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Controlling portfolio skewness and kurtosis without directly optimizing third and fourth moments

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HIGHLIGHTS

- We propose a new mean–variance approach that can control higher moments.
- Our model does not directly impose higher moment terms in the formulation.
- Our model employs robust formulation with a specific choice of uncertainty set.
- We provide analytical proofs showing our model can control higher moments.
- We present empirical results supporting the validity of our model.

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ABSTRACT

In spite of their importance, third or higher moments of portfolio returns are often neglected in portfolio construction problems due to the computational difficulties associated with them. In this paper, we propose a new robust mean–variance approach that can control portfolio skewness and kurtosis without imposing higher moment terms. The key idea is that, if the uncertainty sets are properly constructed, robust portfolios based on the worst-case approach within the mean–variance setting favor skewness and penalize kurtosis.

1. Introduction

The traditional Markowitz model takes the expectation and variance of a portfolio return as the performance and risk measures and determines the optimal portfolio by solving a quadratic program. From the practitioner's perspective, the skewness and kurtosis are also important. When the portfolio return is negatively skewed, it is more likely to have an extreme left-tail event than one in the right-tail. Thus the typical investor prefers more positively skewed return distributions. For instance, a more positively skewed portfolio has better Sortino ratio (Sortino and van der Meer, 1991) and lower semi-deviation. Similarly, a portfolio with smaller kurtosis tends to have less extreme events, thus is preferred by most investors.

Obviously, as the traditional mean–variance approach exclusively deals with the first two moments, a revised approach is required to incorporate and control the third and fourth moments when constructing a portfolio. The most straightforward approach would be to introduce new terms for the portfolio skewness and kurtosis in the objective function. To see this, let us first consider

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the traditional mean–variance formulation as follows.

$$\max_{w \in C} \mu' w - \beta w' \Sigma w \tag{1}$$

where C is a convex subset of the hyperplane $\{w \in \mathbb{R}^n | \sum_i w_i = 1\}$, $\mu \in \mathbb{R}^n$, $\Sigma \in \mathbb{R}^{n\times n}$ are the expected return and covariance matrix, respectively, and $\beta > 0$ is the parameter to reflect investor's risk aversion.

Then, given a coskewness matrix $M_3 = \mathbb{E}[(r-\mathbb{E}r)(r-\mathbb{E}r)' \otimes (r-\mathbb{E}r)'] \in \mathbb{R}^{n\times n^2}$ for the random return vector $r \in \mathbb{R}^n$ and Kronecker product $\otimes$, the third central moment of portfolio return $w'r$ is $S(w, M_3) := \mathbb{E}\left(w'r - Ew'r\right)^3 = w' M_3 (w \otimes w)$. Similarly, for a cokurtosis matrix $M_4 = \mathbb{E}\left[(r-\mathbb{E}r)(r-\mathbb{E}r)' \otimes (r-\mathbb{E}r)' \otimes (r-\mathbb{E}r)'\right] \in \mathbb{R}^{n\times n^3}$, its fourth central moment is $K(w, M_4) := \mathbb{E}\left(w'r - Ew'r\right)^4 = w' M_4 (w \otimes w \otimes w)$.

Consequently, the mean–variance formulation could be modified by adding third and fourth central moments and solving the optimization problem

$$\max_{w \in C} \mu' w - \beta w' \Sigma w + \gamma S(w, M_3) - \delta K(w, M_4)$$

for parameters $\beta, \gamma, \delta \geq 0$. (2)

While problem (2) is intuitive, there are two critical issues. First, the third central moment term $S(w, M_3)$ is a cubic function, thus it makes problem (2) non-convex, which causes a significant increase in computational cost. Second, as the coskewness matrix M_3 is 3-dimensional, the number of parameters is of the order of n^3, which makes it practically impossible to obtain reliable estimators. The same issues apply to the kurtosis term, only with more difficulties. The main objective of this study is to develop an approach that can control higher moments of portfolio returns within the mean–variance framework without directly imposing third and fourth moment terms in the formulation.

2. Models and theories

Let $r \in \mathbb{R}^n$ be the random vector representing returns of n risky assets. We assume it has finite moments up to order four. Also, let $r_{i,j} \in \mathbb{R}^n$ for $i = 1, \ldots, I$, and $j = 1, \ldots, J$ be independent and identically distributed (i.i.d.) samples of r. Let $\hat{\mu}_i = \frac{1}{J}\sum_j r_{i,j}$ and $\hat{\Sigma}_i = \frac{1}{J-1}\sum_j \left(r_{i,j} - \hat{\mu}_i\right)\left(r_{i,j} - \hat{\mu}_i\right)'$ be the sample mean and sample covariance matrix for the i-th sample set, respectively. We also define a joint uncertainty set of $\left(\hat{\mu}, \hat{\Sigma}\right)$, $U_{(\hat{\mu},\hat{\Sigma})} = \left\{\left(\hat{\mu}_1, \hat{\Sigma}_1\right), \ldots, \left(\hat{\mu}_i, \hat{\Sigma}_i\right), \ldots, \left(\hat{\mu}_I, \hat{\Sigma}_I\right)\right\}$ Now, let us consider the robust version of problem (1).

$$\max_{w \in C} \min_{(\hat{\mu},\hat{\Sigma}) \in U_{(\hat{\mu},\hat{\Sigma})}} \hat{\mu}' w - \beta w' \hat{\Sigma} w. \tag{3}$$

For a feasible portfolio w of problem (3), let

$$X_{i,j} = w' r_{i,j},$$

$$\bar{X}_i = \frac{1}{J}\sum_{j=1}^{J} w' r_{i,j} = \frac{1}{J}\sum_{j=1}^{J} X_{i,j},$$

$$S_i^2 = \frac{1}{J-1} w' \left(\sum_{j=1}^{J} \left(r_{i,j} - \hat{\mu}_i\right)\left(r_{i,j} - \hat{\mu}_i\right)'\right) w$$

$$= \frac{1}{J-1}\sum_{j=1}^{J}\left(X_{i,j} - \bar{X}_i\right)^2.$$

In addition, let $\mu = \mathbb{E}X_{i,j}$, $\sigma^2 = \mathbb{E}\left[X_{i,j} - \mu\right]^2$, $\mu_3 = \mathbb{E}\left[X_{i,j} - \mu\right]^3$ and $\mu_4 = \mathbb{E}\left[X_{i,j} - \mu\right]^4$. Also, for $i = 1, \ldots, I$, we define

$$f_i(w) := \hat{\mu}_i' w - \beta w' \hat{\Sigma}_i w = \bar{X}_i - \beta S_i^2,$$

$$f(w) := \min_{(\hat{\mu},\hat{\Sigma}) \in U_{(\hat{\mu},\hat{\Sigma})}} \hat{\mu}' w - \beta w' \hat{\Sigma} w = \min_i f_i(w).$$

We first give a limit result for $f_i(w)$ when $J \to \infty$.

Theorem 2.1. *For every $w \in C$, one has*

$$\sqrt{J}\left[f_i(w) - \mathbb{E}f_i(w)\right] \xrightarrow{d} N(0, \sigma^2 - \beta^2\sigma^4 - 2\beta\mu_3 + \beta^2\mu_4)$$

for $J \to \infty$.

Proof. Consider

$$\begin{aligned}
&\sqrt{J}\left[f_i(w) - \mathbb{E}f_i(w)\right] \\
&= \sqrt{J}\left(\bar{X}_i - \frac{\beta}{J-1}\sum_j \left(X_{i,j} - \bar{X}_i\right)^2\right) - \sqrt{J}\left(\mu - \beta\sigma^2\right) \\
&= \sqrt{J}\left[\bar{X}_i - \frac{\beta}{J}\left\{\sum_j \left[\left(X_{i,j} - \mu\right) - \left(\bar{X}_i - \mu\right)\right]^2 + S_i^2\right\}\right] \\
&\quad - \sqrt{J}\left(\mu - \beta\sigma^2\right) \\
&= \sqrt{J}\left[\bar{X}_i - \frac{\beta}{J}\left\{\sum_j \left(X_{i,j} - \mu\right)^2 - 2J\left(\bar{X}_i - \mu\right)^2\right.\right. \\
&\quad \left.\left. + J\left(\bar{X}_i - \mu\right)^2 + S_i^2\right\}\right] - \sqrt{J}\left(\mu - \beta\sigma^2\right) \\
&= \sqrt{J}\left[\frac{1}{J}\sum_j X_{i,j} - \frac{\beta}{J}\sum_j \left(X_{i,j} - \mu\right)^2\right. \\
&\quad \left. + \beta\left(\bar{X}_i - \mu\right)^2 - \frac{\beta}{J}S_i^2\right] - \sqrt{J}\left(\mu - \beta\sigma^2\right) \\
&= \underbrace{\sqrt{J}\left[\frac{1}{J}\sum_j \{X_{i,j} - \beta(X_{i,j} - \mu)^2\} - (\mu - \beta\sigma^2)\right]}_{(a)} \\
&\quad + \underbrace{\sqrt{J}\beta(\bar{X}_i - \mu)^2}_{(b)} - \underbrace{\frac{\beta}{\sqrt{J}}S_i^2}_{(c)}.
\end{aligned}$$

Since $\mathbb{E}\left[\frac{1}{J}\sum_j \left\{X_{i,j} - \beta\left(X_{i,j} - \mu\right)^2\right\}\right] = \mu - \beta\sigma^2$, by the Central Limit Theorem (CLT), $(a) \xrightarrow{d} N\left(0, \mathbb{V}\mathrm{ar}\left[X_{i,j} - \beta\left(X_{i,j} - \mu\right)^2\right]\right)$. Next, Since $\bar{X}_i - \mu \xrightarrow{P} 0$ by the Weak Law of Large Numbers, and $\sqrt{J}\left(\bar{X}_i - \mu\right) \xrightarrow{d} N\left(0, \sigma^2\right)$ by CLT, by Slutsky's Theorem, $(b) \xrightarrow{d} 0$. Finally, since $\frac{1}{\sqrt{J}} \to 0$ and $S_i^2 \xrightarrow{P} \sigma^2$, $(c) \xrightarrow{P} 0$. Thus, again by Slutsky's Theorem,

$$\sqrt{J}\left[f_i(w) - \mathbb{E}f_i(w)\right] \xrightarrow{d} N\left(0, \mathbb{V}\mathrm{ar}\left[X_{i,j} - \beta\left(X_{i,j} - \mu\right)^2\right]\right).$$

Since $\mathbb{C}\mathrm{ov}\left[X_{i,j}, \left(X_{i,j} - \mu\right)^2\right] = \mu_3$ and $\mathbb{V}\mathrm{ar}\left[\left(X_{i,j} - \mu\right)^2\right] = \mu_4 - \sigma^4$,

$$\begin{aligned}
&\mathbb{V}\mathrm{ar}\left[X_{i,j} - \beta\left(X_{i,j} - \mu\right)^2\right] \\
&= \mathbb{V}\mathrm{ar}\left[X_{i,j}\right] - 2\beta\,\mathbb{C}\mathrm{ov}\left[X_{i,j}, \left(X_{i,j} - \mu\right)^2\right] + \beta^2\,\mathbb{V}\mathrm{ar}\left[\left(X_{i,j} - \mu\right)^2\right] \\
&= \sigma^2 - 2\beta\mu_3 + \beta^2\mu_4 - \beta^2\sigma^4.
\end{aligned}$$

This yields $\sqrt{J}\,[f_i(w) - \mathbb{E}f_i(w)] \xrightarrow{d} N(0, \sigma^2 - \beta^2\sigma^4 - 2\beta\mu_3 + \beta^2\mu_4)$. ■

The following is a limit result for $f(w)$ when $I, J \to \infty$.

Theorem 2.2. *Let* $\mu_f = \mu - \beta\sigma^2$ *and* $\sigma_f = \left[\frac{1}{J}\left(\sigma^2 - \beta^2\sigma^4 - 2\beta\mu_3 + \beta^2\mu_4\right)\right]^{1/2}$. *Then, one has*

$$\lim_{I,J\to\infty} \mathbb{P}\left[\sqrt{2\log I}\left(\frac{f(w)-\mu_f}{\sigma_f}\right) + \log\frac{I^2}{(2\sqrt{\pi}\log I)} \geq x\right] = \exp\left[-\exp(x)\right].$$

Proof. Since $f(w) = \min_i f_i(w)$ and the $f_i(w)$'s are i.i.d., if each $f_i(w)$ were normal, $f(w)$ would be the smallest order statistic of a normal random variable with mean μ_f and variance σ_f^2. Wilks (1948) has shown that the largest order statistic $Z_{(I)}$ from I standard normal samples satisfies the limiting distribution:

$$\lim_{I\to\infty} \mathbb{P}\left[\sqrt{2\log I}Z_{(I)} - \log\frac{I^2}{(2\sqrt{\pi}\log I)} \leq x\right] = \exp\left[-\exp(-x)\right].$$

Consequently, by symmetry, for the smallest order statistic $Z_{(1)}$, we have

$$\lim_{I\to\infty} \mathbb{P}\left[\sqrt{2\log I}Z_{(1)} + \log\frac{I^2}{(2\sqrt{\pi}\log I)} \geq -x\right] = \exp\left[-\exp(-x)\right].$$

By Theorem 2.1, it follows that

$$\mathbb{P}\left[\sqrt{2\log I}\left(\frac{f(w)-\mu_f}{\sigma_f}\right) + \log\frac{I^2}{(2\sqrt{\pi}\log I)} \geq -x\right] \to \exp\left[-\exp(-x)\right]$$

for $I, J \to \infty$, which completes the proof. ■

The following theorem is the main finding of our study and shows that formulation (3) is likely to prefer portfolio returns with higher skewness and lower kurtosis.

Theorem 2.3. *Consider two feasible portfolios w^1 and w^2 for problem (3) whose returns have the same expectation and variance. Moreover, assume that w^1 has larger skewness and smaller kurtosis than w^2. More specifically, for $X^k_{i,j} = \left(w^k\right)' r_{i,j},\ k = 1, 2$, it is assumed that*

$$\mu := \mathbb{E}\left[X^1_{i,j}\right] = \mathbb{E}\left[X^2_{i,j}\right]$$
$$\sigma^2 := \mathbb{E}\left[X^1_{i,j} - EX^1_{i,j}\right]^2 = \mathbb{E}\left[X^2_{i,j} - EX^2_{i,j}\right]^2$$
$$\mu^1_3 := \mathbb{E}\left[X^1_{i,j} - EX^1_{i,j}\right]^3 \geq \mathbb{E}\left[X^2_{i,j} - EX^2_{i,j}\right]^3 := \mu^2_3$$
$$\mu^1_4 := \mathbb{E}\left[X^1_{i,j} - EX^1_{i,j}\right]^4 \leq \mathbb{E}\left[X^2_{i,j} - EX^2_{i,j}\right]^4 := \mu^2_4,$$

where at least one of two inequalities is strict. Then for $\mu_f := \mathbb{E}\left[f\left(w^1\right)\right] = \mathbb{E}\left[f\left(w^2\right)\right] = \mu - \beta\sigma^2$ one has

$$\mathbb{P}\left[f\left(w^1\right) \geq \mu_f\right], \mathbb{P}\left[f\left(w^2\right) \geq \mu_f\right] \to 0 \quad \text{for } I, J \to \infty,$$

and for $y < \mu_f$, and I, J large enough,

$$\mathbb{P}\left[f\left(w^1\right) \geq y\right] > P\left[f\left(w^2\right) \geq y\right].$$

Proof. Denote $\sigma^k_f = \eta^k/\sqrt{J}$, where $\eta^k = \sqrt{\sigma^2 - \beta^2\sigma^4 - 2\beta\mu^k_3 + \beta^2\mu^k_4}$. We know from Theorem 2.2 that for large I and J, $\mathbb{P}\left[f\left(w^1\right) \geq y\right]$ is close to $\exp\left[-\exp\left(x^k_{I,J}\right)\right]$, where

$$x^k_{I,J} := \sqrt{2\log I}\sqrt{J}\left(\frac{y-\mu_f}{\eta^k}\right) + \log\frac{I^2}{2\sqrt{\pi}\log I}.$$

In particular, $\mathbb{P}\left[f\left(w^k\right) \geq \mu_f\right]$ tends to zero for $I, J \to \infty$. Moreover, since $\eta^1 < \eta^2$, one obtains

$$x^2_{I,J} - x^1_{I,J} = \sqrt{2\log I}\sqrt{J}\left(\mu_f - y\right)\left(\frac{1}{\eta^1} - \frac{1}{\eta^2}\right) \quad \text{for } y < \mu_f,$$

from which it follows that $\mathbb{P}\left[f\left(w^1\right) \geq y\right] > P\left[f\left(w^2\right) \geq y\right]$ for large enough I, J. ■

Theorem 2.3 implies that, if the joint uncertainty set $U_{(\hat{\mu},\hat{\Sigma})}$ contains a large number of pairs of sample means and sample covariances, and also each pair of sample mean and sample covariance is obtained from a large number of samples, problem (3) is likely to favor a portfolio with high skewness and low kurtosis, a desired property sought by investors.

Intuitively speaking, as a portfolio w becomes more positively skewed, $f_i(w) = \bar{X}_i - \beta S_i^2$ becomes less dispersed. More specifically, since $\mathbb{C}\mathrm{ov}\left[\bar{X}_i, S_i^2\right] = \mu_3/J$ (Zhang, 2007), a sample variance from a more positively skewed portfolio tends to increase as the realized value of the corresponding sample mean increases. Thus, $\bar{X}_i$ and $-\beta S_i^2$ tend to move in opposite directions, causing a "diversification effect" between the two terms. Thus, the worst case solution of problem (3) gets better as the portfolio becomes more skewed, ultimately enforcing problem (3) more likely to favor portfolios that are more skewed.

Fig. 1 graphically illustrates the reasoning above, that is, the value of problem (3) is likely to be higher for a positively skewed portfolio (left figure) than for a negatively skewed one (right figure), thus is preferred by the robust approach as described above.

The same intuition holds for the kurtosis. The worst case solution tends to get worse as the variance of $f_i(w)$ increases, and because $\mathbb{V}\mathrm{ar}\left[S_i^2\right] = \mu_4 - \sigma^4$, a portfolio with a heavy tailed return distribution is penalized by problem (3).

Theorem 2.3 provides a theoretical explanation of the findings of Martin et al. (2010). They advocate the use of robust statistics in portfolio optimization, arguing that they can provide reliable identification of outliers. Our results show that robust portfolios tend to produce return distributions with thinner tails than their non-robust counterparts and are thus less likely to be affected by the outliers.

We conclude this section with the following theorem which guarantees that one can easily obtain the solution for problem (3).

Theorem 2.4. *If C is defined only with a finite number of linear constraints, the solution for problem (3) can be obtained by solving a convex quadratically constrained quadratic program (QCQP) problem:*

$$\max_{w\in C, z} \quad z$$
$$\text{s.t.} \quad z \leq \hat{\mu}_i' w - \beta w' \hat{\Sigma}_i w \quad \text{for } i = 1, \ldots, I.$$

Proof. Proof is omitted as it is well known that a linear program with quadratic constraints can be efficiently solved by Newton-type algorithms. See Boyd and Vandenberghe (2004). ■

3. Empirical analysis

In our empirical analysis, we construct both mean–variance optimal portfolios and robust portfolios with daily returns of the 10 industry portfolios for the years ranging from 1983 to 2012 used in a series of papers by Fama and French, yielding 7566 days in total. The data set is obtained from the online data library of Kenneth

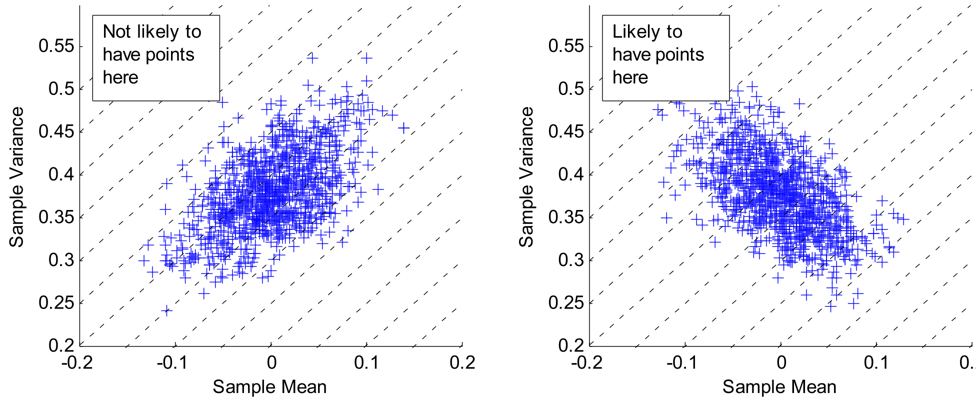

Fig. 1. Sample means and sample variances of skew normal random variables. While the first two moments of the random variables represented by the two figures above are identical, the one on the left figure is positively skewed whereas the one on the right figure is negatively skewed. Dotted lines represent the indifference curves for the objective function of problem (3) when $\beta = 1$. Consequently, the solution for the inner problem of (3) gets worse as more points appear in the north-west region. A positively skewed portfolio (left) is less likely to have points in the north-west region than a negatively skewed portfolio (right). Thus it is likely to have a higher value. See Azzalini and Dalla Valle (1996) and Azzalini and Capitanio (1999) for the details of skew normal distribution.

Table 1
Summary statistics of 10 industry portfolios with market portfolio based on daily returns.

Moments	First ($\times 10^{-4}$)	Second central ($\times 10^{-4}$)	Third central ($\times 10^{-7}$)	Fourth central ($\times 10^{-7}$)
Market	2.97	1.26	−9.02	3.00
Consumer non-durables	4.07	0.93	−6.94	2.04
Consumer durables	2.52	2.24	−11.34	5.67
Manufacturing	3.60	1.34	−13.97	4.05
Oil, gas, and coal extraction and products	4.21	2.14	−8.10	8.67
Business equipment	2.96	2.59	−1.91	7.93
Telephone and television transmission	3.08	1.62	−2.58	4.02
Wholesale, retail, and some services	3.51	1.40	−6.03	2.75
Healthcare, medical equipment, and drugs	3.61	1.35	−9.36	2.90
Utilities	3.00	0.95	−0.37	2.17
Other	2.86	1.72	−5.82	4.91

R. French.[1] To solve problem (3), we solve the equivalent convex QCQP formulation as given in Theorem 2.4. As for the solver, we employ CVX, a package for specifying and solving convex programs by CVX Research, Inc. (2011) and Grant and Boyd (2008). The computation times are not reported because they are trivial in all cases. Table 1 provides summary statistics for the 10 industry portfolios along with the US market portfolio which is also obtained from French's database.

We conduct three sets of empirical tests to see if the claims in the previous section hold. More specifically, we compare third and fourth moments of mean–variance optimal portfolios and robust portfolios by varying (1) β: risk aversion parameter, (2) I: number of pairs of sample means and sample covariance matrices in the uncertainty set, and (3) J: number of daily returns used to obtain the sample mean and sample covariance matrix. The parameters for the base case are set as follows: $\beta = 1$, $I = 100$, and $J = 1$.

Fig. 2 illustrates the results for the base case. We construct 100 robust portfolios along with the mean–variance optimal portfolio whose parameters are obtained from the whole sample data. Each dot in Fig. 2 represents the pair of third and fourth central moments for each robust portfolio estimated over the whole sample period. Of 100 robust portfolios constructed, 100 have higher third central moments and lower fourth central moments, indicating that problem (3) indeed favors skewness and penalizes kurtosis.

Fig. 3 depicts the test results when β, I and J are varied while other parameters are set as the base case. Panel A illustrates that, in most of the cases, the robust portfolios dominate the mean–variance optimal portfolios in both moments (that is, higher third central moment and lower fourth central moment), indicating that

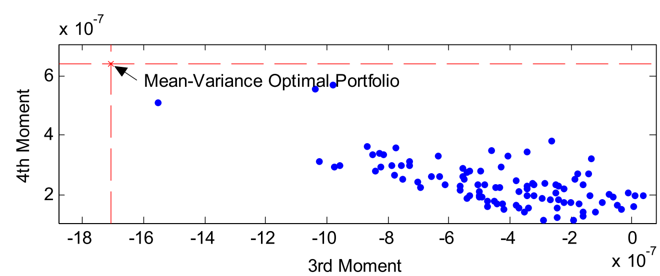

Fig. 2. Test results for based case ($\beta = 1$, $I = 100$, and $J = 1000$).

our argument is insensitive to the changes in β. Panels B and C summarize the results when the values of I and J are varied respectively. These results depict that the robust portfolios are likely to have better third and fourth central moments than the mean–variance as I and J increase, as proved in Theorem 2.3.

4. Conclusions and future directions

In this paper, we demonstrate mathematically that if the uncertainty sets are properly constructed, robust portfolios based on the worst-case approach under the mean–variance setting favors skewness and penalizes kurtosis. Based on this finding, we propose a new mean–variance approach that can control portfolio skewness and kurtosis without imposing higher moment terms. The empirical evidence presented supports this finding.

One promising direction is to link the findings of this study to research utilizing L-moments and trimmed L-moments for financial applications such as Darolles et al. (2009), Qin (2012), and Yanou (2013). One of the main findings of this study is that the per-

[1] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

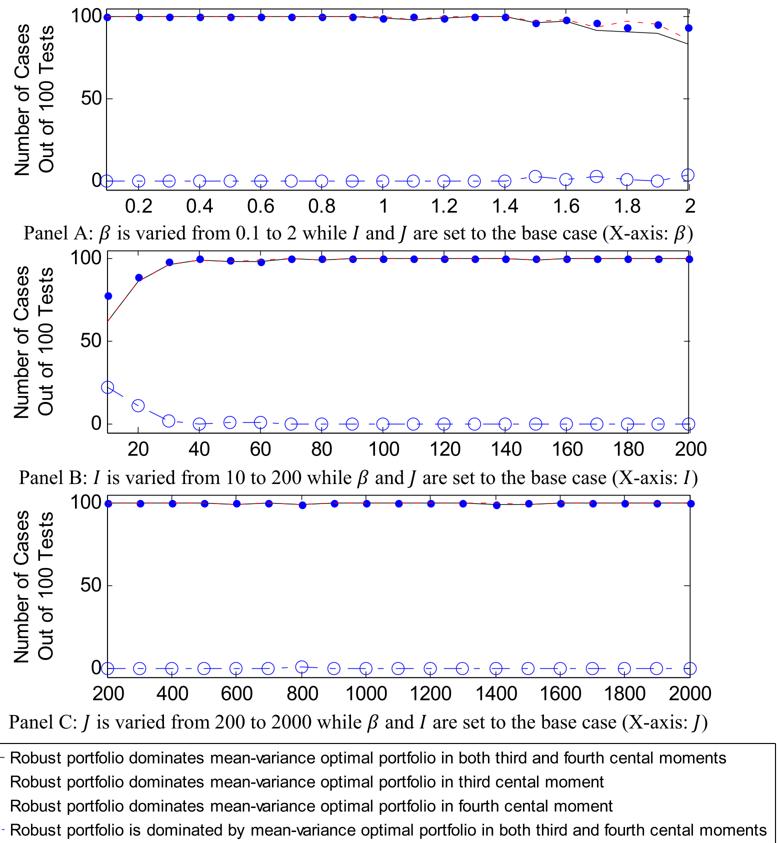

Fig. 3. Test results when β (Panel A), I (Panel B) and J (Panel C) are varied. This figure depicts the test results when β, I and J are varied while other parameters are set as the base case. Similar to Fig. 2, for each value of β (I or J), we construct 100 robust portfolios along with the mean–variance optimal portfolio, and compare their third and fourth central moments. The black solid line exhibits, out of 100 tests, the number of cases that the robust portfolios dominate the mean–variance portfolio in both third and fourth central moments. The blue dots and red dotted line represent the number of cases that the robust portfolios have higher third central moments, and lower fourth central moments, respectively. The blue dotted line with circles illustrates the number of cases that the robust portfolios are dominated by the mean–variance portfolio in both third and fourth central moments.

formance of a robust portfolio is closely related to the distribution of the smallest order statistic. It has been shown that the portfolios constructed with L-moments, which also are closely related to order statistics, yield robust performance. Therefore, investigation of the link between the two would be insightful.

Another direction for research is to extend the main idea of this study to moments higher than the fourth moment. As illustrated by Eeckhoudt and Schlesinger (2006) from a behavioral finance perspective, the derivatives of the expected utility functions alternate in sign. Consequently, if the same argument holds for higher moments in a recursive fashion, it is possible to create a generalized portfolio construction framework in which higher moments can be controlled with less computational cost than traditional approaches.

Finally, the empirical study in this paper can be extended to see if the proposed approach can improve the investment performance in a conventional sense. Our empirical study shows that the robust portfolios have better characteristics than the traditional mean–variance portfolios in terms of higher moments. However, this does not mean that robust portfolios outperform their non-robust counterparts. This warrants a set of comprehensive empirical tests to confirm if the newly proposed approach yields better investment performance.

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