FRACTIONAL ORNSTEIN-UHLENBECK PROCESSES

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Abstract The classical stationary Ornstein-Uhlenbeck process can be obtained in two different ways. On the one hand, it is a stationary solution of the Langevin equation with Brownian motion noise. On the other hand, it can be obtained from Brownian motion by the so called Lamperti transformation. We show that the Langevin equation with fractional Brownian motion noise also has a stationary solution and that the decay of its auto-covariance function is like that of a power function. Contrary to that, the stationary process obtained from fractional Brownian motion by the Lamperti transformation has an auto-covariance function that decays exponentially.

Keywords Fractional Brownian motion, Langevin equation, long-range dependence, selfsimilar processes, Lamperti transformation

AMS subject classification (2000) primary: 60H10; secondary: 60G15, 60G18, 45F05

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition 1.1 *A fractional Brownian motion with Hurst parameter $H \in (0,1]$, is an almost surely continuous, centered Gaussian process $(B_t^H)_{t\in\mathbb{R}}$ with*

$$\operatorname{Cov}\left(B_t^H, B_s^H\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \; t, s \in \mathbb{R}\,. \tag{1.1}$$

[1]The first author would like to thank the Department of Mathematics, Keio University, for having made possible his stay in Yokohama.

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For an in-depth introduction to fractional Brownian motions we refer the reader to Section 7.2 of Samorodnitsky and Taqqu (1994) or Chapter 4 of Embrechts and Maejima (2002). It is clear that for all $H \in (0,1]$, $B_0^H = 0$ almost surely. Moreover, it can be deduced from (1.1) that for all $H \in (0,1]$, $(B_t^H)_{t\in\mathbb{R}}$ has stationary increments and is H-selfsimilar, that is, for every $c>0$, $(B_{ct}^H)_{t\in\mathbb{R}} \overset{d}{=} (c^H B_t^H)_{t\in\mathbb{R}}$, where $\overset{d}{=}$ denotes equality of all finite-dimensional distributions. $(B_t^{\frac{1}{2}})_{t\in\mathbb{R}}$ is a two-sided Brownian motion. In particular, it has independent increments. For $H \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$, $(B_t^H)_{t\geq 0}$ is not a semimartingale and it can be derived from (1.1) that for all $h \in \mathbb{R}$ and $0 < t < s$,

$$\begin{aligned} &\mathrm{Cov}\left(B_{h+t}^H - B_h^H, B_{h+s+t}^H - B_{h+s}^H\right) = \mathrm{Cov}\left(B_t^H, B_{s+t}^H - B_s^H\right) \\ = &\sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!}\left(\prod_{k=0}^{2n-1}(2H-k)\right) s^{2H-2n}, \end{aligned}$$

in particular, for every $N = 1,2,\ldots$, for all $h \in \mathbb{R}$ and $t > 0$,

$$\begin{aligned} &\mathrm{Cov}\left(B_{h+t}^H - B_h^H, B_{h+s+t}^H - B_{h+s}^H\right) \\ = &\sum_{n=1}^{N} \frac{t^{2n}}{(2n)!}\left(\prod_{k=0}^{2n-1}(2H-k)\right) s^{2H-2n} + O(s^{2H-2N-2}), \text{ as } s \to \infty. \end{aligned} \tag{1.2}$$

This shows that for $H \in (\frac{1}{2},1]$,

$$\sum_{n=0}^{\infty} \mathrm{Cov}\left(B_t^H, B_{(n+1)t}^H - B_{nt}^H\right) = \infty,$$

a phenomenon referred to as long-range dependence or long memory of the increments process

$$\left(B_{(n+1)t}^H - B_{nt}^H\right)_{n=0}^{\infty}.$$

The classical Ornstein-Uhlenbeck process with parameters $\lambda > 0$ and $\sigma > 0$ starting at $x \in \mathbb{R}$, is the unique strong solution of the Langevin equation with Brownian motion noise

$$X_t = \xi - \lambda \int_0^t X_s ds + \sigma B_t^{\frac{1}{2}}, \quad t \geq 0, \tag{1.3}$$

with initial condition $\xi = x$. It is given by the almost surely continuous Gaussian Markov process

$$Y_t^{\frac{1}{2},x} := e^{-\lambda t}\left(x + \sigma \int_0^t e^{\lambda u} dB_u^{\frac{1}{2}}\right), \quad t \geq 0.$$

The unique strong solution of (1.3) with initial condition

$$\xi = \sigma \int_{-\infty}^0 e^{\lambda u} dB_u^{\frac{1}{2}},$$

is given by the restriction to non-negative t's of the stationary, almost surely continuous, centered Gaussian Markov process

$$Y_t^{\frac{1}{2}} := \sigma \int_{-\infty}^t e^{-\lambda(t-u)} dB_u^{\frac{1}{2}}, \quad t \in \mathbb{R}.$$

It can easily be checked that

$$\operatorname{Cov}\left(Y_t^{\frac{1}{2}}, Y_{t+s}^{\frac{1}{2}}\right) = \frac{\sigma^2}{2\lambda} e^{-\lambda|s|}, \quad t, s \in \mathbb{R}.$$

This implies that $(Y_t^{\frac{1}{2}})_{t\in\mathbb{R}}$ is ergodic. Moreover, for all $x \in \mathbb{R}$,

$$Y_t^{\frac{1}{2}} - Y_t^{\frac{1}{2},x} = e^{-\lambda t}\left(Y_0^{\frac{1}{2}} - x\right) \to 0, \text{ as } t \to \infty,$$

almost surely. From this it can be derived that if $(Y_t)_{t\geq 0}$ is a stationary process that solves (1.3) with any initial condition $\xi \in L^0(\Omega)$, then $(Y_t)_{t\geq 0} \stackrel{d}{=} (Y_t^{\frac{1}{2}})_{t\geq 0}$.

Now let $\alpha > 0$. Then,

$$Z_t^{\frac{1}{2}} := e^{-\lambda t} B_{\alpha e^{2\lambda t}}^{\frac{1}{2}}, \quad t \in \mathbb{R},$$

is also a stationary, almost surely continuous, centered Gaussian process, and

$$\operatorname{Cov}\left(Z_t^{\frac{1}{2}}, Z_{t+s}^{\frac{1}{2}}\right) = \alpha e^{-\lambda|s|}, \quad t, s \in \mathbb{R}.$$

Hence, for $\alpha = \frac{\sigma^2}{2\lambda}$, $(Y_t^{\frac{1}{2}})_{t\in\mathbb{R}} \stackrel{d}{=} (Z_t^{\frac{1}{2}})_{t\in\mathbb{R}}$.

It is shown in Lamperti (1962) that for every $H > 0$, a stochastic process $(X_t)_{t\geq 0}$ is H-selfsimilar if and only if for all $\lambda, \alpha > 0$, the process

$$\widehat{X}_t = e^{-\lambda t} X_{\alpha \exp\left(\frac{\lambda}{H} t\right)}, \quad t \in \mathbb{R}, \tag{1.4}$$

is stationary. We call (1.4) the Lamperti transformation from selfsimilar processes to stationary processes and $(\widehat{X}_t)_{t\in\mathbb{R}}$ the Lamperti transform of $(X_t)_{t\geq 0}$.

For $H = 1$, fractional Brownian motion can be represented as follows:

$$B_t^1 = t\eta, \quad t \in \mathbb{R},$$

where η is a standard normal random variable. For every initial condition $\xi \in L^0(\Omega)$, the equation,

$$X_t = \xi - \lambda \int_0^t X_s ds + \sigma B_t^1, \quad t \geq 0, \tag{1.5}$$

can path-wise be reduced to the ordinary differential equations,

$$X_t'(\omega) = -\lambda X_t(\omega) + \sigma\eta(\omega), \quad \omega \in \Omega,$$

with initial conditions

$$X_0(\omega) = \xi(\omega), \quad \omega \in \Omega,$$

which have the unique solutions

$$Y_t^{1,\xi}(\omega) := e^{-\lambda t}\left\{\xi(\omega) - \frac{\sigma}{\lambda}\eta(\omega)\right\} + \frac{\sigma}{\lambda}\eta(\omega), \quad t \geq 0, \omega \in \Omega.$$

Equation (1.5) has only a stationary solution for the initial condition $\xi = \frac{\sigma}{\lambda}\eta$. It is given by

$$Y_t^1 := \frac{\sigma}{\lambda}\eta, \quad t \geq 0,$$

which, for all $t \geq 0$, equals the Lamperti transform

$$Z_t^1 := e^{-\lambda t} B^1_{\alpha \exp(\lambda t)} = \alpha \eta \,, \quad t \in \mathbb{R} \,,$$

if $\alpha = \frac{\sigma}{\lambda}$.

This leads us to the question whether for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, the Langevin equation with noise process $(\sigma B_t^H)_{t \geq 0}$ has a stationary solution, if its distribution is unique and if it is equal in some sense to the Lamperti transform

$$Z_t^H := e^{-\lambda t} B^H_{\alpha \exp(\frac{\lambda}{H} t)} \,, \quad t \in \mathbb{R} \,,$$

for an appropriately chosen $\alpha > 0$.

The structure of the paper is as follows. In Section 2 we show that for all $H \in (0, 1]$, the Langevin equation with fractional Brownian motion noise has for all initial conditions $\xi \in L^0(\Omega)$, a unique strong solution $(Y_t^{H,\xi})_{t \geq 0}$. Moreover, there exists a stationary, almost surely continuous, centered Gaussian process $(Y_t^H)_{t \in \mathbb{R}}$ such that $(Y_t^H)_{t \geq 0}$ solves the Langevin equation with fractional Brownian motion noise, and every other stationary solution is equal to $(Y_t^H)_{t \geq 0}$ in distribution. The decay of the auto-covariance function of $(Y_t^H)_{t \in \mathbb{R}}$ is for all $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ similar to that of the increments of $(B_t^H)_{t \in \mathbb{R}}$ (see (1.2)). In particular, $(Y_t^H)_{t \in \mathbb{R}}$ is ergodic, and for $H \in (\frac{1}{2}, 1]$, it exhibits long-range dependence. In Section 3 we show that for all $H \in (0, 1)$ the auto-covariance function of $(Z_t^H)_{t \in \mathbb{R}}$ decays exponentially, which implies that for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, $(Y_t^H)_{t \in \mathbb{R}}$ cannot have the same distribution as $(Z_t^H)_{t \in \mathbb{R}}$.

2 Fractional Ornstein-Uhlenbeck processes

Let λ, $\sigma > 0$ and $\xi \in L^0(\Omega)$. Since the Langevin equation,

$$X_t = \xi - \lambda \int_0^t X_s ds + N_t \,, \quad t \geq 0 \,,$$

only involves an integral with respect to t, it can be solved path-wise for much more general noise processes $(N_t)_{t \geq 0}$ than Brownian motion. For example, it follows from Proposition A.1 that for each $H \in (0, 1]$ and for every $a \in [-\infty, \infty)$,

$$\int_a^t e^{\lambda u} dB_u^H \,, \quad t > a \,,$$

exists as a path-wise Riemann-Stieltjes integral, which is almost surely continuous in t, and

$$Y_t^{H,\xi} := e^{-\lambda t} \left(\xi + \sigma \int_0^t e^{\lambda u} dB_u^H \right) \,, \quad t \geq 0 \,,$$

is the unique almost surely continuous process that solves the equation,

$$X_t = \xi - \lambda \int_0^t X_s ds + \sigma B_t^H \,, \quad t \geq 0 \,. \tag{2.1}$$

In particular, the restriction to positive t's of the almost surely continuous process

$$Y_t^H := \sigma \int_{-\infty}^t e^{-\lambda(t-u)} dB_u^H \,, \quad t \in \mathbb{R}\,,$$

solves (2.1) with initial condition $\xi = Y_0^H$. It is clear that $(Y_t^H)_{t\in\mathbb{R}}$ is a Gaussian process, and it follows immediately from the stationarity of the increments of fractional Brownian motion that it is stationary. Furthermore, as in the Brownian motion case, for every $\xi \in L^0(\Omega)$,

$$Y_t^H - Y_t^{H,\xi} = e^{-\lambda t}\left(Y_0^H - \xi\right) \to 0\,, \text{ as } t \to \infty\,, \text{ almost surely,}$$

which implies that every stationary solution of (2.1) has the same distribution as $(Y_t^H)_{t\geq 0}$. We call $(Y_t^{H,\xi})_{t\geq 0}$ a *fractional Ornstein-Uhlenbeck process with initial condition* ξ and $(Y_t^H)_{t\in\mathbb{R}}$ a *stationary fractional Ornstein-Uhlenbeck process.*

In Pipiras and Taqqu (2000) it is shown that for $H \in (\frac{1}{2}, 1)$ and two real-valued measurable functions

$$f, g \in \left\{ f \,:\, \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(u)|\,|f(v)|\,|u-v|^{2H-2}\, dudv < \infty \right\},$$

the two integrals $\int_{-\infty}^{\infty} f(u)dB_u^H$, $\int_{-\infty}^{\infty} g(u)dB_u^H$ can in a consistent way be defined as limits of integrals of elementary functions, and

$$\mathrm{E}\left[\int_{-\infty}^{\infty} f(u)dB_u^H \int_{-\infty}^{\infty} g(u)dB_u^H\right] = H(2H-1)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)g(v)\,|u-v|^{2H-2}\, dudv\,.$$

For $H \in (0, \frac{1}{2})$, the kernel $|u-v|^{2H-2}$ cannot be integrated over the diagonal. However, if f and g are regular enough and the intersection of their supports is of Lebesgue measure zero, the same holds true. We will only need this result for the case where f and g are given by $f(u) = g(u) = e^{\lambda u}$ and their supports are disjoint intervals. However, the following lemma can easily be generalized.

Lemma 2.1 *Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, $\lambda > 0$ and $-\infty \leq a < b \leq c < d < \infty$. Then*

$$\mathrm{E}\left[\int_a^b e^{\lambda u} dB_u^H \int_c^d e^{\lambda v} dB_v^H\right] = H(2H-1)\int_a^b e^{\lambda u}\left(\int_c^d e^{\lambda v}(v-u)^{2H-2} dv\right) du\,.$$

Proof. We first assume $b = 0 = c$. By Proposition A.1 a) we get

$$\begin{aligned}
& \mathrm{E}\left[\int_a^0 e^{\lambda u} dB_u^H \int_0^d e^{\lambda v} dB_v^H\right] \\
= \; & \mathrm{E}\left[\left(-e^{\lambda a}B_a^H - \lambda\int_a^0 e^{\lambda u}B_u^H du\right)\left(e^{\lambda d}B_d^H - \lambda\int_0^d e^{\lambda v}B_v^H dv\right)\right] \\
= \; & -\frac{1}{2}e^{\lambda a}e^{\lambda d}\left[(-a)^{2H} + d^{2H} - (d-a)^{2H}\right] \\
& +\frac{1}{2}\lambda e^{\lambda a}\int_0^d e^{\lambda v}\left[(-a)^{2H} + v^{2H} - (v-a)^{2H}\right] dv \\
& -\frac{1}{2}\lambda e^{\lambda d}\int_a^0 e^{\lambda u}\left[(-u)^{2H} + d^{2H} - (d-u)^{2H}\right] du \\
& +\frac{1}{2}\lambda^2\int_0^d e^{\lambda v}\left(\int_a^0 e^{\lambda u}\left[(-u)^{2H} + v^{2H} - (v-u)^{2H}\right] du\right) dv\,.
\end{aligned}$$

After partial integration with respect to u, this becomes

$$\begin{aligned}&-He^{\lambda d}\int_a^0 e^{\lambda u}\left[(-u)^{2H-1}-(d-u)^{2H-1}\right]du\\&+H\lambda\int_0^d e^{\lambda v}\left(\int_a^0 e^{\lambda u}\left[(-u)^{2H-1}-(v-u)^{2H-1}\right]du\right)dv\,,\end{aligned}$$

which, by partial integration with respect to v, is equal to

$$H(2H-1)\int_a^0 e^{\lambda u}\left(\int_0^d e^{\lambda v}(v-u)^{2H-2}dv\right)du\,.$$

Now we assume $b=0<c$. It follows from above that

$$\begin{aligned}&\mathrm{E}\left[\int_a^0 e^{\lambda u}dB_u^H\int_c^d e^{\lambda v}dB_v^H\right]=\mathrm{E}\left[\int_a^0 e^{\lambda u}dB_u^H\int_0^d e^{\lambda v}dB_v^H-\int_a^0 e^{\lambda u}dB_u^H\int_0^c e^{\lambda v}dB_v^H\right]\\=\;&H(2H-1)\left[\int_a^0 e^{\lambda u}\left(\int_0^d e^{\lambda v}(v-u)^{2H-2}dv\right)du-\int_a^0 e^{\lambda u}\left(\int_0^c e^{\lambda v}(v-u)^{2H-2}dv\right)du\right]\\=\;&H(2H-1)\int_a^0 e^{\lambda u}\left(\int_c^d e^{\lambda v}(v-u)^{2H-2}dv\right)du\,.\end{aligned}$$

For general $-\infty\le a<b\le c<d<\infty$, the process $\tilde{B}_t^H=B_{t+b}^H-B_b^H$, $t\in\mathbb{R}$, is again a fractional Brownian motion. Therefore,

$$\begin{aligned}&\mathrm{E}\left[\int_a^b e^{\lambda u}dB_u^H\int_c^d e^{\lambda v}dB_v^H\right]=\mathrm{E}\left[\int_{a-b}^0 e^{\lambda(w+b)}d\tilde{B}_w^H\int_{c-b}^{d-b} e^{\lambda(x+b)}d\tilde{B}_x^H\right]\\=\;&H(2H-1)\int_{a-b}^0 e^{\lambda(w+b)}\left(\int_{c-b}^{d-b} e^{\lambda(x+b)}(x-w)^{2H-2}dx\right)dw\\=\;&H(2H-1)\int_a^b e^{\lambda u}\left(\int_c^d e^{\lambda v}(v-u)^{2H-2}dv\right)du\,,\end{aligned}$$

and the proof is complete. □

Lemma 2.2 *Let $\beta<0$. Then for each $N=0,1,2,\dots$,*

$$e^x\int_x^\infty e^{-y}y^\beta dy=x^\beta+\sum_{n=1}^N\left(\prod_{k=0}^{n-1}(\beta-k)\right)x^{\beta-n}+O(x^{\beta-N-1})\,,\quad \textit{as } x\to\infty\,,$$

and

$$e^{-x}\int_1^x e^y y^\beta dy=x^\beta+\sum_{n=1}^N(-1)^n\left(\prod_{k=0}^{n-1}(\beta-k)\right)x^{\beta-n}+O(x^{\beta-N-1})\,,\quad \textit{as } x\to\infty\,,$$

where $\sum_{n=1}^0$ means 0.

Proof. We have

$$\begin{aligned}
& e^x \int_x^\infty e^{-y} y^\beta dy \\
= \;& e^x \left(e^{-x} x^\beta + \beta \int_x^\infty e^{-y} y^{\beta-1} dy \right) = \ldots \\
= \;& x^\beta + \beta x^{\beta-1} + \beta(\beta-1)x^{\beta-2} + \ldots + \beta(\beta-1)\ldots(\beta-N+1)x^{\beta-N} \\
& + e^x \beta(\beta-1)\ldots(\beta-N) \int_x^\infty e^{-y} y^{\beta-N-1} dy \,,
\end{aligned}$$

and

$$e^x \int_x^\infty e^{-y} y^{\beta-N-1} dy \le e^x \int_x^\infty e^{-y} x^{\beta-N-1} dy = x^{\beta-N-1} \,,$$

which proves the first assertion. On the other hand,

$$\begin{aligned}
& e^{-x} \int_1^x e^y y^\beta dy \\
= \;& e^{-x} \left(e^x x^\beta - e - \beta \int_1^x e^y y^{\beta-1} dy \right) = \ldots \\
= \;& x^\beta - \beta x^{\beta-1} + \ldots + (-1)^N \beta(\beta-1)\ldots(\beta-N+1)x^{\beta-N} \\
& - e^{-x} e \left\{ 1 - \beta + \ldots + (-1)^N \beta(\beta-1)\ldots(\beta-N+1) \right\} \\
& - e^{-x} (-1)^N \beta(\beta-1)\ldots(\beta-N) \int_1^x e^y y^{\beta-N-1} dy \,,
\end{aligned}$$

and

$$e^{-x} \int_1^x e^y y^{\beta-N-1} dy \le e^{-x} \left(\int_1^{\frac{x}{2}} e^y dy + \int_{\frac{x}{2}}^x e^y \left(\frac{x}{2}\right)^{\beta-N-1} dy \right) \le e^{-\frac{x}{2}} + \left(\frac{x}{2}\right)^{\beta-N-1} \,.$$

This proves the second part of the lemma. □

Theorem 2.3 *Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ and $N = 1, 2, \ldots$. Then for fixed $t \in \mathbb{R}$ and $s \to \infty$,*

$$\mathrm{Cov}\left(Y_t^H, Y_{t+s}^H\right) = \frac{1}{2}\sigma^2 \sum_{n=1}^N \lambda^{-2n} \left(\prod_{k=0}^{2n-1} (2H-k) \right) s^{2H-2n} + O(s^{2H-2N-2}) \,.$$

Proof. By Lemma 2.1,

$$\begin{aligned}
& \mathrm{Cov}\left(Y_t^H, Y_{t+s}^H\right) = \mathrm{Cov}\left(Y_0^H, Y_s^H\right) \\
= \;& \mathrm{E}\left[\sigma \int_{-\infty}^0 e^{\lambda u} dB_u^H \sigma \int_{-\infty}^s e^{-\lambda(s-v)} dB_v^H \right] \\
= \;& e^{-\lambda s} \mathrm{E}\left[\sigma \int_{-\infty}^0 e^{\lambda u} dB_u^H \sigma \int_{-\infty}^{\frac{1}{\lambda}} e^{\lambda v} dB_v^H \right] \\
& + \sigma^2 H(2H-1) e^{-\lambda s} \int_{-\infty}^0 e^{\lambda u} \left(\int_{\frac{1}{\lambda}}^s e^{\lambda v} (v-u)^{2H-2} dv \right) du
\end{aligned}$$

(by the change of variables: $w = \lambda u$, $x = \lambda v$)

$$= \frac{\sigma^2}{\lambda^{2H}} H(2H-1) e^{-\lambda s} \int_{-\infty}^{0} e^{w} \left(\int_{1}^{\lambda s} e^{x} (x-w)^{2H-2} dx \right) dw + O(e^{-\lambda s})$$

(by the change of variables: $y = x - w$, $z = x + w$)

$$= \frac{\sigma^2}{2\lambda^{2H}} H(2H-1) e^{-\lambda s} \left\{ \int_{1}^{\lambda s} y^{2H-2} \left(\int_{2-y}^{y} e^{z} dz \right) dy \right.$$
$$\left. + \int_{\lambda s}^{\infty} y^{2H-2} \left(\int_{2-y}^{2\lambda s - y} e^{z} dz \right) dy \right\} + O(e^{-\lambda s})$$
$$= \frac{\sigma^2}{2\lambda^{2H}} H(2H-1) e^{-\lambda s}$$
$$\times \left\{ \int_{1}^{\lambda s} e^{y} y^{2H-2} dy + \int_{\lambda s}^{\infty} e^{2\lambda s - y} y^{2H-2} dy - \int_{1}^{\infty} e^{2-y} y^{2H-2} dy \right\} + O(e^{-\lambda s})$$
$$= \frac{\sigma^2}{2\lambda^{2H}} H(2H-1) \left\{ e^{-\lambda s} \int_{1}^{\lambda s} e^{y} y^{2H-2} dy + e^{\lambda s} \int_{\lambda s}^{\infty} e^{-y} y^{2H-2} dy \right\} + O(e^{-\lambda s}) .$$

The proof can now be concluded by applying Lemma 2.2. □

Theorem 2.3 shows that for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, the decay of

$$\mathrm{Cov}\left(Y_t^H, Y_{t+s}^H\right), \quad \text{for } s \to \infty ,$$

is very similar to the decay of

$$\mathrm{Cov}\left(B_{h+t}^H - B_h^H, B_{h+s+t}^H - B_{h+s}^H\right), \quad \text{for } s \to \infty$$

(see (1.2)). In particular, $(Y_t^H)_{t \in \mathbb{R}}$ is ergodic, and for $H \in (\frac{1}{2}, 1]$, it exhibits long-range dependence.

Remark 2.4 Let $s \in \mathbb{R}$. For all $H \in (0,1)$, the functions $f(x) = 1_{\{x \le 0\}} e^{\lambda x}$ and $g(x) = 1_{\{x \le s\}} e^{\lambda x}$ belong to the inner product space $\tilde{\Lambda}_H$ defined on page 289 of Pipiras and Taqqu (2000). Hence, for all $t, s \in \mathbb{R}$, $\mathrm{Cov}\left(Y_t^H, Y_{t+s}^H\right)$ is equal to

$$\sigma^2 e^{-\lambda s} (f,g)_{\tilde{\Lambda}_H} = \sigma^2 \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}sx} \frac{|x|^{1-2H}}{\lambda^2 + x^2} dx . \tag{2.2}$$

Therefore, the expression given in the the statement of Theorem 2.3 is an asymptotic expansion of the right hand side in (2.2) as $s \to \infty$.

The next corollary shows that for the solution $(Y_t^{H,x})_{t \ge 0}$ of (2.1) with deterministic initial value $Y_0^{H,x} = x \in \mathbb{R}$,

$$\mathrm{Cov}\left(Y_t^{H,x}, Y_{t+s}^{H,x}\right), \quad \text{for } s \to \infty ,$$

decays like a power function of the order $2H - 2$ as well.

Corollary 2.5 *Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, $x \in \mathbb{R}$ and $N = 1, 2, \dots$. Then for fixed $t \ge 0$ and $s \to \infty$,*

$$\mathrm{Cov}\left(Y_t^{H,x}, Y_{t+s}^{H,x}\right)$$
$$= \frac{1}{2}\sigma^2 \sum_{n=1}^{N} \lambda^{-2n} \left(\prod_{k=0}^{2n-1} (2H-k) \right) \left\{ s^{2H-2n} - e^{-\lambda t} (t+s)^{2H-2n} \right\} + O(s^{2H-2N-2}) .$$

Proof.

$$
\begin{aligned}
& \operatorname{Cov}\left(Y_t^{H,x}, Y_{t+s}^{H,x}\right) \\
= \ & \mathrm{E}\left[\sigma \int_0^t e^{-\lambda(t-u)} dB_u^H \sigma \int_0^{t+s} e^{-\lambda(t+s-v)} dB_v^H\right] \\
= \ & \mathrm{E}\left[\sigma \int_0^t e^{-\lambda(t-u)} dB_u^H \sigma \left(\int_{-\infty}^{t+s} e^{-\lambda(t+s-v)} dB_v^H - e^{-\lambda s} \int_{-\infty}^0 e^{-\lambda(t-v)} dB_v^H\right)\right] \\
= \ & \mathrm{E}\left[\sigma \left(\int_{-\infty}^t e^{-\lambda(t-u)} dB_u^H - e^{-\lambda t} \int_{-\infty}^0 e^{\lambda u} dB_u^H\right) \left(\int_{-\infty}^{t+s} e^{-\lambda(t+s-v)} dB_v^H\right)\right] \\
& -e^{-\lambda s} \mathrm{E}\left[\sigma \int_0^t e^{-\lambda(t-u)} dB_u^H \sigma \int_{-\infty}^0 e^{-\lambda(t-v)} dB_v^H\right] \\
= \ & \operatorname{Cov}\left(Y_t^H, Y_{t+s}^H\right) - e^{-\lambda t} \operatorname{Cov}\left(Y_0^H, Y_{t+s}^H\right) + O(e^{-\lambda s})\,.
\end{aligned}
$$

Now, the corollary follows from Theorem 2.3. □

3 The Lamperti transform of fractional Brownian motion

Let $\lambda > 0$ and $\alpha > 0$. For each $H \in (0,1]$, we set

$$
Z_t^H := e^{-\lambda t} B^H_{\alpha \exp\left(\frac{\lambda}{H} t\right)}\,, \quad t \in \mathbb{R}\,.
$$

Theorem 3.1 *Let $H \in (0,1]$ and $t, s \in \mathbb{R}$. Then*

$$
\operatorname{Cov}\left(Z_t^H, Z_{t+s}^H\right) = \frac{\alpha^{2H}}{2} \left\{ e^{-\lambda|s|} + \sum_{n=1}^{\infty} (-1)^{n-1} \binom{2H}{n} e^{-\lambda\left(\frac{n}{H}-1\right)|s|} \right\}\,. \tag{3.1}
$$

Proof. Without loss of generality we can assume that $s \geq 0$. Then,

$$
\begin{aligned}
\operatorname{Cov}\left(Z_t^H, Z_{t+s}^H\right) &= e^{-\lambda t} e^{-\lambda(t+s)} \frac{\alpha^{2H}}{2} \left\{ e^{2\lambda(t+s)} + e^{2\lambda t} - \left(e^{\frac{\lambda}{H}(t+s)} - e^{\frac{\lambda}{H} t}\right)^{2H} \right\} \\
&= \frac{\alpha^{2H}}{2} e^{\lambda s} \left\{ 1 + e^{-2\lambda s} - \left(1 - e^{-\frac{\lambda}{H} s}\right)^{2H} \right\} \\
&= \frac{\alpha^{2H}}{2} e^{\lambda s} \left\{ 1 + e^{-2\lambda s} - \sum_{n=0}^{\infty} \binom{2H}{n} \left(-e^{-\frac{\lambda}{H} s}\right)^n \right\} \\
&= \frac{\alpha^{2H}}{2} \left\{ e^{-\lambda s} + \sum_{n=1}^{\infty} (-1)^{n-1} \binom{2H}{n} e^{-\lambda\left(\frac{n}{H}-1\right) s} \right\}\,,
\end{aligned}
$$

which proves the theorem. □

It follows from Theorem 3.1 that for every $N = 1, 2, \ldots$, for each $H \in (0,1)$ and all $t \in \mathbb{R}$,

$$
\operatorname{Cov}\left(Z_t^H, Z_{t+s}^H\right) = \frac{\alpha^{2H}}{2} \left\{ e^{-\lambda|s|} + \sum_{n=1}^{N} (-1)^{n-1} \binom{2H}{n} e^{-\lambda\left(\frac{n}{H}-1\right)|s|} \right\} + O\left(e^{-\lambda\left(\frac{N+1}{H}-1\right)|s|}\right)\,,
$$

as $s \to \infty$. This shows that for all $H \in (0,1)$, the auto-covariance function of $(Z_t^H)_{t\in\mathbb{R}}$ decays exponentially. It follows that for $H \in (0,\frac{1}{2})\cup(\frac{1}{2},1)$, $(Z_t^H)_{t\in\mathbb{R}}$ cannot have the same distribution as $(Y_t^H)_{t\in\mathbb{R}}$. For $H \in (0,\frac{1}{2})$, the leading term in (3.1) for $s \to \infty$, is

$$\frac{\alpha^{2H}}{2} e^{-\lambda|s|} \,,$$

whereas for $H \in (\frac{1}{2},1)$, it is

$$\alpha^{2H} H e^{-\lambda\left(\frac{1}{H}-1\right)|s|} \,.$$

Note that for $H \in (0,\frac{1}{2})$, the leading term of $\mathrm{Cov}\left(Z_t^H, Z_{t+s}^H\right)$ for $s \to \infty$, is positive, whereas the leading term of $\mathrm{Cov}\left(Y_t^H, Y_{t+s}^H\right)$ for $s \to \infty$, is negative (see Theorem 2.3).

Appendix: The Langevin equation

Langevin (1908) pioneered the following approach to the movement of a free particle immersed in a liquid: He described the particle's velocity v by the equation of motion

$$\frac{dv(t)}{dt} = -\frac{f}{m} v(t) + \frac{F(t)}{m} \tag{A.1}$$

where $m > 0$ is the mass of the particle, $f > 0$ a friction coefficient and $F(t)$ a fluctuating force resulting from impacts of the molecules of the surrounding medium. Uhlenbeck and Ornstein (1930) imposed probability hypotheses on $F(t)$ and then derived that for $v(0) = x \in \mathbb{R}$, $v(t)$ is normally distributed with mean $xe^{-\lambda t}$ and variance $\frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda t}\right)$, for $\lambda = \frac{f}{m}$ and $\sigma^2 = \frac{2fkT}{m^2}$, where k is the Boltzmann constant and T the temperature. Doob (1942) noticed that if $v(0)$ is a random variable which is independent of $(F(t))_{t\geq 0}$ and normally distributed with mean zero and variance $\frac{\sigma^2}{2\lambda}$, then the solution $(v(t))_{t\geq 0}$ of (A.1) is stationary and

$$1_{\{t>0\}} t^{\frac{1}{2}} v\left(\frac{1}{2\lambda}\ln t\right) \,, \quad t \geq 0 \,,$$

is a Brownian motion, from which he concluded that every solution of (A.1) has almost surely continuous paths which are nowhere differentiable. To avoid the "embarrassing situation" that the equation (A.1) involves the derivative of v but leads to solutions v that do not have a derivative, he gave a rigorous meaning to stochastic differential equations of the form

$$dX_t = -\lambda X_t dt + dN_t \,, \tag{A.2}$$

for the case that N is a Lévy process and showed that for all $x \in \mathbb{R}$, the equation (A.2) with initial condition $X_0 = x \in \mathbb{R}$, has the unique solution

$$X_t^x = e^{-\lambda t}\left(x + \int_0^t e^{\lambda u} dN_u\right) \,, \quad t \geq 0 \,.$$

In the modern theory of stochastic differential equations (see e.g. Protter (1990)) the equation (A.2) with initial condition $X_0 = \xi \in L^0(\Omega)$ is understood as the integral equation

$$X_t = \xi - \lambda \int_0^t X_s ds + N_t \,, \quad t \geq 0 \,, \tag{A.3}$$

and it can be shown that the unique strong solution of (A.3) is given by

$$X_t^\xi := e^{-\lambda t}\left(\xi + \int_0^t e^{\lambda u} dN_u\right), \quad t \geq 0,$$

whenever $(N_t)_{t\geq 0}$ is a semimartingale with respect to the filtration generated by $(N_t)_{t\geq 0}$ and ξ.

Proposition A.1 *Let $(B_t^H)_{t\in\mathbb{R}}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1]$ and $\xi \in L^0(\Omega)$. Let $-\infty \leq a < \infty$ and λ, $\sigma > 0$. Then for almost all $\omega \in \Omega$, we have the following:*
a) *For all $t > a$,*

$$\int_a^t e^{\lambda u} dB_u^H(\omega)$$

exists as a Riemann-Stieltjes integral and is equal to

$$e^{\lambda t} B_t^H(\omega) - e^{\lambda a} B_a^H(\omega) - \lambda \int_a^t B_u^H(\omega) e^{\lambda u} du\,.$$

b) *The function*

$$\int_a^t e^{\lambda u} dB_u^H(\omega)\,, \quad t > a\,,$$

is continuous in t.
c) *The unique continuous function y that solves the equation,*

$$y(t) = \xi(\omega) - \lambda \int_0^t y(s) ds + \sigma B_t^H(\omega)\,, \quad t \geq 0\,. \tag{A.4}$$

is given by

$$y(t) = e^{-\lambda t}\left\{\xi(\omega) + \sigma \int_0^t e^{\lambda u} dB_u^H(\omega)\right\}, \quad t \geq 0\,. \tag{A.5}$$

In particular, the unique continuous solution of the equation,

$$y(t) = \sigma \int_{-\infty}^0 e^{\lambda u} dB_u^H(\omega) - \lambda \int_0^t y(s) ds + \sigma B_t^H(\omega)\,, \quad t \geq 0\,,$$

is given by

$$y(t) = \sigma \int_{-\infty}^t e^{-\lambda(t-u)} dB_u^H(\omega)\,, \quad t \geq 0\,.$$

Proof. It can easily be checked that

$$\tilde{B}_s^H := 1_{\{s<0\}}(-s)^{2H} B_{\frac{1}{s}}^H + 1_{\{s>0\}} s^{2H} B_{\frac{1}{s}}^H\,, \quad s \in \mathbb{R}\,,$$

is again a fractional Brownian motion. It follows from the Kolmogorov-Čentsov theorem (see e.g. Theorem 2.2.8 of Karatzas and Shreve (1991)) that there exists a measurable null set

$N \subset \Omega$, such that for every $\omega \in \Omega \setminus N$, $B^H_s(\omega)$ and $\tilde{B}^H_s(\omega)$ are continuous in s, and for all $\beta < H$,

$$\lim_{s \to 0} \frac{\tilde{B}^H_s(\omega)}{|s|^\beta} = 0\,.$$

This implies that for all $\gamma > H$,

$$\lim_{|s| \to \infty} \frac{B^H_s(\omega)}{|s|^\gamma} = 0\,.$$

Hence, for all $t > a$,

$$\int_a^t B^H_u(\omega) e^{\lambda u} du$$

exists as a Riemann integral, which, by Theorem 2.21 of Wheeden and Zygmund (1977), implies that the Riemann-Stieltjes integral

$$\int_a^t e^{\lambda u} dB^H_u(\omega)$$

exists too and is equal to

$$e^{\lambda t} B^H_t(\omega) - e^{\lambda a} B^H_a(\omega) - \lambda \int_a^t B^H_u(\omega) e^{\lambda u} du\,.$$

This proves a).

b) follows from a) and the fact that the function

$$e^{\lambda t} B^H_t(\omega) - \lambda \int_a^t B^H_u(\omega) e^{\lambda u} du\,, \quad t > a\,,$$

is continuous in t.

A continuous function y solves (A.4) if and only if the function

$$z(t) = \int_0^t y(s) ds\,, \quad t \geq 0\,,$$

solves the linear differential equation:

$$z'(t) = -\lambda z(t) + \xi(\omega) + \sigma B^H_t(\omega)\,, \quad z(0) = 0\,. \tag{A.6}$$

Since the unique solution of (A.6) is given by

$$z(t) = e^{-\lambda t} \int_0^t e^{\lambda u} \left(\xi(\omega) + \sigma B^H_u(\omega) \right) du\,, \quad t \geq 0\,,$$

the unique continuous function y that solves (A.4) is given by

$$-\lambda e^{-\lambda t} \int_0^t e^{\lambda u} \left(\xi(\omega) + \sigma B^H_u(\omega) \right) du + \xi(\omega) + \sigma B^H_t(\omega)\,, \quad t \geq 0\,,$$

which, by a), is equal to (A.5). This shows c). □

Remark A.2 Equation (A.3) can be solved path-wise for all stochastic processes $(N_t)_{t\geq 0}$ that have almost all paths in

$$\mathcal{L}^1_{\text{loc}}(\mathbb{R}_+) := \left\{ h : \mathbb{R}_+ \to \mathbb{R} \,:\, h \text{ is measurable and } \forall t \geq 0\,,\ \int_0^t |h(s)|\, ds < \infty \right\}\,,$$

and even when the constant λ is replaced by a stochastic process with almost all paths in

$$\mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_+) := \left\{ g : \mathbb{R}_+ \to \mathbb{R} \,:\, g \text{ is measurable and } \forall t \geq 0\,,\ \sup_{0\leq s\leq t} |g(s)| < \infty \right\}\,.$$

Indeed, if $h \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}_+)$ and $g \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_+)$, then it can easily be checked that the function

$$f(t) := h(t) + \int_0^t g(s) e^{\int_s^t g(u)du} h(s) ds\,, \quad t \geq 0\,, \tag{A.7}$$

is in $\mathcal{L}^1_{\text{loc}}(\mathbb{R}_+)$ and solves the integral equation

$$f(t) = \int_0^t g(s) f(s) ds + h(t)\,, \quad t \geq 0\,. \tag{A.8}$$

On the other hand, if $\tilde{f} \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}_+)$ is a solution of (A.8), then

$$f(t) - \tilde{f}(t) = \int_0^t g(s) \left[f(t) - \tilde{f}(t) \right] ds\,, \quad t \geq 0\,,$$

and it follows from a variant of Gronwall's lemma that

$$f(t) - \tilde{f}(t) = 0\,, \quad t \geq 0\,.$$

Hence, (A.7) is the only function in $\mathcal{L}^1_{\text{loc}}(\mathbb{R}_+)$ that solves (A.8).

If the functions g and h are both in $\mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_+)$ and continuous on $\mathbb{R}_+ \setminus C$, where C is of Lebesgue measure zero, then it can be deduced from Theorems 5.54 and 2.21 of Wheeden and Zygmund (1977) that f can be written as follows:

$$f(t) = e^{\int_0^t g(u)du} \left(h(0) + \int_0^t e^{-\int_0^s g(u)du} dh(s) \right)\,, \quad t \geq 0\,,$$

where $\int_0^t e^{-\int_0^s g(u)du} dh(s)$ is a Riemann-Stieltjes integral.

Note that almost all paths of a semimartingale are right-continuous and have left limits, in particular, they are in $\mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_+)$ and have at most countably many discontinuities.

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