# Exercises in Convex Optimization
# Lecture 5

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**The self-conjugate function** Prove that the only function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ that is equal to its conjugate is $f(x) = ||x||_2^2/2$.

`Lecture_5/SelfConjugate`

**Subgradients** (i) Let $f_1(x,y) = -\sqrt{2 - x^2 - (y-1)^2}, f_2(x,y) = -\sqrt{2 - x^2 - (y+1)^2}$ be the functions representing the lower half of two balls of radius $\sqrt{2}$ centered in $(0,1,0)$ and $(0,-1,0)$ respectively. We set $f_1, f_2$ to be equal to $+\infty$ whenever the radical would not be real. We want to consider the function $f$ whose epigraph is the intersection of epi$f_1$ and epi$f_2$, i.e. $f = \max\{f_1, f_2\}$ over dom$f$ = dom$f_1 \cap$ dom$f_2$.

Compute the subgradient $\partial f(x,y)$ of $f$ for every point of its domain: can you already guess where this set will be empty, where it will be a singleton and where it will contain more than one element? Why?

(ii) Developing duality theory using affine increasing functions we obtained the Lagrange dual problem

$$\max_{u \succcurlyeq 0} \underbrace{\min_{x \in X} L(x,u)}_{=:\, h(u)}, \quad \text{where } L(x,u) := f(x) + u^T (b - g(x)) \, .$$

Let $\bar{u} \succcurlyeq 0$ be fixed and $\bar{x} \in X$ be an optimal solution of $h(\bar{u}) = \min_{x \in X} L(x, \bar{u})$.

Show that $b - g(\bar{x})$ is a *super*gradient of $h$ at $u = \bar{u}$.

`Lecture_5/Subgradients`

**Strong convexity and Lipschitz continuity of the gradient** Let $\omega : \mathbb{R} \to \mathbb{R}_+$ be a convex function that equals zero only in 0. A proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is $\omega$-*strongly convex* if for every $x \in \text{relint}(\text{dom}f), y \in \text{dom}f$, we have:

$$f(y) - f(x) - \langle f'(x), y - x\rangle \geq \omega(||y - x||),$$

where $f'(x) \in \partial f(x)$. Show that the conjugate $f_*$ satisfies:

$$f_*(v) - f_*(u) - \langle f_*'(u), v - u\rangle \leq \omega_*(||v - u||_*) \quad \forall u, v \in \text{dom}f_*,$$

where $\omega_*$ is the conjugate of $\omega$.

Note: this exercise proved to have important consequences. It is at the basis of *Smoothing Techniques*, which are now largely used to solve efficiently some large-scale nondifferentiable convex problems.

A frequently used particular case of the above exercise is given when we take $\omega(x) = \frac{\sigma}{2}||x||_2^2$ for some $\sigma > 0$: the conjugate of a $\sigma$-strongly convex function is a $\frac{1}{\sigma}$-smooth function (because $\omega^*(x) = \frac{1}{2\sigma}||x||_2^2$). The case $\omega(x) = \sum_i x_i \log(x_i)$ for $x_i \geq 0$ and $\sum_i x_i = 1$, $\omega(x) = +\infty$ otherwise plays an important role in Smoothing Techniques.

`Lecture_5/StrongConvexity`