Exercises in Convex Optimization
Lecture 4

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Strong duality in convex optimization Let $X \subseteq \mathbb{R}^n$ be a *convex compact* set, $f : X \to \mathbb{R}$ be a *continuous convex* function, $g = (g_1, \dots, g_m)^T$, where each component $g_1, \dots, g_m : X \to \mathbb{R}$ is a *continuous concave* function (i.e. $-g_i$ is convex) and $b \in \mathbb{R}^m$. We consider the following optimization problem:

$$\begin{aligned} p^* := \min \ & f(x) \\ \text{s.t. } & g(x) \geq b \\ & x \in X \end{aligned}$$

where "$u \geq v$" means that $u_i \geq v_i$ for every component $1 \leq i \leq m$. We also consider the following dual:

$$d^* := \max_{u \geq 0} \min_{x \in X} f(x) + u^T(b - g(x))$$

Prove that strong duality holds, i.e. prove that $p^* = d^*$.

`Lecture_4/StrongDuality`

A convex problem in which strong duality fails (Boyd, Ex. 5.21) Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & \exp(-x_1) \\ \text{subject to} \quad & x_1^2/x_2 \leq 0 \\ & x \in D \end{aligned}$$

where $D := \{(x_1, x_2) \in \mathbb{R}^2 | x_2 > 0\}$.

(a) Verify that this is a convex optimization problem and find the optimal value.

(b) Give the Lagrange dual problem, i.e., $d^* = \sup_{u \geq 0} \inf_{x \in D} L(x, u)$ where $L(x, u)$ is the Lagrangian, and find the optimal dual solution u^* and the optimal value d^* of the dual problem. What is the duality gap?

(c) Does Slater's condition hold for this problem?

`Lecture_4/StrongDualityFails`

A nonconvex problem with strong duality Let $A \in \mathbb{S}^n$ and $b \in \mathbb{R}^n$. Consider the problem

$$\begin{aligned} \inf\ & x^T A x + b^T x \\ \text{s.t.}\ & x^T x - 1 \leq 0, \\ & x \in \mathbb{R}^n. \end{aligned} \tag{1}$$

Observe that this problem is convex for $A \in \mathbb{S}^n_+$, whereas for $A \notin \mathbb{S}^n_+$, it is nonconvex. Prove that for (1) the strong duality holds. We suggest you to follow these steps:

i) Prove the result for $A \in \mathbb{S}^n_+$.

ii) If A is not positive definite, prove that the primal optimal value does not change if we replace the inequality in the constraint by an equality.

iii) By observing, for any real α, that the equality

$$\min\{x^T A x + b^T x \ : \ x^T x = 1\} = -\alpha + \min\{x^T (A + \alpha I) x + b^T x \ : \ x^T x = 1\}$$

holds, prove the desired result (for A not necessarily positive semidefinite).

`Lecture_4/NonconvexProblemWithStrongDuality`