Exercises in Convex Optimization
Lecture 11

Michel Baes, Patrick Cheridito

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Derivation of the Barrier Method for Linear Programming Consider the following LP:

$$\begin{aligned} \min \quad & x_1 + x_2 + x_3 \\ \text{s.t.} \quad & -x_1 + x_2 = 1 \\ & x_3 = 1 \\ & x \geq 0 \end{aligned}$$

(a) What is the optimal solution x^* of this problem?

(b) Show that the corresponding dual problem is:

$$\begin{aligned} \max \quad & u_1 + u_2 \\ \text{s.t.} \quad & -u_1 \leq 1 \\ & u_1 \ \leq 1 \\ & u_2 \ \leq 1. \end{aligned}$$

Find the optimal dual solution u^* and verify that strong duality holds.

(c) We want now to build a (self-concordant) barrier function $F_Q(u)$ for the closed, convex feasible set $Q := \{u \mid -u_1 \leq 1, u_1 \leq 1, u_2 \leq 1\}$ (careful: we first need to rewrite this dual problem in minimization form, hence the problem becomes $\min\{f(u) := -(u_1 + u_2) \mid u \in Q\}$). We claim that a valid self-concordant barrier for Q is $F_Q(u) := -\ln(1 + u_1) - \ln(1 - u_1) - \ln(1 - u_2)$. As seen in the lecture, we now want to approximate the exact problem $\min\{f(u) \mid u \in Q\} = \min_u f(u) + \chi_Q(u)$ by the problem $\min_u f(u) + \mu F_Q(u)$ for $\mu \to 0$, i.e. by

$$\min_u \phi_\mu(u) := \underbrace{-(u_1 + u_2)}_{f(u)} + \mu \underbrace{\left(-\ln(1 + u_1) - \ln(1 - u_1) - \ln(1 - u_2)\right)}_{F_Q(u)}.$$

Show that for fixed $\mu > 0$, the function $\phi_\mu(u)$ is indeed self-concordant and compute its unique minimum $u^*(\mu)$.

(d) For fixed $\mu > 0$, consider now the primal problem in "barrier form":

$$\begin{aligned} \min \quad & x_1 + x_2 + x_3 + \mu\left(-\ln(x_1) - \ln(x_2) - \ln(x_3)\right) \\ \text{s.t.} \quad & -x_1 + x_2 = 1 \\ & x_3 = 1 \end{aligned}$$

(notice that we only encoded the inequality constraints in the barrier!) Write down the KKT optimality conditions for this problem and check that $x_1^*(\mu) = \frac{\mu}{1+u_1^*(\mu)}$, $x_2^*(\mu) = \frac{\mu}{1-u_1^*(\mu)}$, $x_3^*(\mu) = \frac{\mu}{1-u_2^*(\mu)}$ is an optimal point (i.e. satisfies the KKT conditions). Show that $(x^*(\mu), u^*(\mu))$ converges to primal and dual optimal solutions (x^*, u^*) for $\mu \longrightarrow 0$. The curve $\mu \longmapsto (x^*(\mu), u^*(\mu))$ is called the *primal-dual central path.*

(e) For fixed $\mu > 0$, compute the difference between the primal objective value evaluated at $x^*(\mu)$ and the dual objective value evaluated at $u^*(\mu)$, i.e. evaluate the duality gap at $(x^*(\mu), u^*(\mu))$, and show that it goes to 0.

`Lecture_12/DerivationBarrierMethodforLinearPrograming`

Newton Decrement **(Boyd)** Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a twice differentiable μ-strongly convex function with respect to the Euclidean norm, with $\mu > 0$. For every $x \in \mathrm{dom} f$, we define the *local norm* of $d \in \mathbb{R}^n$ as:

$$||d||_x := \langle f''(x)d, d\rangle^{1/2},$$

where $\langle \cdot, \cdot \rangle$ represents the standard dot product.

1. Show that the dual norm of $|| \cdot ||_x$ satisfies for all $d \in \mathbb{R}^n$:

$$||d||_{x*} := \langle f''(x)^{-1}d, d\rangle^{1/2}.$$

2. The *Newton decrement* of a point x is defined as

$$\lambda(x) := ||x_+ - x||_x,$$

where $x_+ := x - f''(x)^{-1}f'(x)$ is the result of a Newton step made from x. Show that these alternative definitions of the Newton decrement are equivalent.

$$\lambda(x) = ||f'(x)||_{x*} = \sup_{\langle f''(x)v,v\rangle=1} -\langle f'(x), v\rangle = \sup_{v\neq 0} \frac{-\langle f'(x), v\rangle}{\langle f''(x)v, v\rangle^{1/2}}.$$

Give an interpretation of the last equality.

`Lecture_12/NewtonDecrement`