# Exercises in Convex Optimization
# Lecture 8

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$L_{20}$ **optimization** Optimization and approximation methods that use an $L_2$-norm (or its square), an $L_1$-norm, or an $L_\infty$-norm (minimize worst-case residuals) are currently very popular in statistics, machine learning, and signal and image processing. In this problem we study a natural method for blending $L_2$-norm and $L_\infty$-norm, by using the $L_{20}$-norm, defined as

$$||z||_{20} = \left(\sum_{k=1}^{n} |z_i|^{20}\right)^{1/20} \quad \text{for } z \in \mathbb{R}^n.$$

We will consider the simplest approximation or regression problem: minimize $||Ax - b||_{20}$, with variable $x \in \mathbb{R}^n$, and problem data $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. We will assume that $m > n$ and that $A$ has full rank. The hope is that this $L_{20}$-optimal approximation problem should share some of the good features of $L_2$ and $L_\infty$ approximation.

Explain how to formulate the $L_{20}$-norm approximation problem as an SDP and write a Matlab function taking as arguments $A$ and $b$, and returning an approximate solution $x$ to the corresponding $L_{20}$-norm approximation problem.

(Hint: Actually, you will get an SOCP. This exercise applies the fact that $S_{p_0,p_1,\ldots,p_n} := \{(x_0, x_1, \ldots, x_n) : x_0^{p_0} \leq x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}\}$, is an CQr set for any nonnegative integers $(p_0, p_1, \ldots, p_n)$ with $p_0 \geq p_1 + \cdots + p_n$. To prove this general statement and to be able to rewrite the $L_{20}$ problem correctly, we suggest to prove it in small steps, each of which involving only one new trick. Verify it first for $p = (2, 1, 1)$, then for $p = (4, 1, 1, 1, 1)$ (Hint: use 3 second-order cones). Observe that the trick can be applied inductively for $p_0 = 2^N = n$, $p_i = 1$ for all $i > 0$. Then prove it for $p = (4, 1, 1, 1, 0)$ , then for $p = (4, 2, 1, 1)$, then for $p = (3, 1, 1, 1)$. With that, you have all the tools to prove that $S_p$ is CQr for $p$ as above. Now, verify that $\sqrt[p]{\sum_{i=1}^n |z_i|^p} \leq t$ iff $t, s_i \geq 0$, $-s_i^{1/p} t^{1-1/p} \leq z_i \leq s_i^{1/p} t^{1-1/p}$ and $\sum_{i=1}^n s_i \leq t$. It remains to put all the pieces together.)

`Lecture_8/SedumiL20Optimization`

**Efficiency of the Maximum Volume Inscribed Ellipsoid** In this exercise we want to prove the following geometrical result. Suppose that $P$ is a polyhedron in $\mathbb{R}^n$, bounded, symmetric around the origin and described as

$$P = \left\{x \mid -1 \leq a_i^T x \leq 1, i = 1, \ldots, p\right\}.$$

Let

$$\mathcal{E} = \left\{x \mid x^T Q^{-1} x \leq 1\right\},$$

where $Q \in \mathbb{S}^n_{++}$, be the maximum volume ellipsoid centered in the origin, inscribed in $P$.

**<u>Theorem:</u>** The ellipsoid $\sqrt{n}\mathcal{E} = \left\{x \mid x^T Q^{-1} x \leq n\right\}$ (i.e. the ellipsoid $\mathcal{E}$ scaled by a factor $\sqrt{n}$ around the origin) then *contains* $P$.

To prove this, proceed as follows:

(a) Show the equivalent characterization: $\mathcal{E} \subseteq P \Longleftrightarrow a_i^T Q a_i \leq 1, \forall i = 1, \ldots, p$.

(b) It is well-known that the volume of $\mathcal{E}$ is proportional to $\sqrt{\det Q}$, so we can find the maximum volume ellipsoid $\mathcal{E}$ inside $P$ by solving the following convex optimization problem:

$$\begin{array}{ll} \min & \log\left(\det\left(Q^{-1}\right)\right) \\ \text{s.t.} & a_i^T Q a_i \leq 1, \forall i = 1, \ldots, p \end{array}$$

The variable is the matrix $Q \in \mathbb{S}^n$ and the domain of the objective function is $\mathbb{S}^n_{++}$. Derive the Lagrange dual problem $\max\limits_{u \succcurlyeq 0} \min\limits_{Q \succ 0} L(Q, u)$.

(*Hint:* the gradient of the convex objective function $Q \mapsto \log\left(\det\left(Q^{-1}\right)\right)$ is given by $-Q^{-1}$).

(c) Note that Slater's condition holds for the primal problem above (e.g. $a_i^T Q a_i < 1$ for $Q = \varepsilon I, \varepsilon > 0$ small enough), so we have strong duality, and the KKT conditions are necessary and sufficient for optimality. What are the KKT conditions for this primal problem?

Suppose $Q^*$ is optimal (i.e. describing our maximal volume ellipsoid $\mathcal{E} \subseteq P$). Use the KKT conditions to show that $x \in P \Longrightarrow x^T (Q^*)^{-1} x \leq n$, which is the desired result.

`Lecture_9/MaximumVolumeInscribedEllipsoid`