1.1) with driver $f' \geq f$ and terminal condition $\xi' \geq \xi$ such that Y' is bounded, one has $Y'_t \geq Y_t$ for all t .

Remark 2.3 If (Y_t, Z_t, A_t) is a supersolution of the BSDE (1.1) such that Y is bounded and satisfies bounded comparison from above, one has $Y'_t \geq Y_t$, $0 \leq t \leq T$, for every other supersolution (Y'_t, Z'_t, A'_t) of (1.1) such that Y' is bounded. So (Y_t, Z_t, A_t) is the minimal bounded supersolution of (1.1). If in addition, $A \equiv 0$, (Y_t, Z_t) is the minimal bounded solution.

Denote by $|\cdot|$ the Euclidean norm on \mathbb{R}^d . For most of our results we need the driver to satisfy some or all of the following properties:

- (f1) $f(t, w, y, z)$ is convex in z
- (f2) $\sup_{t,w} |f(t, w, 0, 0)| < \infty$
- (f3) There exists a constant $K \in \mathbb{R}_+$ such that

$$|f(t, w_1, y_1, z) - f(t, w_2, y_2, z)| \leq K \left(\sup_{0 \leq s \leq t} |w_1(s) - w_2(s)| + |y_1 - y_2| \right)$$

for all t, w_1, w_2, y_1, y_2, z .

- (f4) For every $a \in \mathbb{R}_+$ there exists a $b \in \mathbb{R}_+$ such that

$$|f(t, w, y, z_1) - f(t, w, y, z_2)| \leq b |z_1 - z_2|$$

for all t, w, y and $z_1, z_2 \in \mathbb{R}^d$ with $|z_1| \vee |z_2| \leq a$.

- (f5) $\inf_{w \in C^d[0, T], y \in [-c, c], z \in \mathbb{R}^d, t \in [0, T]} f(t, w, y, z) > -\infty$ for all $c \in \mathbb{R}_+$.

Our first result shows that the BSDE (1.1) has a unique solution such that Z is bounded when f satisfies (f1)–(f4) and the terminal condition is Lipschitz-continuous in the underlying Brownian motion W . We prove it by discretizing equation (1.1) in time and then passing to the continuous-time limit. To do that we approximate W by a sequence W^N , $N \in \mathbb{N}$, of d -dimensional square-integrable martingales starting at 0 with independent increments satisfying the following conditions:

(W1) For every $N \in \mathbb{N}$ there exists a finite sequence $0 = t_0^N < t_1^N < t_2^N \cdots < t_{i_N}^N = T$ such that

$$\lim_{N \rightarrow \infty} \max_i |t_{i+1}^N - t_i^N| = 0$$

and W_t^N is constant on the intervals $[t_i^N, t_{i+1}^N)$.

(W2)

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\sup_{0 \leq t \leq T} |W_t^N - W_t|^2 \right] = 0.$$

(W3) For all N and i , $\Delta W_{t_i^N}^N := W_{t_i^N}^N - W_{t_{i-1}^N}^N$ takes only finitely many different values.

(W4) For all N , i and $k \neq l$,

$$\mathbb{E} \left[\Delta W_{t_i^N}^{N,k} \Delta W_{t_i^N}^{N,l} \right] = 0 \quad \text{and} \quad \Delta \langle W^{N,k} \rangle_{t_i^N} = \Delta \langle W^{N,l} \rangle_{t_i^N} = \Delta t_i^N := t_i^N - t_{i-1}^N > 0.$$

(W5)

$$\sup_{N,i,k} \frac{\left\| \Delta W_{t_i^N}^{N,k} \right\|_\infty}{\sqrt{\Delta t_i^N}} < \infty.$$

One can, for instance, set $t_i^N = iT/N$, $i = 0, \dots, N$ and let the W^N be d -dimensional Bernoulli random walks with increments $\pm\sqrt{T/N}$, that is, the increments $W_{t_i^N}^{N,k} - W_{t_{i-1}^N}^{N,k}$, $i = 1, \dots, N$, $k = 1, \dots, d$, are independent and have distribution $\mathbb{P} \left[W_{t_i^N}^{N,k} - W_{t_{i-1}^N}^{N,k} = \pm\sqrt{T/N} \right] = 1/2$ (see Cheridito and Stadje [9] for details on how to construct d -dimensional Bernoulli random walks on the same probability space as W such that one has the convergence of (W2)).

We set $\langle W^N \rangle_t := \langle W^{N,1} \rangle_t = \cdots = \langle W^{N,d} \rangle_t = t_i^N$ for $t_i^N \leq t < t_{i+1}^N$. Let (\mathcal{F}_t^N) be the filtration generated by W^N . To define the approximating BSDEs, we construct two continuous approximations to W^N . The process

$$\bar{W}_t^N = W_{t_{i-1}^N}^N + \frac{t - t_{i-1}^N}{t_i^N - t_{i-1}^N} (W_{t_i^N}^N - W_{t_{i-1}^N}^N) \quad \text{for } t_{i-1}^N \leq t \leq t_i^N \quad (2.2)$$

is continuous but not adapted to (\mathcal{F}_t^N) . To make it (\mathcal{F}_t^N) -adapted, we shift it by $h^N := \max_i |t_i^N - t_{i-1}^N|$ and define

$$\hat{W}_t^N = \begin{cases} 0 & \text{for } 0 \leq t \leq h^N \\ \bar{W}_{t-h^N}^N & \text{for } h^N \leq t \leq T. \end{cases} \quad (2.3)$$

Introduce the left-continuous, piecewise constant process \hat{f}^N on $C^d[0, T]$ by $\hat{f}^N(0, w, y, z) := f(0, w, y, z)$ and

$$\hat{f}^N(t, w, y, z) := \frac{\int_{t_i^N}^{t_{i+1}^N} f(s, w, y, z) ds}{\Delta t_{i+1}^N} \quad \text{for } t_i^N < t \leq t_{i+1}^N. \quad (2.4)$$

Since the approximating processes W^N do in general not have the predictable representation property, solutions to the discretized equations involve orthogonal martingales. More precisely, a solution to the N -th BSDE (backward stochastic difference equation) corresponding to an \mathcal{F}_T^N -measurable terminal condition ξ^N consists of a triple of (\mathcal{F}_t^N) -adapted processes (Y_t^N, Z_t^N, M_t^N) taking values in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ such that (Y_t^N) is constant on the intervals $[t_i^N, t_{i+1}^N)$, (Z_t^N) is constant on the intervals $(t_i^N, t_{i+1}^N]$, (M_t^N) is a martingale starting at 0 and orthogonal to (W_t^N) that is constant on the intervals $[t_i^N, t_{i+1}^N)$ and

$$Y_t^N = \xi^N + \int_{(t,T]} \hat{f}^N(s, \hat{W}^N, Y_{s-}^N, Z_s^N) d\langle W^N \rangle_s - \int_{(t,T]} Z_s^N dW_s^N - (M_T^N - M_t^N), \quad t \in [0, T]. \quad (2.5)$$

Since the process (W_t^N) is piece-wise constant, it is completely determined by the finite sequence $(W_{t_1^N}^N, \dots, W_T^N)$, and equation (2.5) can be written as

$$Y_{t_{i+1}^N}^N = Y_{t_i^N}^N + f^N(t_{i+1}^N, W^N, Y_{t_i^N}^N, Z_{t_i^N}^N) \Delta t_{i+1}^N - Z_{t_{i+1}^N}^N \Delta W_{t_{i+1}^N}^N - \Delta M_{t_{i+1}^N}^N \quad (2.6)$$

$$Y_T^N = \xi^N, \quad (2.7)$$

for functions

$$f^N : \{t_1^N, \dots, T\} \times \mathbb{R}^{d \times i_N} \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}.$$

If f satisfies (f1)–(f4), f^N has the following properties:

(f1') $f^N(t_i^N, w, y, z)$ is convex in z

(f2') $\sup_{N,i} |f^N(t_i^N, W^N, 0, 0)| < \infty$

(f3') There exists a constant $K \in \mathbb{R}_+$ such that

$$|f^N(t_i^N, w_1, y_1, z) - f^N(t_i^N, w_2, y_2, z)| \leq K \left(\sup_{j \leq i-1} |w_1(t_j^N) - w_2(t_j^N)| + |y_1 - y_2| \right)$$

for all $N, i, w_1, w_2, y_1, y_2, z$.

(f4') For every $a \in \mathbb{R}_+$ there exists a $b \in \mathbb{R}_+$ such that

$$|f^N(t_i^N, w, y, z_1) - f^N(t_i^N, w, y, z_2)| \leq b |z_1 - z_2|$$

for all N, i, w, y and $z_1, z_2 \in \mathbb{R}^d$ satisfying $|z_1| \vee |z_2| \leq a$.

We endow $C^d[0, T]$ with the supremum norm $\|w\|_\infty := \sup_{0 \leq t \leq T} |w(t)|$. Our first result assumes that the terminal condition is of the form $\xi = \varphi(W)$ for a Lipschitz-continuous function $\varphi : C^d[0, T] \rightarrow \mathbb{R}$, that is, there exists a constant $L \in \mathbb{R}$ such that $|\varphi(w_1) - \varphi(w_2)| \leq L \|w_1 - w_2\|_\infty$ for all $w_1, w_2 \in C^d[0, T]$.

Theorem 2.4 *Assume f satisfies (f1)–(f4) and ξ is of the form $\xi = \varphi(W)$ for a Lipschitz-continuous function $\varphi : C^d[0, T] \rightarrow \mathbb{R}$. Then the BSDE (1.1) has a unique solution (Y, Z) such that Z is bounded. Moreover, if $\xi^N = \varphi(\hat{W}^N)$, then for N large enough, there exist unique solutions (Y^N, Z^N, M^N) to the corresponding BSDEs (2.5) and*

$$\sup_t \left(|Y_t^N - Y_t| + \left| \int_0^t Z_s^N dW_s^N - \int_0^t Z_s dW_s \right| + |M_t^N| \right) \rightarrow 0 \quad \text{in } L^2 \quad (2.8)$$

as well as

$$\sup_t \left(\sum_{k=1}^d \left| \int_0^t Z_s^{N,k} d\langle W^N \rangle_s - \int_0^t Z_s^k ds \right|^2 + \left| \int_0^t |Z_s^N|^2 d\langle W^N \rangle_s - \int_0^t |Z_s|^2 ds \right| \right) \rightarrow 0 \quad \text{in } L^1. \quad (2.9)$$

If (Y', Z') is the solution with bounded Z' of the BSDE (1.1) corresponding to a driver $f' \geq f$ satisfying (f1)–(f4) and terminal condition $\xi' \geq \xi$ of the form $\xi' = \varphi'(W)$ for a Lipschitz-continuous function $\varphi' : C^d[0, T] \rightarrow \mathbb{R}$, then $Y'_t \geq Y_t$ for all $t \in [0, T]$. In particular, if φ is bounded, then Y is bounded as well.

Next, we consider terminal conditions that can be uniformly approximated by Lipschitz-continuous terminal conditions. We call a d -dimensional (\mathcal{F}_t) -predictable process $(\mu_t)_{t \in [0, T]}$ BMO if there exists a constant $C \in \mathbb{R}_+$ such that

$$\mathbb{E} \left[\int_\tau^T |\mu_s|^2 ds \middle| \mathcal{F}_\tau \right] \leq C$$

for all stopping times τ taking values in $[0, T]$. By choosing $\tau = 0$, one obtains that a BMO process μ satisfies $\mathbb{E} \left[\int_0^T |\mu_s|^2 ds \right] < \infty$.

Theorem 2.5 *Assume f satisfies (f1)–(f5) and $\varphi^n : C^d[0, T] \rightarrow \mathbb{R}$ is a sequence of bounded Lipschitz-continuous functions such that $\|\varphi^n - \varphi\|_\infty \rightarrow 0$ for a bounded function $\varphi : C^d[0, T] \rightarrow \mathbb{R}$. Denote by (Y^n, Z^n) the solution of the BSDE (1.1) with terminal condition $\xi^n = \varphi^n(W)$ such that Z^n is bounded. Then*

$$\left\| \sup_t |Y_t^n - Y_t| \right\|_{L^\infty} \rightarrow 0 \quad \text{and} \quad \mathbb{E} \left[\sqrt{\int_0^T |Z_s^n - Z_s|^2 ds} \right] \rightarrow 0,$$

where (Y, Z, A) is a supersolution of (1.1) such that Y is bounded and continuous and Z is a BMO process. If moreover, f is increasing or decreasing in y , then (Y, Z, A) satisfies bounded comparison from above and hence, is the minimal bounded supersolution of the BSDE (1.1).

Since the uniform limits of bounded Lipschitz-continuous functions on $C^d[0, T]$ are all the bounded uniformly continuous functions on $C^d[0, T]$, the following corollary is an immediate consequence of Theorem 2.5:

Corollary 2.6 *If f satisfies (f1)–(f5) and the terminal condition is of the form $\xi = \varphi(W)$ for a bounded uniformly continuous function $\varphi : C^d[0, T] \rightarrow \mathbb{R}$, then the BSDE (1.1) has a supersolution (Y, Z, A) such that Y is bounded and continuous and Z is a BMO process. If in addition, f is increasing or decreasing in y , then (1.1) has a minimal bounded supersolution (Y, Z, A) . It satisfies bounded comparison from above, Y is continuous and Z is a BMO process.*

The next result is about terminal conditions that can be approximated pointwise from below by Lipschitz-continuous terminal conditions:

Theorem 2.7 *Assume f satisfies (f1)–(f5) and is increasing in y . Let $\varphi^n : C^d[0, T] \rightarrow \mathbb{R}$ be a sequence of bounded Lipschitz-continuous functions such that $\varphi^n \uparrow \varphi$ pointwise for a bounded function $\varphi : C^d[0, T] \rightarrow \mathbb{R}$. Denote by (Y^n, Z^n) the solution of the BSDE (1.1) corresponding to the terminal condition $\xi^n = \varphi^n(W)$ such that Z^n is bounded. Then $Y_t^n \uparrow Y_t$ a.s. for all t , where (Y, Z, A) is a supersolution of (1.1) satisfying bounded comparison from above such that Y is bounded and Z is a BMO process.*

Note that every bounded function $\varphi : C^d[0, T] \rightarrow \mathbb{R}$ that is the pointwise limit of an increasing sequence of bounded Lipschitz-continuous functions $\varphi^n : C^d[0, T] \rightarrow \mathbb{R}$ is lower semicontinuous. On the other hand, for every bounded lower semicontinuous function $\varphi : C^d[0, T] \rightarrow \mathbb{R}$, the functions

$$\varphi^n(w) := \inf_{v \in C^d[0, T]} \varphi(v) + n \|v - w\|_\infty$$

are bounded Lipschitz-continuous and increase pointwise to φ . This gives the following corollary to Theorem 2.7:

Corollary 2.8 *If f satisfies (f1)–(f5) and is increasing in y , then for every bounded lower semicontinuous function $\varphi : C^d[0, T] \rightarrow \mathbb{R}$, the BSDE (1.1) with terminal condition $\xi = \varphi(W)$ has a minimal bounded supersolution (Y, Z, A) . It satisfies bounded comparison from above and Z is a BMO process.*

3 Solutions of BSΔEs and their properties

Lemma 3.1 *If the BSΔE (2.6)–(2.7) has a solution (Y^N, Z^N, M^N) , then*

$$Y_{t_i^N}^N - f^N(t_{i+1}^N, W^N, Y_{t_i^N}^N, Z_{t_{i+1}^N}^N) \Delta t_{i+1}^N = \mathbb{E} \left[Y_{t_{i+1}^N}^N \mid \mathcal{F}_{t_i^N}^N \right] \quad (3.1)$$

$$Z_{t_{i+1}^N}^{N,k} = \frac{\mathbb{E} \left[Y_{t_{i+1}^N}^N \Delta W_{t_{i+1}^N}^{N,k} \mid \mathcal{F}_{t_i^N}^N \right]}{\Delta t_{i+1}^N} \quad (3.2)$$

$$\Delta M_{t_{i+1}^N}^N = Y_{t_{i+1}^N}^N - \mathbb{E} \left[Y_{t_{i+1}^N}^N \mid \mathcal{F}_{t_i^N}^N \right] - Z_{t_{i+1}^N}^N \Delta W_{t_{i+1}^N}^N \quad (3.3)$$

for all $i \leq i_N - 1$.

Proof. (3.1) follows from equation (2.6) by taking conditional expectation with respect to $\mathcal{F}_{t_i^N}^N$. (3.2) is obtained by first multiplying (2.6) with $\Delta W_{t_{i+1}^N}^{N,k}$ and then taking conditional expectation with respect to $\mathcal{F}_{t_i^N}^N$. (3.3) is a consequence of (2.6) and (3.1). \square

By condition (W1), there exists $N_0 \in \mathbb{N}$ such that $\max_i \Delta t_i^N < 1/K$ for all $N \geq N_0$. So it follows from the following proposition that for large enough N , the BSΔE (2.6)–(2.7) has a unique solution for every terminal condition.

Proposition 3.2 *If $\max_i \Delta t_i^N < 1/K$, the N -th BSΔE has for every terminal condition ξ^N a unique solution (Y^N, Z^N, M^N) .*

Proof. We show the proposition by backward induction. One must have $Y_T^N = \xi^N$, and if $Y_{t_{i+1}^N}^N$ is given, the only possible choice for $Z_{t_{i+1}^N}^N$ is (3.2). Since $\Delta t_i^N K < 1$, one obtains from (f3) that for every possible realization $(w(t_1^N), \dots, w(T)) \in \mathbb{R}^{d \times i_N}$ of $(W_{t_1^N}^N, \dots, W_T^N)$, there exists a unique $y \in \mathbb{R}$ such that

$$y - f^N(t_{i+1}^N, w, y, Z_{t_{i+1}^N}^N) \Delta t_{i+1}^N = \mathbb{E} \left[Y_{t_{i+1}^N}^N \mid \mathcal{F}_{t_i^N}^N \right].$$

This gives an $\mathcal{F}_{t_i^N}$ -measurable $Y_{t_i^N}^N$ random variable satisfying (3.1). Finally, one defines (M_t^N) through $M_0^N = 0$ and (3.3). Then (Y_t^N, Z_t^N, M_t^N) is the unique solution of the BSΔE (2.6)–(2.7). \square

Let us denote by g^N the convex conjugate of f^N with respect to z , given by

$$g^N(t, w, y, \mu) = \sup_{z \in \mathbb{R}^d} \{z\mu - f^N(t, w, y, z)\}, \quad \mu \in \mathbb{R}^d.$$

g^N is a mapping from $[0, T] \times \mathbb{R}^{d \times i_N} \times \mathbb{R} \times \mathbb{R}^d$ to $\mathbb{R} \cup \{\infty\}$, which inherits condition (f3') from f^N , that is,

$$|g^N(t_i, w_1, y_1, \mu) - g^N(t_i, w_2, y_2, \mu)| \leq K \left(\sup_{j \leq i-1} |w_1(t_j^N) - w_2(t_j^N)| + |y_1 - y_2| \right)$$

Our goal is to derive a convex dual representation of Y_t^N in terms of g^N . We need the following notation: Let (μ_t) be an (\mathcal{F}_t^N) -adapted \mathbb{R}^d -valued process constant on the intervals $(t_i^N, t_{i+1}^N]$ such that

$$\mu_{t_i} \Delta W_{t_i}^N > -1 \quad \text{for all } i. \quad (3.4)$$

Then

$$\frac{d\mathbb{P}^\mu}{d\mathbb{P}} = \prod_{i=1}^{i_N} (1 + \mu_{t_i} \Delta W_{t_i}^N) \quad (3.5)$$

defines a probability measure \mathbb{P}^μ equivalent to \mathbb{P} under which the processes

$$W_{t_i}^{N, \mu, k} := W_{t_i}^{N, k} - \sum_{j=1}^i \mu_{t_j}^k \Delta t_j^N, \quad i = 1, \dots, i_N, \quad k = 1, \dots, d,$$

are (\mathcal{F}_t^N) -martingales. Note that M^N is still a martingale under \mathbb{P}^μ . The following result gives an implicit convex dual representation of Y^N in terms of g^N . In Lemma 5.3 below we show an analogue in continuous time.

Lemma 3.3 *For every constant $C > 0$, there exists an $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ the following holds: If there is an $i \leq i_N - 1$ such that the N -th BS ΔE has a solution (Y^N, Z^N, M^N) satisfying $|Z_{t_j^N}^N| \leq C$ for all $j \geq i + 1$, then*

$$Y_{t_i}^N = \sup_{\mu} \mathbb{E}^\mu \left[\xi^N - \sum_{j=i+1}^{i_N} g^N(t_j^N, W^N, Y_{t_{j-1}^N}^N, \mu_{t_j^N}) \Delta t_j^N \mid \mathcal{F}_{t_i}^N \right], \quad (3.6)$$

where the supremum is taken over all (\mathcal{F}_t^N) -adapted \mathbb{R}^d -valued processes (μ_t) that are constant on the intervals $(t_j^N, t_{j+1}^N]$ and satisfy (3.4). Furthermore, the supremum is attained for some process (μ_t^*) .

Proof. First assume (Y^N, Z^N, M^N) is a solution of the N -th BS ΔE and μ is an (\mathcal{F}_t^N) -adapted \mathbb{R}^d -valued process that is constant on the intervals $(t_j^N, t_{j+1}^N]$ and satisfies (3.4). Since $(W_t^{N, \mu, k})$ is

for all $k = 1, \dots, d$, a martingale under \mathbb{P}^μ , one obtains for every $i \leq i_N - 1$,

$$\begin{aligned}
Y_{t_i^N}^N &= \mathbb{E}^\mu \left[\xi^N + \sum_{j=i+1}^{i_N} f^N(t_j^N, W^N, Y_{t_{j-1}^N}^N, Z_{t_j^N}^N) \Delta t_j^N \right. \\
&\quad \left. - \sum_{j=i+1}^{i_N} Z_{t_j^N}^N \Delta W_{t_j^N}^N - (M_T^N - M_{t_i^N}^N) \mid \mathcal{F}_{t_i^N}^N \right] \\
&= \mathbb{E}^\mu \left[\xi^N + \sum_{j=i+1}^{i_N} \left(f^N(t_j^N, W^N, Y_{t_{j-1}^N}^N, Z_{t_j^N}^N) - Z_{t_j^N}^N \mu_{t_j^N}^N \right) \Delta t_j^N \right. \\
&\quad \left. - \sum_{j=i+1}^{i_N} Z_{t_j^N}^N \left(\Delta W_{t_j^N}^N - \mu_{t_j^N}^N \Delta t_j^N \right) \mid \mathcal{F}_{t_i^N}^N \right] \\
&= \mathbb{E}^\mu \left[\xi^N + \sum_{j=i+1}^{i_N} \left(f^N(t_j^N, W^N, Y_{t_{j-1}^N}^N, Z_{t_j^N}^N) - Z_{t_j^N}^N \mu_{t_j^N}^N \right) \Delta t_j^N \mid \mathcal{F}_{t_i^N}^N \right] \\
&\geq \mathbb{E}^\mu \left[\xi^N - \sum_{j=i+1}^{i_N} g^N(t_j^N, W^N, Y_{t_{j-1}^N}^N, \mu_{t_j^N}^N) \Delta t_j^N \mid \mathcal{F}_{t_i^N}^N \right]. \tag{3.7}
\end{aligned}$$

Now let $C > 0$. By condition (f4'), there exists a constant $b \in \mathbb{R}_+$ such that

$$|f^N(t_i^N, w, y, z_1) - f^N(t_i^N, w, y, z_2)| \leq b|z_1 - z_2| \tag{3.8}$$

for all N, i, w, y , and $z_1, z_2 \in \mathbb{R}^d$ with $|z_1| \vee |z_2| \leq 2C$. Due to (W5) there exists a $D \in \mathbb{R}_+$ such that

$$\sup_{N, i, k} \frac{\left\| \Delta W_{t_i^N}^{N, k} \right\|_\infty}{\sqrt{\Delta t_i^N}} \leq D,$$

and by (W1), there is an $N_0 \in \mathbb{N}$ such that

$$\sup_i b \sqrt{d} D \sqrt{\Delta t_i^N} < 1 \quad \text{for all } N \geq N_0. \tag{3.9}$$

Fix $N \geq N_0$ and $i \in \{0, \dots, i_N - 1\}$. Assume (Y^N, Z^N, M^N) is a solution of the N -th BSDE such that $|Z_{t_j^N}^N| \leq C$ for all $j \geq i+1$. To see that inequality (3.7) is actually an equality for some process (μ_t^*) , note that the subdifferential $\partial f(t, w, y, z)$ of f with respect to z is non-empty for all (t, w, y, z) . For every $j \geq i+1$, the filtration $\mathcal{F}_{t_{j-1}^N}^N$ has only finitely many atoms B_1, \dots, B_m . On every atom B_l choose a vector $z_l \in \partial f(t_j^N, W^N, Y_{t_{j-1}^N}^N, Z_{t_j^N}^N)$. Set $\mu_{t_j^N}^* := z_l$ for $t \in (t_{j-1}^N, t_j^N]$ and $\omega \in B_l$ and $\mu_t^* := 0$ for $t \leq t_i^N$. Then (μ_t^*) is an (\mathcal{F}_t^N) -adapted \mathbb{R}^d -valued process constant on the intervals $(t_i^N, t_{i+1}^N]$ such that

$$Z_{t_j^N}^N \mu_{t_j^N}^* = f^N(t_j^N, W^N, Y_{t_{j-1}^N}^N, Z_{t_j^N}^N) + g^N(t_j^N, W^N, W^N, Y_{t_{j-1}^N}^N, \mu_{t_j^N}^*) \quad \text{for all } j \geq i+1.$$

It remains to show that μ^* satisfies condition (3.4). Then \mathbb{P}^{μ^*} is a probability measure equivalent to \mathbb{P} and for $\mu = \mu^*$, the inequality in (3.7) becomes an equality. But since $|Z_{t_j^N}^N| \leq C$ for all $j \geq i+1$, it follows from (3.8) that $|\mu_{t_j^N}^*| \leq b$, and one obtains

$$|\mu_{t_j^N}^* \Delta W_{t_j^N}^N| \leq b \sqrt{d} D \sqrt{\Delta t_j^N} < 1 \quad \text{for all } j \geq i+1.$$

□

4 Proof of Theorem 2.4

We need the following discrete-time version of Gronwall's lemma:

Lemma 4.1 *For every $B \in \mathbb{R}_+$ there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ the following holds: If $(X_t^N)_{t \in [0, T]}$ is a stochastic process that is constant on the intervals $[t_{i-1}^N, t_i^N)$ and satisfies*

$$|X_T^N| \leq A \quad \text{as well as} \quad |X_{t_i^N}^N| \leq A + B \sum_{j=i+1}^{i_N} |X_{t_{j-1}^N}^N| \Delta t_j^N, \quad i \leq i_N - 1 \quad \text{for some } A \in \mathbb{R}_+,$$

then

$$|X_t^N| \leq 2A \exp\{B(T-t)\}, \quad \text{for all } t \in [0, T].$$

Proof. If N is so large that $B \max_i \Delta t_i^N < 1$, then the unique process that is constant on the intervals $[t_{i-1}^N, t_i^N)$ and solves the deterministic backward equation

$$\hat{X}_T^N = A, \quad \hat{X}_{t_i^N}^N = A + B \sum_{j=i+1}^{i_N} \hat{X}_{t_{j-1}^N}^N \Delta t_j^N, \quad i \leq i_N - 1,$$

is given by

$$\hat{X}_t^N = A \quad \text{and} \quad \hat{X}_t^N = A \prod_{j: t_j^N > t} (1 - B \Delta t_j^N)^{-1} \quad \text{for } t < T.$$

Since $\prod_{j: t_j^N > t} (1 - B \Delta t_j^N)^{-1}$ is converging uniformly in t to $\exp(B(T-t))$, there exists $N_0 \in \mathbb{N}$ such that $B \max_i \Delta t_i^N < 1$ and $\prod_{j: t_j^N > t} (1 - B \Delta t_j^N)^{-1} \leq 2 \exp(B(T-t))$ for all $N \geq N_0$ and $t \in [0, T]$. Therefore, $\hat{X}_t^N \leq 2A \exp(B(T-t))$ for all $N \geq N_0$ and $t \in [0, T]$. It remains to show that $|X_t^N| \leq \hat{X}_t^N$. But this follows by backward induction from

$$|X_{t_i^N}^N| \leq \frac{A + B \sum_{j=i+2}^{i_N} |X_{t_{j-1}^N}^N| \Delta t_j^N}{1 - B \Delta t_{i+1}^N} \leq \frac{A + B \sum_{j=i+2}^{i_N} \hat{X}_{t_{j-1}^N}^N \Delta t_j^N}{1 - B \Delta t_{i+1}^N} = \hat{X}_{t_i^N}^N.$$

□

Lemma 4.2 *Assume all ξ^N are of the form $\xi^N = \varphi(\hat{W}^N)$ for a function $\varphi : C^d[0, T] \rightarrow \mathbb{R}$ for which there exists a constant $L \in \mathbb{R}_+$ such that*

$$|\varphi(w_1) - \varphi(w_2)| \leq L \|w_1 - w_2\|_\infty \tag{4.1}$$

for all $w_1, w_2 \in C^d[0, T]$. Then there exists an $N_0 \in \mathbb{N}$ such that for $N \geq N_0$, every solution (Y^N, Z^N, M^N) of the N -th BSΔE satisfies

$$\sup_{0 \leq t \leq T} |Z_t^N| \leq 2\sqrt{d}(L + KT) \exp(KT). \tag{4.2}$$

Proof. Choose $N_0 \in \mathbb{N}$ so large that the statement of Lemma 3.3 holds for $C = 2\sqrt{d}(L + KT) \exp(KT)$ and the statement of Lemma 4.1 holds for $B = K$. Assume (Y^N, Z^N, M^N) is a solution of the N -th BSΔE for some $N \geq N_0$. We prove (4.2) by backward induction. Fix $i \in \{1, \dots, i_N\}$ and if $i \leq i_N - 1$, assume

$$|Z_{t_j^N}^N| \leq 2\sqrt{d}(L + KT) \exp(KT) \quad \text{for all } j \geq i + 1.$$

There exist functions $\varphi^N : \mathbb{R}^{d \times i_N} \rightarrow \mathbb{R}$ such that $\varphi^N(W_{t_1^N}^N, \dots, W_T^N) = \varphi(\hat{W}^N)$ and

$$|\varphi^N(w_1, \dots, w_{i_N}) - \varphi^N(w'_1, \dots, w'_{i_N})| \leq L \sup_{i=1, \dots, i_N} |w_i - w'_i| \quad (4.3)$$

for all $w_1, \dots, w_{i_N}, w'_1, \dots, w'_{i_N} \in \mathbb{R}^d$. Choose $x_1, \dots, x_i \in \mathbb{R}^d$ such that

$$\mathbb{P}[(W_{t_1^N}^N, \dots, W_{t_i^N}^N) = (x_1, \dots, x_i)] > 0$$

and denote by $(Y_t^{N,x}, Z_t^{N,x}, M_t^{N,x})_{t \geq t_i^N}$ the solution (Y_t^N, Z_t^N, M_t^N) conditioned on

$$(W_{t_1^N}^N, \dots, W_{t_i^N}^N) = (x_1, \dots, x_i).$$

It is adapted to the filtration $(\tilde{\mathcal{F}}_t)_{t \geq t_i^N}$ generated by the d -dimensional Brownian motion

$$\tilde{W}_t^N := W_t^N - W_{t_i^N}^N, \quad t \geq t_i^N,$$

and solves the BSΔE

$$\begin{aligned} Y_{t_j^N}^{N,x} &= Y_{t_{j+1}^N}^{N,x} + f^N(t_{j+1}^N, x_1, \dots, x_{i-1}, x_i + \tilde{W}^N, Y_{t_j^N}^{N,x}, Z_{t_{j+1}^N}^{N,x}) \Delta t_{j+1}^N \\ &\quad - Z_{t_{j+1}^N}^{N,x} \Delta \tilde{W}_{t_{j+1}^N}^N - (M_{t_{j+1}^N}^{N,x} - M_{t_j^N}^{N,x}) \end{aligned} \quad (4.4)$$

$$Y_T^{N,x} = \xi^{N,x}, \quad (4.5)$$

where

$$\xi^{N,x} = \varphi^N(x_1, \dots, x_{i-1}, x_i + \tilde{W}^N).$$

Now let $x'_i \in \mathbb{R}^d$ such that

$$\mathbb{P}[(W_{t_1^N}^N, \dots, W_{t_i^N}^N) = (x_1, \dots, x_{i-1}, x'_i)] > 0$$

and denote $x' = (x_1, \dots, x_{i-1}, x'_i)$. If $i = i_N$, one obtains directly from (4.3) that

$$|Y_T^{N,x'} - Y_T^{N,x}| = |\varphi^N(x_1, \dots, x'_{i_N}) - \varphi^N(x_1, \dots, x_{i_N})| \leq L|x'_{i_N} - x_{i_N}|.$$

If $i \leq i_N - 1$, note that $|\max[a_1, a_2] - \max[b_1, b_2]| \leq \max[|a_1 - b_1|, |a_2 - b_2|]$ for all $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Therefore, one obtains from Lemma 3.3 for all $j \geq i$,

$$\begin{aligned} &|Y_{t_j^N}^{N,x'} - Y_{t_j^N}^{N,x}| \\ &= \left| \max_{\mu \in \{\mu', \mu^*\}} \mathbb{E}^\mu \left[\xi^{N,x'} - \sum_{l=j+1}^{i_N} g^N(t_l^N, x_1, \dots, x_{i-1}, x'_i + \tilde{W}^N, Y_{t_{l-1}^N}^{N,x'}, \mu_{t_l^N}) \Delta t_l^N \middle| \tilde{\mathcal{F}}_{t_j^N}^N \right] \right. \\ &\quad \left. - \max_{\mu \in \{\mu', \mu^*\}} \mathbb{E}^\mu \left[\xi^{N,x} - \sum_{l=j+1}^{i_N} g^N(t_l^N, x_1, \dots, x_{i-1}, x_i + \tilde{W}^N, Y_{t_{l-1}^N}^{N,x}, \mu_{t_l^N}) \Delta t_l^N \middle| \tilde{\mathcal{F}}_{t_j^N}^N \right] \right| \\ &\leq \max_{\mu \in \{\mu', \mu^*\}} \mathbb{E}^\mu \left[|\xi^{N,x'} - \xi^{N,x}| + \sum_{l=j+1}^{i_N} |g^N(t_l^N, x_1, \dots, x_{i-1}, x'_i + \tilde{W}^N, Y_{t_{l-1}^N}^{N,x'}, \mu_{t_l^N}) \right. \\ &\quad \left. - g^N(t_l^N, x_1, \dots, x_{i-1}, x_i + \tilde{W}^N, Y_{t_{l-1}^N}^{N,x}, \mu_{t_l^N})| \Delta t_l^N \middle| \tilde{\mathcal{F}}_{t_j^N}^N \right] \\ &\leq (L + KT)|x'_i - x_i| + K \sum_{l=j+1}^{i_N} \left\| Y_{t_{l-1}^N}^{N,x'} - Y_{t_{l-1}^N}^{N,x} \right\|_\infty \Delta t_l^N. \end{aligned}$$

It follows from Lemma 4.1 that

$$\left\| Y_{t_i^N}^{N,x'} - Y_{t_i^N}^{N,x} \right\|_\infty \leq 2(L + KT) \exp(KT) |x' - x|. \quad (4.6)$$

To see that this implies $|Z_{t_i^N}^N| \leq 2\sqrt{d}(L + KT) \exp(KT)$, note that because $Y_{t_i^N}^N$ is $(\mathcal{F}_{t_i^N}^N)$ -measurable, there exist functions $y_{t_i^N}^N : \mathbb{R}^{i \times d} \rightarrow \mathbb{R}$ such that

$$Y_{t_i^N}^N = y_{t_i^N}^N(W_{t_1^N}^N, \dots, W_{t_i^N}^N).$$

So the components of $Z_{t_i^N}^N$ satisfy

$$\begin{aligned} |Z_{t_i^N}^{N,k}| &= \frac{\left| \mathbb{E} \left[Y_{t_i^N}^N \Delta W_{t_i^N}^{N,k} \middle| \mathcal{F}_{t_{i-1}^N}^N \right] \right|}{\Delta t_i^N} \\ &= \frac{\left| \mathbb{E} \left[\left(y_{t_i^N}^N(W_{t_1^N}^N, \dots, W_{t_{i-1}^N}^N, W_{t_i^N}^N) - y_{t_i^N}^N(W_{t_1^N}^N, \dots, W_{t_{i-1}^N}^N, W_{t_{i-1}^N}^N) \right) \Delta W_{t_i^N}^{N,k} \middle| \mathcal{F}_{t_{i-1}^N}^N \right] \right|}{\Delta t_i^N} \\ &\leq \frac{2(L + KT) \exp(KT)}{\Delta t_i^N} \mathbb{E} \left[|\Delta W_{t_i^N}^{N,k}|^2 \right] = 2(L + KT) \exp(KT), \end{aligned}$$

which entails $|Z_{t_i^N}^N| \leq 2\sqrt{d}(L + KT) \exp(KT)$. \square

Lemma 4.3 *Assume (Y, Z) is a solution of the BSDE (1.1) corresponding to a bounded terminal condition such that Z is bounded and f satisfies (f2)–(f4). Then Y is bounded.*

Proof. Since Z is bounded, one can assume without loss of generality that the driver f is Lipschitz in y and z with Lipschitz-constant $b \in \mathbb{R}_+$. By condition (f2), there exists a constant $a \in \mathbb{R}_+$ such that $f(t, w, 0, 0) \leq a$ for all t and w . Therefore,

$$f(t, w, y, z) \leq f'(t, w, y, z) := a + b(|y| + |z|).$$

Since f' is Lipschitz in y and z , it follows from Pardoux and Peng (1990) that the BSDE with driver f' and terminal condition $\hat{\xi} := \|\xi\|_\infty$ has a unique solution (\hat{Y}, \hat{Z}) , which is easily verified to be

$$\hat{Y}_t = \left(\hat{\xi} + \frac{a}{b} \right) e^{b(T-t)} - \frac{a}{b}, \quad \hat{Z}_t = 0,$$

and it follows from the comparison result shown in El Karoui et al. [13] that $Y_t \leq \hat{Y}_t$ for all t . Similarly, one obtains that Y is bounded from below. \square

Proof of Theorem 2.4.

By assumption, there exists a constant $L \in \mathbb{R}_+$ such that $|\varphi(w_1) - \varphi(w_2)| \leq L \|w_1 - w_2\|_\infty$ for all $w_1, w_2 \in C^d[0, T]$. It follows from Proposition 3.2 and Lemma 4.2 that there exists an $N_0 \in \mathbb{N}$ such that for every $N \geq N_0$, the N -th BSDE has a unique solution (Y^N, Z^N, M^N) and

$$\sup_{0 \leq t \leq T} |Z_t^N| \leq 2\sqrt{d}(L + KT) \exp(KT).$$

One can choose a function $\tilde{f} : [0, T] \times C^d[0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ that agrees with f for $|z| \leq 2\sqrt{d}(L + KT) \exp(KT)$, satisfies (f1)–(f3) and is Lipschitz-continuous in z . From Pardoux and Peng [17] one

obtains that the BSDE (1.1) with driver \tilde{f} has a unique solution (Y, Z) , and it is a consequence of Theorem 12 of Briand et al. [5] that

$$\sup_t \left(|Y_t^N - Y_t| + \left| \int_0^t Z_s^N dW_s^N - \int_0^t Z_s dW_s \right| + |M_t^N| \right) \rightarrow 0 \quad \text{in } L^2,$$

and

$$\sup_t \left(\sum_{k=1}^d \left| \int_0^t Z_s^{N,k} d\langle W^N \rangle_s - \int_0^t Z_s^k ds \right|^2 + \left| \int_0^t |Z_s^N|^2 d\langle W^N \rangle_s - \int_0^t |Z_s|^2 ds \right| \right) \rightarrow 0 \quad \text{in } L^1. \quad (4.7)$$

(Briand et al. [5] prove this result for the case where the Brownian motion W is one-dimensional and drivers are RCLL. But we show in the Appendix that it also holds in our setup.) It follows from (4.7) that

$$|Z_t| \leq 2\sqrt{d}(L + KT) \exp(KT) \quad dt \times d\mathbb{P}\text{-almost everywhere.}$$

So (Y, Z) is also a solution of the BSDE (1.1) with driver f .

If one replaces f by a driver $f' \geq f$ satisfying (f1)–(f4) and ξ by a terminal condition $\xi' \geq \xi$ of the form $\xi' = \varphi'(W)$ for a Lipschitz-continuous function $\varphi' : C^d[0, T] \rightarrow \mathbb{R}$, the BSDE (1.1) has a solution (Y', Z') such that Z' is bounded by a constant $C \in \mathbb{R}_+$. So one can modify f and f' for

$$|z| > C \vee 2\sqrt{d}(L + KT) \exp(KT)$$

such that they satisfy (f1)–(f3) and are Lipschitz-continuous in z . But then it follows from the comparison result proved in El Karoui et al. [13] that $Y'_t \geq Y_t$ for all t . In particular, (Y, Z) is the only solution of (1.1) such that Z is bounded. Finally, if φ is bounded, one obtains from Lemma 4.3 that Y is bounded as well. \square

5 Convex duality and comparison

As in the discrete-time case we exploit the convexity of f to obtain convex dual representations for solutions of BSDEs (see Lemma 5.3 below). To do that we denote the set of all d -dimensional BMO processes μ by BMO . The norm $\|\mu\|_{\text{BMO}}$ is the smallest number c such that

$$\sqrt{\mathbb{E} \left[\int_{\tau}^T |\mu_s|^2 ds \middle| \mathcal{F}_{\tau} \right]} \leq c$$

for all stopping times τ taking values in $[0, T]$. It is well-known from Kazamaki [15] that for every $\mu \in \text{BMO}$,

$$\Gamma_t^{\mu} = \exp \left(\int_0^t \mu_s dW_s - \frac{1}{2} \int_0^t |\mu_s|^2 ds \right), \quad 0 \leq t \leq T,$$

is a martingale. By Girsanov's theorem, $\mathbb{P}^{\mu} = \Gamma_T^{\mu} \cdot \mathbb{P}$ defines a probability measure equivalent to \mathbb{P} under which $W_t^{\mu} = W_t - \int_0^t \mu_s ds$ is a d -dimensional Brownian motion. Moreover, every BMO process with respect to \mathbb{P} is also a BMO process with respect to \mathbb{P}^{μ} .

Before we can turn to convex dual representations, we need the following technical

Lemma 5.1 *Let Y^n , $n \in \mathbb{N}$, be a sequence of (\mathcal{F}_t) -semimartingales with canonical decompositions*

$$Y_t^n = Y_0^n + U_t^n + V_t^n.$$

Assume the Y^n are uniformly bounded by a constant $C \in \mathbb{R}_+$ and there exists $b \in \mathbb{R}_+$ such that for all $n \in \mathbb{N}$, $V_t^n + bt$ is increasing. Then there exist BMO processes Z^n such that $U_t^n = \int_0^t Z_s^n dW_s$ and

$$\mathbb{E} \left[\int_{\tau}^T |Z_s^n|^2 ds \mid \mathcal{F}_{\tau} \right] + \mathbb{E} \left[\int_0^T |dV_s^n| \right] \leq 4e^{2C+2|b|T} + |b|T \quad (5.1)$$

for all stopping times τ and $n \in \mathbb{N}$. In particular, $\sup_n \|Z^n\|_{\text{BMO}} < \infty$.

Proof. The canonical decomposition of the semimartingale $\tilde{Y}_t^n = Y_t^n + bt$ is

$$\tilde{Y}_t^n = Y_0^n + U_t^n + \tilde{V}_t^n,$$

for the increasing finite variation process $\tilde{V}_t^n = V_t^n + bt$. Since (W_t) has the predictable representation property, there exist \mathbb{R}^d -valued (\mathcal{F}_t) -predictable processes Z^n such that $U_t^n = \int_0^t Z_s^n dW_s$. In particular, U_t^n is continuous. Hence, $\Delta \tilde{Y}_t^n = \Delta \tilde{V}_t^n \geq 0$ for all t . For fixed $n \in \mathbb{N}$, let $\sigma_m, m \in \mathbb{N}$, be an increasing sequence of $[0, T]$ -valued stopping times such that $\mathbb{P}[\sigma_m = T] \uparrow 1$ and $U_{t \wedge \sigma_m}^n$ is a martingale for every m . It follows from Itô's formula that for every $[0, T]$ -valued stopping time τ ,

$$\begin{aligned} \exp(\tilde{Y}_{\sigma_m}^n) &= \exp(\tilde{Y}_{\tau \wedge \sigma_m}^n) + \int_{\tau}^T \exp(\tilde{Y}_s^n) dU_{s \wedge \sigma_m}^n + \frac{1}{2} \int_{\tau}^T \exp(\tilde{Y}_s^n) d\langle U^n \rangle_{s \wedge \sigma_m} \\ &+ \int_{\tau+}^T \exp(\tilde{Y}_{s-}^n) d\tilde{V}_{s \wedge \sigma_m}^n + \sum_{\tau < s \leq \sigma_m} \Delta \exp(\tilde{Y}_s^n) - \exp(\tilde{Y}_{s-}^n) \Delta \tilde{Y}_s^n. \end{aligned}$$

Since $\sum_{\tau < s \leq \sigma_m} \Delta \exp(\tilde{Y}_s^n) - \exp(\tilde{Y}_{s-}^n) \Delta \tilde{Y}_s^n \geq 0$, one can take conditional expectation to obtain

$$\mathbb{E}_{\mathcal{F}_{\tau \wedge \sigma_m}} \left[\exp(\tilde{Y}_{\sigma_m}^n) \right] \geq \mathbb{E}_{\mathcal{F}_{\tau \wedge \sigma_m}} \left[\int_{\tau}^T \frac{1}{2} \exp(\tilde{Y}_s^n) d\langle U^n \rangle_{s \wedge \sigma_m} + \int_{\tau+}^T \exp(\tilde{Y}_{s-}^n) d\tilde{V}_{s \wedge \sigma_m}^n \right].$$

But since $\langle U^n \rangle$ and \tilde{V}^n are increasing and \tilde{Y}^n is bounded by $\tilde{C} = C + |b|T$, one obtains

$$\exp(\tilde{C}) \geq \frac{1}{2} \exp(-\tilde{C}) \mathbb{E}_{\mathcal{F}_{\tau \wedge \sigma_m}} \left[\langle U^n \rangle_{\sigma_m} - \langle U^n \rangle_{\tau \wedge \sigma_m} + \tilde{V}_{\sigma_m}^n - \tilde{V}_{\tau \wedge \sigma_m}^n \right],$$

and therefore,

$$\mathbb{E}_{\mathcal{F}_{\tau \wedge \sigma_m}} \left[\langle U^n \rangle_{\sigma_m} - \langle U^n \rangle_{\tau \wedge \sigma_m} + \tilde{V}_{\sigma_m}^n - \tilde{V}_{\tau \wedge \sigma_m}^n \right] \leq 2e^{2\tilde{C}}. \quad (5.2)$$

By choosing $\tau = 0$ and letting m converge to infinity, one obtains from Beppo Levi's monotone convergence theorem that

$$\mathbb{E} [\langle U^n \rangle_T] \leq 2e^{2\tilde{C}},$$

which, by the Burkholder–Davis–Gundy inequality, implies that U is a square-integrable martingale. So one may choose $\sigma_m = T$, and it follows from (5.2) that

$$\mathbb{E}_{\mathcal{F}_{\tau}} \left[\langle U^n \rangle_T - \langle U^n \rangle_{\tau} + \tilde{V}_T^n - \tilde{V}_{\tau}^n \right] \leq 2e^{2\tilde{C}}. \quad (5.3)$$

Using $\langle U^n \rangle_T - \langle U^n \rangle_{\tau} = \int_{\tau}^T |Z_s^n|^2 ds$ and the fact that \tilde{V} is increasing, one obtains

$$\mathbb{E}_{\mathcal{F}_{\tau}} \left[\int_{\tau}^T |Z_s^n|^2 ds \right] + \mathbb{E} \left[\int_0^T |d\tilde{V}_s^n| \right] \leq 4e^{2\tilde{C}},$$

which implies (5.1). \square

Remark 5.2 By replacing Y with $-Y$, one sees that Lemma 5.1 also holds if there exist constants C and b such that for every $n \in \mathbb{N}$, Y^n is bounded by C and the process $A_t^n + bt$ is decreasing.

Let us denote by g the convex conjugate of f with respect to z , that is,

$$g(t, w, y, \mu) = \sup_z \{z\mu - f(t, w, y, z)\}, \quad \mu \in \mathbb{R}^d.$$

g maps $[0, T] \times C^d[0, T] \times \mathbb{R} \times \mathbb{R}^d$ to $\mathbb{R} \cup \{\infty\}$ and inherits condition (f3) from f :

$$|g(t, w_1, y_1, \mu) - g(t, w_2, y_2, \mu)| \leq K \left(\sup_{0 \leq s \leq t} |w_1(s) - w_2(s)| + |y_1 - y_2| \right)$$

for all $t, w_1, w_2, y_1, y_2, \mu$.

In the next lemma we give a convex dual representation result for solutions of the continuous-time BSDE (1.1) analogous to the discrete-time representation of Lemma 3.3. If f does not depend on y , the representation is explicit and coincides with the ones in Barrieu and El Karoui [2] and Delbaen et al. [12, 10]. But if f depends on y , it is implicit as in Lemma 3.3. For an explicit convex dual representation in the case where f depends on y we refer to Delbaen et al. [11].

Lemma 5.3 *Suppose that (Y, Z, A) is a supersolution of the BSDE (1.1) such that Y is bounded. If f satisfies (f5) or Z is BMO, then*

$$Y_\sigma \geq \mathbb{E}_{\mathcal{F}_\sigma}^\mu \left[Y_\tau - \int_\sigma^\tau g(s, W, Y_s, \mu_s) ds \right] \quad (5.4)$$

for every $\mu \in \text{BMO}$ and all stopping times $0 \leq \sigma \leq \tau \leq T$. If (Y, Z) is a solution of the BSDE such that Z is bounded and f satisfies (f1), (f4) and (f5), there exists a bounded \mathbb{R}^d -valued (\mathcal{F}_t) -predictable process μ^* such that

$$Y_\sigma = \mathbb{E}_{\mathcal{F}_\sigma}^{\mu^*} \left[Y_\tau - \int_\sigma^\tau g(s, W, Y_s, \mu_s^*) ds \right] \quad (5.5)$$

for all stopping times $0 \leq \sigma \leq \tau \leq T$.

Proof. If Y is bounded and f satisfies condition (f5), one obtains from Lemma 5.1 and Remark 5.2 applied to $Y_t^n = Y_t$, $U_t^n = \int_0^t Z_s dW_s$ and $V_t^n = -\int_0^t f(s, W, Y_s, Z_s) ds - A_t$ that Z is BMO. But then it is also BMO with respect to \mathbb{P}^μ for every $\mu \in \text{BMO}$; see Section 3.3 of Kazamaki [15]. It follows that

$$\begin{aligned} Y_\sigma &= \mathbb{E}_{\mathcal{F}_\sigma}^\mu \left[Y_\tau + \int_\sigma^\tau f(s, W, Y_s, Z_s) ds - \int_\sigma^\tau Z_s dW_s + A_\tau - A_\sigma \right] \\ &\geq \mathbb{E}_{\mathcal{F}_\sigma}^\mu \left[Y_\tau - \int_\sigma^\tau [\mu_s Z_s - f(s, W, Y_s, Z_s)] ds - \int_\sigma^\tau Z_s (dW_s - \mu_s ds) \right] \end{aligned} \quad (5.6)$$

$$\begin{aligned} &= \mathbb{E}_{\mathcal{F}_\sigma}^\mu \left[Y_\tau - \int_\sigma^\tau [\mu_s Z_s - f(s, W, Y_s, Z_s)] ds \right] \\ &\geq \mathbb{E}_{\mathcal{F}_\sigma}^\mu \left[Y_\tau - \int_\sigma^\tau g(s, W, Y_s, \mu_s) ds \right]. \end{aligned} \quad (5.7)$$

Of course, (5.6) becomes an equality if Y is not only a supersolution but a true solution. Furthermore, if f satisfies (f1), it follows from Cheridito et al. [8] that there exists an \mathbb{R}^d -valued (\mathcal{F}_t) -predictable process (μ_t^*) such that μ_t^* is in the subgradient $\partial f(t, W, Y_t, Z_t)$ of f with respect to z for $dt \times d\mathbb{P}$ -almost all (t, ω) . If Z_t is bounded, it follows from (f4) that μ^* is bounded too. So \mathbb{P}^{μ^*} is a well-defined probability measure, and inequality (5.7) becomes an equality for $\mu = \mu^*$. \square

Definition 5.4 We say a supersolution (Y, Z, A) of the BSDE (1.1) satisfies assumption (A) if for every constant $\varepsilon > 0$, there exists a $\mu \in \text{BMO}$ such that

$$Y_t \leq \mathbb{E}_{\mathcal{F}_t}^\mu \left[\xi - \int_t^T g(s, W, Y_s, \mu_s) ds \right] + \varepsilon \quad \text{for all } 0 \leq t \leq T. \quad (5.8)$$

Note that if (Y, Z, A) is a supersolution of the BSDE (1.1) satisfying assumption (A) such that Y is bounded, then

$$Y_t \leq \text{ess sup}_{\mu \in \text{BMO}} \mathbb{E}_{\mathcal{F}_t}^\mu \left[\xi - \int_t^T g(s, W, Y_s, \mu_s) ds \right] \quad \text{for all } 0 \leq t \leq T.$$

The following proposition gives a comparison result:

Proposition 5.5 Assume f is increasing in y and (Y, Z, A) is a supersolution of the BSDE (1.1) such that Y is bounded and fulfils assumption (A). Then if (Y', Z', A') is a supersolution of (1.1) with bounded terminal condition $\xi' \geq \xi$ and driver $f' \geq f$ satisfying (f5) such that Y' is bounded, one has $Y'_t \geq Y_t$ for all $0 \leq t \leq T$.

Proof. Fix $\varepsilon > 0$. There exists a BMO process μ such that for all $t \in [0, T]$,

$$Y_t \leq \mathbb{E}_{\mathcal{F}_t}^\mu \left[\xi - \int_t^T g(s, W, Y_s, \mu_s) ds \right] + \varepsilon \quad \text{for all } 0 \leq t \leq T.$$

Define

$$g'(t, w, y, \mu) = \sup_{z \in \mathbb{R}^d} \{ \mu z - f'(t, w, y, z) \}, \quad \mu \in \mathbb{R}^d.$$

Since $f' \geq f$, one has $g' \leq g$, and therefore,

$$Y'_t - \mathbb{E}_{\mathcal{F}_t}^\mu \left[\xi' - \int_t^T g(s, W, Y'_s, \mu_s) ds \right] \geq Y'_t - \mathbb{E}_{\mathcal{F}_t}^\mu \left[\xi' - \int_t^T g'(s, W, Y'_s, \mu_s) ds \right] \geq 0,$$

where the last inequality follows from Lemma 5.3. Since f is increasing in y , g is decreasing in y . So $g(t, w, y_1, z) - g(t, w, y_2, z) \leq 0$ for all $y_1 \geq y_2$. On the other hand, if $y_1 \leq y_2$, one has

$$0 \leq g(t, w, y_1, z) - g(t, w, y_2, z) \leq K(y_2 - y_1).$$

Hence,

$$(g(t, w, y_1, z) - g(t, w, y_2, z))^+ \leq K(y_2 - y_1)^+ \quad \text{for all } y_1, y_2 \in \mathbb{R}. \quad (5.9)$$

It follows that

$$\begin{aligned} & (Y_t - Y'_t)^+ \\ & \leq \left(\varepsilon + \mathbb{E}_{\mathcal{F}_t}^\mu \left[\xi - \int_t^T g(s, W, Y_s, \mu_s) ds \right] - \mathbb{E}_{\mathcal{F}_t}^\mu \left[\xi' - \int_t^T g(s, W, Y'_s, \mu_s) ds \right] \right)^+ \\ & \leq \varepsilon + \mathbb{E}_{\mathcal{F}_t}^\mu \left[\int_t^T (g(s, W, Y'_s, \mu_s) - g(s, W, Y_s, \mu_s))^+ ds \right] \\ & \leq \varepsilon + \mathbb{E}_{\mathcal{F}_t}^\mu \left[\int_t^T K(Y_s - Y'_s)^+ ds \right]. \end{aligned}$$

In particular,

$$\mathbb{E}^\mu [(Y_t - Y'_t)^+] \leq \varepsilon + K \int_t^T \mathbb{E}^\mu [(Y_s - Y'_s)^+] ds \quad \text{for all } t,$$

and one obtains from Gronwall's Lemma that

$$\mathbb{E}^\mu [(Y_t - Y'_t)^+] \leq \varepsilon \exp\{K(T - t)\}, \quad 0 \leq t \leq T.$$

Since $\varepsilon > 0$ can be chosen arbitrarily, one gets $Y_t \leq Y'_t$ for all t . \square

The following proposition gives a comparison result for the case when f is decreasing in y :

Proposition 5.6 *Assume f is decreasing in y and (Y, Z, A) is a supersolution of the BSDE (1.1) such that Y is bounded and satisfies assumption (A). If (Y', Z', A') is a supersolution of (1.1) with bounded terminal condition $\xi' \geq \xi$ and driver $f' \geq f$ satisfying (f5) such that Y' is bounded, then $Y'_t \geq Y_t$ for all $0 \leq t \leq T$.*

Proof. We prove this proposition by contradiction. Set

$$g'(t, w, y, \mu) = \sup_z \{\mu z - f'(t, w, y, z)\}, \quad \mu \in \mathbb{R}^d.$$

Since $f' \geq f$, one has $g' \leq g$. Assume that there exists $t \in [0, T]$ such that $\mathbb{P}[Y'_t < Y_t] > 0$ and define $\tau := \inf\{s > t : Y'_s \geq Y_s\}$. Since $Y'_T = \xi' \geq \xi = Y_T$, one has $t \leq \tau \leq T$. By conditioning on $\{Y'_t < Y_t\}$, one can assume that $\mathbb{P}[Y'_t < Y_t] = 1$. Then

$$\begin{aligned} & \text{ess sup}_{\mu \in \text{BMO}} \mathbb{E}_{\mathcal{F}_t}^\mu \left[Y'_\tau - \int_t^\tau g(s, W, Y'_s, \mu_s) ds \right] \\ & \leq \text{ess sup}_{\mu \in \text{BMO}} \mathbb{E}_{\mathcal{F}_t}^\mu \left[Y'_\tau - \int_t^\tau g'(s, W, Y'_s, \mu_s) ds \right] \leq Y'_t \\ & < Y_t \leq \text{ess sup}_{\mu \in \text{BMO}} \mathbb{E}_{\mathcal{F}_t}^\mu \left[Y_\tau - \int_t^\tau g(s, W, Y_s, \mu_s) ds \right]. \end{aligned}$$

However, since f is decreasing in y , g is increasing in y . By the definition of τ , one has $Y'_s \leq Y_s$ for $t \leq s < \tau$ and hence,

$$\int_t^\tau g(s, W, Y'_s, \mu_s) ds \leq \int_t^\tau g(s, W, Y_s, \mu_s) ds.$$

On the other hand, $Y_\tau \leq Y'_\tau$, and therefore,

$$\mathbb{E}_{\mathcal{F}_t}^\mu \left[Y'_\tau - \int_t^\tau g(s, W, Y'_s, \mu_s) ds \right] \geq \mathbb{E}_{\mathcal{F}_t}^\mu \left[Y_\tau - \int_t^\tau g(s, W, Y_s, \mu_s) ds \right]$$

for all $\mu \in \text{BMO}$, a contradiction. \square

6 Proofs of Theorems 2.5 and 2.7

For $p \in [1, \infty]$, denote by \mathcal{S}^p the space of all (\mathcal{F}_t) -semimartingales X such that

$$\|X\|_{\mathcal{S}^p} := \left\| \sup_{0 \leq t \leq T} |X_t| \right\|_{L^p} < \infty,$$

and by \mathcal{H}^p the space of all special (\mathcal{F}_t) -semimartingales X with canonical decomposition $X = X_0 + U + V$ satisfying

$$\|X\|_{\mathcal{H}^p} := \|X_0\|_{L^p} + \left\| [U, U]_T^{1/2} \right\|_{L^p} + \left\| \int_0^T |dV_s| \right\|_{L^p} < \infty.$$

Lemma 6.1 *Let (Y^n, Z^n) , $n = 1, 2$, be solutions of the BSDE (1.1) corresponding to bounded terminal conditions ξ^n such that Z^n are bounded and f satisfies (f1)–(f5). Then Y^1 and Y^2 are bounded and*

$$\|Y^1 - Y^2\|_{\mathcal{S}^\infty} \leq \exp\{KT\} \|\xi^1 - \xi^2\|_{L^\infty}.$$

Proof. By Lemma 4.3, Y^1 and Y^2 are bounded. So it follows from Lemma 5.3 that there exist bounded \mathbb{R}^d -valued (\mathcal{F}_t) -predictable processes μ^n , $n = 1, 2$, such that

$$\begin{aligned} Y_t^n &= \operatorname{ess\,sup}_{\mu \in \text{BMO}} \mathbb{E}_{\mathcal{F}_t}^\mu \left[\xi^n - \int_t^T g(s, W, Y_s^n, \mu_s) ds \right] \\ &= \mathbb{E}_{\mathcal{F}_t}^{\mu^n} \left[\xi^n - \int_t^T g(s, W, Y_s^n, \mu_s^n) ds \right], \end{aligned}$$

and one obtains as in the proof of Lemma 4.2 that

$$\begin{aligned} |Y_t^1 - Y_t^2| &\leq \sup_{\mu \in \{\mu^1, \mu^2\}} \mathbb{E}_{\mathcal{F}_t}^\mu \left[|\xi^1 - \xi^2| + \int_t^T |g(s, W, Y_s^1, \mu_s) - g(s, W, Y_s^2, \mu_s)| ds \right] \\ &\leq \|\xi^1 - \xi^2\|_\infty + K \int_t^T \|Y_s^1 - Y_s^2\|_\infty ds. \end{aligned}$$

Now the lemma follows from Gronwall's lemma. \square

We need the following result of Barlow and Protter [1]:

Theorem 6.2 (Barlow and Protter [1])

Let $(Y_t^n)_{0 \leq t \leq T}$, $n \in \mathbb{N}$, be a sequence of semimartingales in \mathcal{H}^1 over a filtered probability space with canonical decompositions $Y^n = Y_0^n + U^n + V^n$ such that

$$\sup_n \|U^n\|_{\mathcal{S}^1} \leq K \quad \text{and} \quad \sup_n \|V^n\|_{\mathcal{H}^1} \leq K \quad \text{for some } K \in \mathbb{R}_+ \quad (6.1)$$

and Y a RCLL process on the same probability space such that

$$\lim_{n \rightarrow \infty} \|Y^n - Y\|_{\mathcal{S}^1} = 0.$$

Then Y is a semimartingale in \mathcal{H}^1 with canonical decomposition $Y = Y_0 + U + V$ satisfying

$$\|U\|_{\mathcal{S}^1} \leq K, \quad \|V\|_{\mathcal{H}^1} \leq K$$

and

$$\lim_{n \rightarrow \infty} \|U^n - U\|_{\mathcal{H}^1} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|V^n - V\|_{\mathcal{S}^1} = 0.$$

Now we are ready for the proof of Theorem 2.5:

Proof of Theorem 2.5. It follows from Theorem 2.4 that Y^n is bounded for all n and from Lemma 6.1 that

$$\|Y^m - Y^n\|_{\mathcal{S}^\infty} \leq \exp\{KT\} \|\xi^m - \xi^n\|_{L^\infty}.$$

Hence, Y^n is a Cauchy sequence in \mathcal{S}^∞ . So there exists a continuous process $Y \in \mathcal{S}^\infty$ such that $\|Y^n - Y\|_{\mathcal{S}^\infty} \rightarrow 0$ for $n \rightarrow \infty$. It follows that $Y_T = \xi$. To see that Y is a supersolution of the BSDE (1.1), note that Y^n is a continuous semimartingale with canonical decomposition $Y^n = Y_0^n + U^n + V^n$, where

$$U_t^n = \int_0^t Z_s^n dW_s \quad \text{and} \quad V_t^n = - \int_0^t f(s, W, Y_{n,s}, Z_s^n) ds.$$

Due to (f5) and the fact that the Y^n are uniformly bounded it follows from Lemma 5.1 and Remark 5.2 that there exists a constant C such that

$$\mathbb{E} \left[\int_{\tau}^T |Z_s^n|^2 ds \mid \mathcal{F}_{\tau} \right] + \mathbb{E} \left[\int_0^T |f(s, W, Y_s^n, Z_s^n)| ds \right] \leq C$$

for all n and every stopping time τ . In particular, $\sup_n \|Z^n\|_{\text{BMO}} < \infty$ and $\sup_n \|V^n\|_{\mathcal{H}^1} < \infty$. It follows that $\sup_n \|U^n\|_{\mathcal{H}^2} < \infty$, which implies that $Y^n \in \mathcal{H}^1$ and $\sup_n \|U^n\|_{\mathcal{S}^1} < \infty$. So the assumptions of Theorem 6.2 are satisfied, and it follows that Y is a semimartingale in \mathcal{H}^1 with canonical decomposition $Y_t = Y_0 + U_t + V_t$ such that $U^n \rightarrow U$ in \mathcal{H}^1 and $V^n \rightarrow V$ in \mathcal{S}^1 . By the predictable representation property of (W_t) , there exists a d -dimensional (\mathcal{F}_t) -predictable process Z such that $U_t = \int_0^t Z_s dW_s$ and

$$\mathbb{E} \left[\sqrt{\int_0^T |Z_s^n - Z_s|^2 ds} \right] \rightarrow 0.$$

By passing to a subsequence, one can assume that

$$\int_0^T |Z_s^n - Z_s|^2 ds \rightarrow 0 \quad \text{almost surely.} \quad (6.2)$$

For every stopping time τ and $B \in \mathcal{F}_{\tau}$, one obtains from Fatou's lemma that

$$\mathbb{E} \left[1_B \int_{\tau}^T |Z_s|^2 ds \right] \leq \liminf_n \mathbb{E} \left[1_B \int_{\tau}^T |Z_s^n|^2 ds \right],$$

which shows that Z belong to BMO. It follows from (6.2) that for almost all ω , one can pass to another subsequence such that $Z_s^n(\omega) \rightarrow Z_s(\omega)$ for Lebesgue-almost all $s \in [0, T]$. Hence, due to condition (f5), one can deduce from Fatou's lemma that

$$\begin{aligned} -V_t(\omega) + V_r(\omega) &= \lim_n -V_t^n(\omega) + V_r^n(\omega) \\ &= \lim_n \int_r^t f(s, W(\omega), Y_s^n(\omega), Z_s^n(\omega)) ds \geq \int_r^t f(s, W(\omega), Y_s(\omega), Z_s(\omega)) ds \end{aligned}$$

for all $r < t$. So $A_t = -V_t - \int_0^t f(s, W, Y_s, Z_s) ds$ is a continuous increasing process starting at 0 such that

$$Y_t = \xi + \int_t^T f(s, W, Y_s, Z_s) ds - \int_t^T Z_s dW_s + A_T - A_t.$$

This shows that (Y, Z, A) is a supersolution of the BSDE (1.1) such that Y is bounded and continuous.

Now assume that f is increasing or decreasing in y . To see that then Y satisfies bounded comparison from above, note that one obtains from the second part of Lemma 5.3 that

$$Y_t^n = \text{ess sup}_{\mu \in \text{BMO}} \mathbb{E}_{\mathcal{F}_t}^{\mu} \left[\xi^n - \int_t^T g(s, W, Y_s^n, \mu_s) ds \right] = \mathbb{E}_{\mathcal{F}_t}^{\mu^n} \left[\xi^n - \int_t^T g(s, W, Y_s^n, \mu_s^n) ds \right]$$

for a sequence $\mu^n \in \text{BMO}$. We will show that this implies that Y satisfies assumption (A). For given $\varepsilon > 0$, choose $n \in \mathbb{N}$ so large that

$$\|Y^n - Y\|_{\mathcal{S}^{\infty}} \leq \min \left(\frac{\varepsilon}{3KT}, \frac{\varepsilon}{3} \right).$$

Then

$$\begin{aligned}
\frac{\varepsilon}{3} &\geq \|Y^n - Y\|_{\mathcal{S}^\infty} \\
&= \left\| \sup_t \left| \mathbb{E}_{\mathcal{F}_t}^{\mu^n} \left[\xi^n - \int_t^T g(s, W, Y_s^n, \mu_s^n) ds \right] - Y_t \right| \right\|_{L^\infty} \\
&\geq \left\| \sup_t \left| \mathbb{E}_{\mathcal{F}_t}^{\mu^n} \left[\xi - \int_t^T g(s, W, Y_s^n, \mu_s^n) ds \right] - Y_t \right| \right\|_{L^\infty} - \|Y_T^n - Y_T\|_{L^\infty} \\
&\geq \left\| \sup_t \left| \mathbb{E}_{\mathcal{F}_t}^{\mu^n} \left[\xi - \int_t^T g(s, W, Y_s, \mu_s^n) ds \right] - Y_t \right| \right\|_{L^\infty} - K \left\| \int_0^T |Y_s^n - Y_s| ds \right\|_{L^\infty} - \frac{\varepsilon}{3} \\
&\geq \left\| \sup_t \left| \mathbb{E}_{\mathcal{F}_t}^{\mu^n} \left[\xi - \int_t^T g(s, W, Y_s, \mu_s^n) ds \right] - Y_t \right| \right\|_{L^\infty} - \frac{2\varepsilon}{3}.
\end{aligned}$$

In particular,

$$Y_t \leq \mathbb{E}_{\mathcal{F}_t}^{\mu^n} \left[\xi - \int_t^T g(s, W, Y_s, \mu_s^n) ds \right] + \varepsilon \quad \text{for all } t \in [0, T].$$

This shows that Y satisfies assumption (A). Now if (Y', Z') is a solution to the BSDE (1.1) with bounded terminal condition $\xi' \geq \xi$ and driver $f' \geq f$ such that Y' is bounded, then f' satisfies condition (f5). So it follows from Proposition 5.5 or Proposition 5.6 that $Y'_t \geq Y_t$ for all t . \square

Proof of Theorem 2.7. We know from Theorem 2.4 that Y^n is an increasing sequence. By Lemma 6.1, it is bounded in \mathcal{S}^∞ . So it converges pointwise to a bounded predictable process Y . By Lemma 5.3, one has for all t ,

$$\begin{aligned}
Y_t &= \sup_n Y_t^n = \sup_n \operatorname{ess\,sup}_{\mu \in \mathcal{B}} \mathbb{E}_{\mathcal{F}_t}^\mu \left[\xi^n - \int_t^T g(s, W, Y_s^n, \mu_s) ds \right] \\
&= \operatorname{ess\,sup}_{\mu \in \mathcal{B}} \sup_n \mathbb{E}_{\mathcal{F}_t}^\mu \left[\xi^n - \int_t^T g(s, W, Y_s^n, \mu_s) ds \right] \\
&= \operatorname{ess\,sup}_{\mu \in \mathcal{B}} \mathbb{E}_{\mathcal{F}_t}^\mu \left[\xi - \int_t^T g(s, W, Y_s, \mu_s) ds \right],
\end{aligned}$$

where the last equality follows from Beppo Levi's monotone converge theorem. Now one deduces as in Proposition 2.1 and Theorem 2.1 of Delbaen et al. [10] that there exists a martingale of the form $U_t = \int_0^t Z_s dW_s$ and a RCLL predictable process $V_t \geq \int_0^t f(s, W, Y_s, Z_s) ds$ starting at 0 such that

$$Y_t = Y_0 + U_t + V_t.$$

By Lemma 5.1, Z is in BMO and $\|V\|_{\mathcal{H}^1} < \infty$. Defining $A_t := V_t - \int_0^t f(s, W, Y_s, Z_s) ds$ shows the existence of a supersolution.

To see that the supersolution satisfies bounded comparison from above, assume (Y', Z', A') is a supersolution of the BSDE (1.1) with terminal condition $\xi' \geq \varphi(W)$ and driver $f' \geq f$ such that Y' is bounded. Then it follows from Theorem 2.5 that $Y'_t \geq Y_t^n$ for all t and n . Therefore, $Y'_t \geq Y_t$ for all t . \square

A Appendix: The validity of Theorem 12 of Briand et al. [5] in our setting

The purpose of this appendix is to show that Theorem 12 of Briand et al. [5] still holds in the context of the proof of Theorem 2.4. Most of their arguments go through in our setup. But where

they apply Proposition 11 we use Lemma A.2 below. Assume that (W1)–(W5) hold and f is a driver satisfying

$$\sup_t |f(t, 0, 0, 0)| < \infty$$

and

$$|f(t, w_1, y_1, z_1) - f(t, w_2, y_2, z_2)| \leq K \left(\sup_{0 \leq s \leq t} |w_1(s) - w_2(s)| + |y_1 - y_2| + |z_1 - z_2| \right) \quad (\text{A.1})$$

for some constant $K \in \mathbb{R}_+$. As in Theorem 2.4, $\varphi : C^d[0, T] \rightarrow \mathbb{R}$ is assumed to be a Lipschitz-continuous function. In particular, $\varphi(W)$ is square-integrable. Under these assumptions it follows from Pardoux and Peng [17] that the BSDE (1.1) has a unique solution (Y, Z) , and we know from Proposition 3.2 that for N so large that $\max_i \Delta t_i^N < 1/K$, the N -th BSDE has a unique solution (Y^N, Z^N, M^N) . We are showing the following version of Theorem 12 of Briand et al. [5]:

Theorem A.1 *For $N \rightarrow \infty$, one has*

$$\sup_t \left(|Y_t^N - Y_t| + \left| \int_0^t Z_s^N dW_s^N - \int_0^t Z_s dW_s \right| + |M_t^N| \right) \rightarrow 0 \quad \text{in } L^2,$$

and

$$\sup_t \left(\sum_{k=1}^d \left| \int_0^t Z_s^{N,k} d\langle W^N \rangle_s - \int_0^t Z_s^k ds \right|^2 + \left| \int_0^t |Z_s^N|^2 d\langle W^N \rangle_s - \int_0^t |Z_s|^2 ds \right| \right) \rightarrow 0 \quad \text{in } L^1.$$

Proof. Set

$$(Y^{\infty,0}, Z^{\infty,0}) := (0, 0) \quad \text{and} \quad (Y^{N,0}, Z^{N,0}, M^{N,0}) := (0, 0, 0).$$

For $p \in \mathbb{N}$, define $(Y^{\infty,p+1}, Z^{\infty,p+1})$ as follows: $Z^{\infty,p+1}$ is the unique d -dimensional (\mathcal{F}_t) -predictable process satisfying

$$\begin{aligned} \int_0^t Z_s^{\infty,p+1} dW_s &= \mathbb{E}_{\mathcal{F}_t} \left[\varphi(W) + \int_0^T f(s, W, Y_s^{\infty,p}, Z_s^{\infty,p}) ds \right] \\ &\quad - \mathbb{E} \left[\varphi(W) + \int_0^T f(s, W, Y_s^{\infty,p}, Z_s^{\infty,p}) ds \right] \end{aligned}$$

and

$$Y_t^{\infty,p+1} = \varphi(W) + \int_t^T f(s, W, Y_s^{\infty,p}, Z_s^{\infty,p}) ds - \int_t^T Z_s^{\infty,p+1} dW_s.$$

Similarly, decompose

$$\begin{aligned} &\mathbb{E}_{\mathcal{F}_t^N} \left[\varphi(\hat{W}^N) + \int_{(0,T]} \hat{f}^N(s, \hat{W}^N, Y_{s-}^{N,p}, Z_s^{N,p}) d\langle W^N \rangle_s \right] \\ &- \mathbb{E} \left[\varphi(\hat{W}^N) + \int_{(0,T]} \hat{f}^N(s, \hat{W}^N, Y_{s-}^{N,p}, Z_s^{N,p}) d\langle W^N \rangle_s \right] \end{aligned}$$

into a martingale of the form $\int_{(0,t]} Z_s^{N,p+1} dW_s^N$ and a martingale $M^{N,p+1}$ orthogonal to W^N . Then set

$$\begin{aligned} Y_t^{N,p+1} &= \varphi(\hat{W}^N) + \int_{(t,T]} \hat{f}^N(s, \hat{W}^N, Y_{s-}^{N,p}, Z_s^{N,p}) d\langle W^N \rangle_s \\ &\quad - \int_{(t,T]} Z_s^{N,p+1} dW_s^N - (M_T^{N,p+1} - M_t^{N,p+1}). \end{aligned}$$

It is well-known from Pardoux and Peng [17] that

$$\mathbb{E} \left[\sup_t |Y_t^{\infty,p} - Y_t|^2 + \int_0^T |Z_s^{\infty,p} - Z_s|^2 ds \right] \rightarrow 0 \quad \text{for } p \rightarrow \infty,$$

and it follows as in Corollary 10 of Briand et al. [5] that there exists an N_0 such that

$$\sup_{N \geq N_0} \mathbb{E} \left[\sup_t |Y_t^{N,p} - Y_t^N|^2 + \int_0^T |Z_s^{N,p} - Z_s^N|^2 d\langle W^N \rangle_s + |M_T^{N,p} - M_T^N|^2 \right] \rightarrow 0 \quad \text{for } p \rightarrow \infty.$$

So it is enough to show that for fixed p and $N \rightarrow \infty$, one has

$$\sup_t \left(|Y_t^{N,p} - Y_t^{\infty,p}| + \left| \int_0^t Z_s^{N,p} dW_s^N - \int_0^t Z_s^{\infty,p} dW_s \right| + |M_t^{N,p}| \right) \rightarrow 0 \quad \text{in } L^2 \quad (\text{A.2})$$

and

$$\sup_t \left(\sum_{k=1}^d \left| \int_0^t Z_s^{N,p,k} d\langle W^N \rangle_s - \int_0^t Z_s^{\infty,p,k} ds \right|^2 + \left| \int_0^t |Z_s^{N,p}|^2 d\langle W^N \rangle_s - \int_0^t |Z_s^{\infty,p}|^2 ds \right| \right) \rightarrow 0 \quad (\text{A.3})$$

in L^1 . This can be proven by induction over p . Assume that it holds for p . Then by Lemma A.2 below,

$$\sup_t \left| \int_{(0,t]} \hat{f}^N(s, \hat{W}^N, Y_{s-}^{N,p}, Z_s^{N,p}) d\langle W^N \rangle_s - \int_0^t f(s, W, Y_s^{\infty,p}, Z_s^{\infty,p}) ds \right| \rightarrow 0 \quad \text{in } L^2.$$

Moreover, one obtains as in Briand et al. [5] that

$$\mathbb{E}_{\mathcal{F}_t^N} \left[\varphi(\hat{W}^N) \right] = Y_t^{N,p+1} - \mathbb{E}_{\mathcal{F}_t^N} \left[\int_{(t,T]} \hat{f}^N(s, \hat{W}^N, Y_{s-}^{N,p}, Z_s^{N,p}) d\langle W^N \rangle_s \right]$$

converges in \mathcal{S}^2 to

$$\mathbb{E}_{\mathcal{F}_t} \left[\varphi(\hat{W}) \right] = Y_t^{\infty,p+1} - \mathbb{E}_{\mathcal{F}_t} \left[\int_t^T f(s, W, Y_s^{\infty,p}, Z_s^{\infty,p}) ds \right].$$

So $Y^{N,p+1} \rightarrow Y^{\infty,p+1}$ in \mathcal{S}^2 . Finally, since the martingale

$$\mathbb{E}_{\mathcal{F}_t^N} \left[Y_T^{N,p+1} - Y_0^{N,p+1} + \int_{(0,T]} \hat{f}^N(s, W^N, Y_{s-}^{N,p}, Z_s^{N,p}) d\langle W^N \rangle_s \right] = \int_{(0,t]} Z_s^{N,p+1} dW_s^N + M_t^{N,p+1}$$

converges in \mathcal{S}^2 to

$$\mathbb{E}_{\mathcal{F}_t} \left[Y_T^{\infty,p+1} - Y_0^{\infty,p+1} + \int_0^T f(s, W, Y_s^{\infty,p}, Z_s^{\infty,p}) ds \right] = \int_0^t Z_s^{p+1} dW_s,$$

(A.2)–(A.3) follow from Theorem 5 in Briand et al. [5]. \square

Lemma A.2 Fix $p \in \mathbb{N}$ and assume that

$$\begin{aligned} & \sup_t |Y_t^{N,p} - Y_t^{\infty,p}|^2 + \sum_{k=1}^d \left| \int_{(0,t]} Z_s^{N,p,k} d\langle W^N \rangle_s - \int_0^t Z_s^{\infty,p,k} ds \right|^2 \\ & + \left| \int_{(0,t]} |Z_s^{N,p}|^2 d\langle W^N \rangle_s - \int_0^t |Z_s^{\infty,p}|^2 ds \right| \rightarrow 0 \quad \text{in } L^1 \quad \text{for } N \rightarrow \infty. \end{aligned}$$

Then

$$\sup_t \left| \int_{(0,t]} \hat{f}^N(s, \hat{W}^N, Y_{s-}^{N,p}, Z_s^{N,p}) d\langle W^N \rangle_s - \int_0^t f(s, W, Y_s^{\infty,p}, Z_s^{\infty,p}) ds \right| \rightarrow 0 \quad \text{in } L^2$$

for $N \rightarrow \infty$.

Proof. By definition (2.4), one has

$$\begin{aligned} & \int_{(t_i^N, t_{i+1}^N]} \hat{f}^N(s, \hat{W}^N, Y_{s-}^{N,p}, Z_s^{N,p}) d\langle W^N \rangle_s = \int_{t_i^N}^{t_{i+1}^N} f(s, \hat{W}^N, Y_{t_i^N}^{N,p}, Z_{t_{i+1}^N}^{N,p}) ds \\ & = \int_{t_i^N}^{t_{i+1}^N} f(s, \hat{W}^N, Y_s^{N,p}, Z_s^{N,p}) ds, \end{aligned}$$

and therefore,

$$\begin{aligned} & \sup_t \left| \int_{(0,t]} \hat{f}^N(s, \hat{W}^N, Y_s^{N,p}, Z_s^{N,p}) d\langle W^N \rangle_s - \int_{(0,t]} f(s, \hat{W}^N, Y_s^{N,p}, Z_s^{N,p}) ds \right|^2 \quad (\text{A.4}) \\ & = \max_i \sup_{t_i^N < t \leq t_{i+1}^N} \left| \int_{t_i^N}^t f(s, \hat{W}^N, Y_s^{N,p}, Z_s^{N,p}) ds \right|^2 \\ & \leq \max_i \Delta t_{i+1}^N \int_{t_i^N}^{t_{i+1}^N} |f(s, \hat{W}^N, Y_s^{N,p}, Z_s^{N,p})|^2 ds \\ & \leq 4 \max_i (\Delta t_{i+1}^N)^2 \left\{ \sup_t |f(t, 0, 0, 0)|^2 + K^2 \left(\sup_t |W_t|^2 + |Y_{t_i^N}^{N,p}|^2 + |Z_{t_{i+1}^N}^{N,p}|^2 \right) \right\} \rightarrow 0, \end{aligned}$$

in L^1 for $N \rightarrow \infty$, where we used (A.1) and $(a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2)$. Next, observe that it follows from the assumptions that for all $k = 1, \dots, d$,

$$Z^{N,p,k} \rightarrow Z^{\infty,p,k} \quad \text{weakly in } L^2([0, T] \times \Omega)$$

as well as

$$\left\| Z^{N,p,k} \right\|_{L^2([0, T] \times \Omega)} \rightarrow \left\| Z^{\infty,p,k} \right\|_{L^2([0, T] \times \Omega)}.$$

This gives

$$\int_0^T |Z_s^{N,p} - Z_s^{\infty,p}|^2 ds \rightarrow 0 \quad \text{in } L^1 \quad \text{as } N \rightarrow \infty,$$

which, together with $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$, shows that

$$\begin{aligned} & \sup_t \left| \int_0^t f(s, \hat{W}^N, Y_s^{N,p}, Z_s^{N,p}) ds - \int_0^t f(s, W, Y_s^{\infty,p}, Z_s^{\infty,p}) ds \right|^2 \\ & \leq T \int_0^T \left| f(s, \hat{W}^N, Y_s^{N,p}, Z_s^{N,p}) - f(s, W, Y_s^{\infty,p}, Z_s^{\infty,p}) \right|^2 ds \\ & \leq 3T^2 K^2 \sup_t \left(|\hat{W}_t^N - W_t|^2 + |Y_t^{N,p} - Y_t^{\infty,p}|^2 \right) + 3TK^2 \int_0^T |Z_s^{N,p} - Z_s^{\infty,p}|^2 ds \\ & \rightarrow 0 \quad \text{in } L^1 \quad \text{for } N \rightarrow \infty. \end{aligned} \quad (\text{A.5})$$

Combining (A.4) and (A.5), one obtains

$$\begin{aligned} & \sup_t \left| \int_{(0,t]} \hat{f}^N(s, \hat{W}^N, Y_{s-}^{N,p}, Z_s^{N,p}) d\langle W^N \rangle_s - \int_0^t f(s, W, Y_s^{\infty,p}, Z_s^{\infty,p}) ds \right|^2 \\ & \leq 2 \sup_t \left\{ \left| \int_{(0,t]} \hat{f}^N(s, \hat{W}^N, Y_{s-}^{N,p}, Z_s^{N,p}) d\langle W^N \rangle_s - \int_0^t f(s, \hat{W}^N, Y_s^{N,p}, Z_s^{N,p}) ds \right|^2 \right. \\ & \quad \left. + \left| \int_0^t f(s, \hat{W}^N, Y_s^{N,p}, Z_s^{N,p}) ds - \int_0^t f(s, W, Y_s^{\infty,p}, Z_s^{\infty,p}) ds \right|^2 \right\} \rightarrow 0 \quad \text{in } L^1 \quad \text{as } N \rightarrow \infty. \end{aligned}$$

□

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