Gaussian moving averages, semimartingales and option pricing

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Abstract: We provide a characterization of the Gaussian processes with stationary increments that can be represented as a moving average with respect to a two-sided Brownian motion. For such a process we give a necessary and sufficient condition to be a semimartingale with respect to the filtration generated by the two-sided Brownian motion. Furthermore, we show that this condition implies that the process is either of finite variation or a multiple of a Brownian motion with respect to an equivalent probability measure. As an application we discuss the problem of option pricing in financial models driven by Gaussian moving averages with stationary increments. In particular, we derive option prices in a regularized fractional version of the Black–Scholes model.

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1 Introduction

Let $(\Omega, \mathcal{A}, P)$ be a probability space equipped with a two-sided Brownian motion $(W_t)_{t \in \mathbb{R}}$, that is, a continuous centred Gaussian process with covariance

$$\operatorname{Cov}(W_t, W_s) = \frac{1}{2}\left(|t| + |s| - |t-s|\right), \quad t, s \in \mathbb{R}.$$

For a function $\varphi : \mathbb{R} \to \mathbb{R}$ that is zero on the negative real axis and satisfies for all $t > 0$,

$$\varphi(t - .) - \varphi(-.) \in L^2(\mathbb{R}),$$

one can define the centred Gaussian process with stationary increments,

$$Y_t^{\varphi} = \int_{\infty}^{t} \left[\varphi(t-u) - \varphi(-u)\right] dW_u, \quad t \in \mathbb{R}. \tag{1.1}$$

The purpose of this paper is the study of processes of the form (1.1) with a view towards financial modelling.

If $(X_t)_{t \geq 0}$ is a stochastic process on $(\Omega, \mathcal{A}, P)$, we denote by $(\bar{\mathcal{F}}_t^X)_{t \geq 0}$ the smallest filtration that satisfies the usual assumptions and contains the filtration

$$\mathcal{F}_t^X := \sigma(X_s : 0 \leq s \leq t), \quad t \geq 0.$$

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By $(\bar{\mathcal{F}}^W_t)_{t\geq 0}$ we denote the smallest filtration that satisfies the usual assumptions and contains the filtration

$$\mathcal{F}^W_t := \sigma(W_s : -\infty < s \leq t)\,, \quad t \geq 0\,.$$

Since $(W_t)_{t\geq 0}$ is a strong Markov process, it follows from Proposition 2.7.7 of Karatzas and Shreve (1991) that

$$\bar{\mathcal{F}}^W_t = \sigma(\mathcal{F}^W_t, \mathcal{N})\,, \quad t \geq 0\,,$$

where

$$\mathcal{N} := \left\{ N \subset \Omega \, : \, N \subset M \text{ for some } M \in \sigma(\bigcup_{t>0} \mathcal{F}^W_t) \text{ with } P[M] = 0 \right\}.$$

The structure of the paper is as follows. In Section 2 we recall a result of Karhunen (1950), which gives necessary and sufficient conditions for a stationary centred Gaussian process to be representable in the form

$$\int_{\mathbb{R}} \xi(t-u)dW_u\,, \quad t \in \mathbb{R}\,, \tag{1.2}$$

where $\xi \in L^2(\mathbb{R})$. In Section 3 we give a characterization of those processes of the form (1.1) that are $\bar{\mathcal{F}}^W$-semimartingales and we show that they are either finite variation processes, or for every $T \in (0,\infty)$, there exists an equivalent probability measure under which $(Y^\varphi_t)_{t\in[0,T]}$ is a multiple of a Brownian motion. In Section 4 we apply a transformation introduced in Masani (1972) to establish a one-to-one correspondence between stationary centred Gaussian processes and centred Gaussian processes with stationary increments that are zero for $t = 0$. This allows us to extend Karhunen's result to centred Gaussian processes with stationary increments and to show that every process of the form (1.1) can be approximated by semimartingales of the form (1.1). By transferring the results from Section 3 back to the framework of stationary centred Gaussian processes, we obtain an extension of Theorem 6.5 of Knight (1992), which gives a necessary and sufficient condition for a process of the form (1.2) to be an $\bar{\mathcal{F}}^W$-semimartingale. In Section 5 we discuss the problem of option pricing in financial models driven by processes of the form (1.1). As an example we price a European call option in a regularized fractional Black–Scholes model.

2 Stationary Gaussian moving averages

Definition 2.1 *A stochastic process $(X_t)_{t\in\mathbb{R}}$ is stationary if for all $t_0 \in \mathbb{R}$,*

$$(X_{t+t_0})_{t\in\mathbb{R}} \overset{\text{(d)}}{=} (X_t)_{t\in\mathbb{R}}\,,$$

where $\overset{\text{(d)}}{=}$ denotes equality of all finite-dimensional distributions.

Definition 2.2 *By Φ_{S} we denote the set of functions $\xi \in L^2(\mathbb{R})$ such that $\xi(t) = 0$ for all $t < 0$.*

If $\xi \in \Phi_{\mathrm{S}}$, we can for all $t \in \mathbb{R}$, define

$$X_t^\xi := \int_{\mathbb{R}} \xi(t-u) dW_u$$

in the L^2-sense. It is clear that $(X_t^\xi)_{t\in\mathbb{R}}$ is a stationary centred Gaussian process. If possible, we choose a right-continuous version.

Example 2.3 Let $\xi(t) = 1_{[0,\infty)}(t) \exp(-\lambda t)\,,\ t \in \mathbb{R}$, for a $\lambda > 0$. Then, $\xi \in \Phi_{\mathrm{S}}$, and $(X_t^\xi)_{t\in\mathbb{R}}$ is a stationary Ornstein-Uhlenbeck process.

Remark 2.4 Let $\xi \in \Phi_{\mathrm{S}}$. It can be shown by approximating ξ with continuous functions with compact support, that

$$\lim_{t\to 0} \mathrm{E}\left[(X_t^\xi - X_0^\xi)^2\right] = \lim_{t\to 0} \int_{\mathbb{R}} (\xi(t-u) - \xi(-u))^2 du = 0\,.$$

Hence, $t \mapsto X_t^\xi$ is a continuous mapping from $\mathbb{R}$ to $L^2(\Omega)$. Moreover,

$$\bigcap_{t\in\mathbb{R}} \overline{\mathrm{sp}} \left\{X_s^\xi \,:\, -\infty < s \le t\right\} \subset \bigcap_{t\in\mathbb{R}} \overline{\mathrm{sp}} \left\{W_{s_2} - W_{s_1} \,:\, -\infty < s_1, s_2 \le t\right\} \subset \{0\}\ ,$$

where $\overline{\mathrm{sp}}$ denotes the L^2-closure of the linear span of a set of square-integrable random variables.

The following theorem follows from Satz 5 in Karhunen (1950).

Theorem 2.5 (Karhunen (1950))
Let $(X_t)_{t\in\mathbb{R}}$ be a stationary centred Gaussian process such that

$$\begin{array}{l} \lim_{t\to 0} \mathrm{E}\left[(X_t - X_0)^2\right] = 0 \quad \textit{and} \\ \bigcap_{t\in\mathbb{R}} \overline{\mathrm{sp}} \left\{X_s \,:\, -\infty < s \le t\right\} = \{0\}\ . \end{array}$$

Then there exists a $\xi \in \Phi_{\mathrm{S}}$ such that

$$(X_t)_{t\in\mathbb{R}} \overset{(\mathrm{d})}{=} (X_t^\xi)_{t\in\mathbb{R}}\,.$$

3 Gaussian moving averages with stationary increments

Definition 3.1 *We say that a stochastic process $(Y_t)_{t\in\mathbb{R}}$ has stationary increments if for all $t_0 \in \mathbb{R}$,*

$$(Y_{t+t_0} - Y_{t_0})_{t\in\mathbb{R}} \overset{(\mathrm{d})}{=} (Y_t - Y_0)_{t\in\mathbb{R}}\,.$$

Obviously, every stationary process has stationary increments.

Definition 3.2 *By Φ_{SI} we denote the set of all measurable functions $\varphi : \mathbb{R} \to \mathbb{R}$ such that*

$$\text{for all } t < 0 : \qquad \varphi(t) = 0 \quad \text{ and } \tag{3.1}$$

$$\text{for all } t > 0 : \qquad \int_{\mathbb{R}} [\varphi(t-u) - \varphi(-u)]^2 \, du < \infty \,. \tag{3.2}$$

By $\Phi_{\mathrm{SI}}^{\mathrm{I}}$ we denote the set of real-valued functions that can be written in the form

$$\varphi(t) = \begin{cases} v + \int_0^t \psi(s) ds \,, & t \geq 0 \\ 0 \,, & t < 0 \end{cases}$$

for a $v \in \mathbb{R}$ and a $\psi \in L^2(\mathbb{R}_+)$.

It can easily be checked that $\Phi_{\mathrm{S}} \subset \Phi_{\mathrm{SI}}$ and $\Phi_{\mathrm{SI}}^{\mathrm{I}} \subset \Phi_{\mathrm{SI}}$.

For $\varphi \in \Phi_{\mathrm{SI}}$ we define the centred Gaussian process with stationary increments

$$Y_t^{\varphi} := \int_{\mathbb{R}} [\varphi(t-u) - \varphi(-u)] \, dW_u \,, \quad t \in \mathbb{R} \,, \tag{3.3}$$

where we choose a right-continuous version whenever possible.

Examples 3.3

a) The function $\varphi(t) = 1_{[0,\infty)}(t)$ is in $\Phi_{\mathrm{SI}}^{\mathrm{I}}$, and $Y_t^{\varphi} = W_t$, $t \in \mathbb{R}$.

b) Let $H \in (0,1)$ and set $\varphi(t) = 1_{(0,\infty)}(t) t^{H-\frac{1}{2}}$, $t \in \mathbb{R}$. Then, $\varphi \in \Phi_{\mathrm{SI}} \setminus \Phi_{\mathrm{SI}}^{\mathrm{I}}$, and

$$\mathrm{Cov}\,(Y_t^{\varphi}, Y_s^{\varphi}) = c_H^2 \frac{1}{2} \left(|t|^{2H} + |s|^{2H} - |t-s|^{2H} \right) \,, \quad t, s \in \mathbb{R} \,,$$

where

$$c_H = \left(\frac{1}{2H} + \int_0^{\infty} \left[(1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}} \right] du \right)^{\frac{1}{2}} \,.$$

These processes were studied by Kolmogorov (1940) in a Hilbert space framework. Mandelbrot and Van Ness (1968) represented them in the form (3.3) and gave them the name ‘fractional Brownian motions’ (fBm). More information on fBm and further references can be found in Section 7.2 of Samorodnitsky and Taqqu (1994) or Chapter 4 of Embrechts and Maejima (2002).

Lemma 3.4 *Let $\varphi \in \Phi_{\mathrm{SI}}$. Then*

$$\lim_{t \to 0} \int_{\mathbb{R}} [\varphi(t-u) - \varphi(-u)]^2 \, du = 0 \,. \tag{3.4}$$

Proof. By condition (3.2),

$$\int_0^T \varphi^2(u)du + \int_0^\infty [\varphi(T+u) - \varphi(u)]^2 \, du = \int_{\mathbb{R}} [\varphi(T-u) - \varphi(-u)]^2 \, du < \infty \,,$$

for all $T > 0$. This shows that $1_{[0,T]}\varphi \in L^2(\mathbb{R})$ for all $T > 0$. Therefore, it can be shown by approximating $1_{[0,T]}\varphi$ with continuous functions with compact support that

$$\text{for all } T > 0, \quad \lim_{t \to 0} \int_{\mathbb{R}} \left[1_{[0,T]}(t+u)\varphi(t+u) - 1_{[0,T]}(u)\varphi(u)\right]^2 du = 0\,. \tag{3.5}$$

Now, assume that (3.4) does not hold. Then there exists a $c > 0$ such that

$$\limsup_{t \searrow 0} \int_{\mathbb{R}} [\varphi(t+u) - \varphi(u)]^2 \, du \geq 5c^2\,. \tag{3.6}$$

We set $t_0 := 0$ and $S_1 := -1$. It follows from (3.6) that there exists a $t_1 \in (0, \frac{1}{2}]$ and a $T_1 > S_1$ such that

$$\int_{S_1}^{T_1} [\varphi(t_1+u) - \varphi(u)]^2 \, du \geq 4c^2\,.$$

Since $\varphi \in \Phi_{\mathrm{SI}}$, the function $\varphi(t_1 + .) - \varphi(.)$ is in $L^2(\mathbb{R})$. Therefore, there exists an $S_2 \geq T_1$ such that

$$\int_{S_2}^{\infty} [\varphi(t_1+u) - \varphi(u)]^2 \, du \leq \frac{1}{4}c^2\,.$$

It follows from (3.5) and (3.6) that there exists a $t_2 \in (t_1, t_1 + \frac{1}{4}]$ and a $T_2 > S_2$ such that

$$\int_{S_1}^{S_2} [\varphi(t_2+u) - \varphi(t_1+u)]^2 \, du \leq \frac{1}{16}c^2$$

and

$$\int_{S_2}^{T_2} [\varphi(t_2+u) - \varphi(t_1+u)]^2 \, du \geq 4c^2\,.$$

Moreover, there exists an $S_3 \geq T_2$ such that

$$\int_{S_3}^{\infty} [\varphi(t_2+u) - \varphi(t_1+u)]^2 \, du \leq \frac{1}{16}c^2\,.$$

Continuing like this, one can inductively construct sequences of increasing numbers $\{t_n\}_{n=1}^\infty$, $\{S_n\}_{n=1}^\infty$ and $\{T_n\}_{n=1}^\infty$ such that for all $n \geq 1$, $t_n \in (t_{n-1}, t_{n-1} + 2^{-n}]$, $S_{n+1} \geq T_n > S_n$,

$$\int_{S_1}^{S_n} [\varphi(t_n+u) - \varphi(t_{n-1}+u)]^2 \, du \leq 4^{-n}c^2\,,$$

$$\int_{S_n}^{T_n} [\varphi(t_n+u) - \varphi(t_{n-1}+u)]^2 \, du \geq 4c^2 \quad \text{and}$$

$$\int_{S_{n+1}}^{\infty} [\varphi(t_n+u) - \varphi(t_{n-1}+u)]^2 \, du \leq 4^{-n}c^2\,.$$

We set $t := \lim_{n\to\infty} t_n \in (0,1]$. It follows that for all $n \geq 1$,

$$\begin{aligned} & ||\varphi(t+.) - \varphi(.)||_{L^2[S_n,T_n]} \\ \geq \; & ||\varphi(t_n+.) - \varphi(t_{n-1}+.)||_{L^2[S_n,T_n]} - \sum_{j\neq n} ||\varphi(t_j+.) - \varphi(t_{j-1}+.)||_{L^2[S_n,T_n]} \\ \geq \; & 2c - \sum_{j\neq n} 2^{-j} c \geq c\,. \end{aligned}$$

Hence,

$$||\varphi(t+.) - \varphi(.)||^2_{L^2(\mathbb{R})} \geq \sum_{n=1}^{\infty} ||\varphi(t+.) - \varphi(.)||^2_{L^2[S_n,T_n]} = \infty\,.$$

This contradicts (3.2). Hence, (3.6) cannot be true, and the lemma is proved. □

Proposition 3.5 *Let $\varphi \in \Phi_{\mathrm{SI}}$. Then*

(i) $\lim_{t\to 0} \mathrm{E}\left[(Y_t^\varphi)^2\right] = 0$ *and*
(ii) $\bigcap_{t\in\mathbb{R}} \overline{\mathrm{sp}}\,\{Y^\varphi_{s_2} - Y^\varphi_{s_1} \,:\, -\infty < s_1, s_2 \leq t\} = \{0\}$.

Proof. Property (i) follows immediately from Lemma 3.4. Property (ii) follows from

$$\begin{aligned} & \bigcap_{t\in\mathbb{R}} \overline{\mathrm{sp}}\left\{Y^\varphi_{s_2} - Y^\varphi_{s_1} \,:\, -\infty < s_1, s_2 \leq t\right\} \\ \subset \; & \bigcap_{t\in\mathbb{R}} \overline{\mathrm{sp}}\,\{W_{s_2} - W_{s_1} \,:\, -\infty < s_1, s_2 \leq t\} \subset \{0\}\,. \end{aligned}$$

□

For the proof of our main result, Theorem 3.9, we need the subsequent technical lemma and the following proposition.

Lemma 3.6 *Let $k \in L^2(\mathbb{R}^2)$ such that $||k||_2^2 < 1$. Then*

$$\mathrm{E}\left[\exp\left\{\frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} k(s,u)dW_u\right)^2 ds\right\}\right] \leq \frac{1}{\sqrt{1-||k||_2^2}}\,.$$

Proof. There exists a sequence $\{k_n\}_{n=1}^\infty \subset L^2(\mathbb{R}^2)$ such that

$$\lim_{n\to\infty} ||k - k_n||_2 = 0$$

and all k_n are of the form

$$k_n(s,u) = \sum_{j=-n^2}^{n^2-1} 1_{(\frac{j}{n},\frac{j+1}{n}]}(s) k_{n,j}(u)\,,$$

where

$$k_{n,j} \in L^2(\mathbb{R})\,, \quad j = -n^2, \ldots, n^2-1\,.$$

For all $n \in \mathbb{N}$ and $j = -n^2, \ldots, n^2-1$ we set

$$Z_j^n := \frac{1}{\sqrt{n}} \int_{\mathbb{R}} k_{n,j}(u)dW_u\,.$$

There exists an orthogonal $2n^2 \times 2n^2$-matrix U_n such that $V^n := U_n Z^n$ is a centred Gaussian vector with independent components. Therefore, for all $n \in \mathbb{N}$, such that $||k_n||_2 < 1$,

$$\begin{aligned}
&\mathrm{E}\left[\exp\left\{\frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} k_n(s,u)dW_u\right)^2 ds\right\}\right] = \mathrm{E}\left[\exp\left\{\frac{1}{2}\sum_{j=-n^2}^{n^2-1}\left(Z_j^n\right)^2\right\}\right] \\
= \;& \mathrm{E}\left[\exp\left\{\frac{1}{2}\sum_{j=-n^2}^{n^2-1}\left(V_j^n\right)^2\right\}\right] = \prod_{j=-n^2}^{n^2-1}\mathrm{E}\left[\exp\left\{\frac{1}{2}\left(V_j^n\right)^2\right\}\right] \\
= \;& \prod_{j=-n^2}^{n^2-1}\left(1-\mathrm{E}\left[\left(V_j^n\right)^2\right]\right)^{-\frac{1}{2}} \le \left(1-\sum_{j=-n^2}^{n^2-1}\mathrm{E}\left[\left(V_j^n\right)^2\right]\right)^{-\frac{1}{2}} \\
= \;& \left(1-\sum_{j=-n^2}^{n^2-1}\mathrm{E}\left[\left(Z_j^n\right)^2\right]\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-||k_n||_2^2}}\,.
\end{aligned} \tag{3.7}$$

Furthermore, it follows from

$$\lim_{n\to\infty}\mathrm{E}\left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\{k(s,u)-k_n(s,u)\}\,dW_u\right)^2 ds\right] = \lim_{n\to\infty}||k-k_n||_2^2 = 0\,,$$

that there exists a subsequence $(n_l)_{l=1}^{\infty}$ such that

$$\lim_{l\to\infty}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\{k(s,u)-k_{n_l}(s,u)\}\,dW_u\right)^2 ds = 0 \quad \text{almost surely}\,.$$

This implies that

$$\lim_{l\to\infty}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} k_{n_l}(s,u)dW_u\right)^2 ds = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} k(s,u)dW_u\right)^2 ds \quad \text{almost surely}\,.$$

Hence, it follows from Fatou's lemma and (3.7) that

$$\begin{aligned}
&\mathrm{E}\left[\exp\left\{\frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} k(s,u)dW_u\right)^2 ds\right\}\right] \\
\le \;& \liminf_{l\to\infty}\mathrm{E}\left[\exp\left\{\frac{1}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} k_{n_l}(s,u)dW_u\right)^2 ds\right\}\right] \\
\le \;& \liminf_{l\to\infty}\frac{1}{\sqrt{1-||k_{n_l}||_2^2}} = \frac{1}{\sqrt{1-||k||_2^2}}\,,
\end{aligned}$$

which concludes the proof. □

Proposition 3.7 *Let $k : \{(s,u) \in \mathbb{R}_+ \times \mathbb{R} : u \le s\} \to \mathbb{R}$, be a measurable function such that for all $t > 0$,*

$$\int_0^t \int_{-\infty}^s k^2(s,u)duds < \infty\,.$$

Then

$$\exp\left(\int_0^t \int_{-\infty}^s k(s,u)dW_u dW_s - \frac{1}{2}\int_0^t \left(\int_{-\infty}^s k(s,u)dW_u\right)^2 ds\right)\,, \quad t \ge 0\,,$$

is a martingale on $\left(\Omega, \left(\bar{\mathcal{F}}_t^W\right)_{t\ge 0}, P\right)$, and for all $T \in (0,\infty)$, the process

$$W_t^k := W_t - \int_0^t \int_{-\infty}^s k(s,u)dW_u ds\,, \quad t \in [0,T]\,,$$

is a Brownian motion on $\left(\Omega, \left(\bar{\mathcal{F}}_t^W\right)_{t\in[0,T]}, P_T^k\right)$, where

$$P_T^k := \exp\left(\int_0^T \int_{-\infty}^s k(s,u)dW_u dW_s - \frac{1}{2}\int_0^T \left(\int_{-\infty}^s k(s,u)dW_u\right)^2 ds\right)\cdot P\,,$$

that is, under P_T^k, the law of $(W_t^k)_{t\in[0,T]}$ is the Wiener measure, and for all $0 \le t_0 \le t_1 < t_2$, $W_{t_2}^k - W_{t_1}^k$ is independent of $\bar{\mathcal{F}}_{t_0}^W$.

Proof. The second claim follows from the first one by Girsanov's theorem. To prove the first claim it is enough to show that, for all $t > 0$,

$$\mathrm{E}\left[\exp\left(\int_0^t \int_{-\infty}^s k(s,u)dW_u dW_s - \frac{1}{2}\int_0^t \left(\int_{-\infty}^s k(s,u)dW_u\right)^2 ds\right)\right] = 1\,, \tag{3.8}$$

because

$$\exp\left(\int_0^t \int_{-\infty}^s k(s,u)dW_u dW_s - \frac{1}{2}\int_0^t \left(\int_{-\infty}^s k(s,u)dW_u\right)^2 ds\right), t \ge 0\,,$$

is a positive local martingale and therefore also a supermartingale. To prove (3.8), let $t > 0$. There exists an $n \in \mathbb{N}$ such that, for all $j = 1, \dots, n$,

$$\int_{\frac{j-1}{n}t}^{\frac{j}{n}t} \int_{-\infty}^s k^2(s,u)duds < 1\,.$$

Therefore, by Lemma 3.6, for all $j = 1, \dots, n$,

$$\mathrm{E}\left[\exp\left\{\frac{1}{2}\int_{\frac{j-1}{n}t}^{\frac{j}{n}t} \left(\int_{-\infty}^s k(s,u)dW_u\right)^2 ds\right\}\right] < \infty\,.$$

Hence, on all intervals $[\frac{j-1}{n}t, \frac{j}{n}t]$, $j = 1, \dots, n$, the Novikov condition is satisfied, which implies (3.8) (see e.g. Corollary 3.5.14 of Karatzas and Shreve (1991)). □

Remark 3.8 Proposition 3.7 is a generalization of Theorem 2 in Hitsuda (1968). Whereas our proof is based on Lemma 3.6 and the Novikov condition, Hitsuda's proof uses results from the theory of Volterra integral equations.

Theorem 3.9
a) *Let $\varphi \in \Phi^{\mathrm{I}}_{\mathrm{SI}}$. Then the corresponding process $(Y^\varphi_t)_{t\geq 0}$ is a continuous semimartingale on $(\Omega, (\bar{\mathcal{F}}^W_t)_{t\geq 0}, P)$ with canonical decomposition*

$$Y^\varphi_t = \varphi(0)W_t + \int_0^t \int_{-\infty}^s \psi(s-u)dW_u ds\,, \quad \textit{where} \;\; \varphi(t) = \varphi(0) + \int_0^t \psi(s)ds\,,\, t > 0\,. \quad (3.9)$$

In particular, $(Y^\varphi_t)_{t\geq 0}$ is a finite variation process if $\varphi(0) = 0$.
If $\varphi(0) \neq 0$, then for all $T \in (0,\infty)$,

$$Q^\varphi_T = \exp\left\{ -\int_0^T \int_{-\infty}^s \frac{\psi(s-u)}{\varphi(0)} dW_u dW_s - \frac{1}{2}\int_0^T \left(\int_{-\infty}^s \frac{\psi(s-u)}{\varphi(0)} dW_u \right)^2 ds \right\} \cdot P \quad (3.10)$$

is a probability measure on $(\Omega, \bar{\mathcal{F}}^W_T)$ and $(\frac{1}{\varphi(0)} Y^\varphi_t)_{t\in[0,T]}$ is a Brownian motion on $(\Omega, (\bar{\mathcal{F}}^W_t)_{t\in[0,T]}, Q^\varphi_T)$.
b) *Let $\varphi \in \Phi_{\mathrm{SI}}$. If there exists a $T \in (0,\infty)$ such that $(Y^\varphi_t)_{t\in[0,T]}$ is a semimartingale on $(\Omega, (\bar{\mathcal{F}}^W_t)_{t\in[0,T]}, P)$, then $\varphi \in \Phi^{\mathrm{I}}_{\mathrm{SI}}$.*

Proof.
a) If $\varphi \in \Phi^{\mathrm{I}}_{\mathrm{SI}}$ with $\varphi(t) = \varphi(0) + \int_0^t \psi(s)ds\,,\, t > 0$, then

$$\begin{aligned} Y^\varphi_t &= \int_{-\infty}^t [\varphi(t-u) - \varphi(-u)]\, dW_u \\ &= \int_{-\infty}^0 [\varphi(t-u) - \varphi(-u)]\, dW_u + \int_0^t \varphi(t-u) dW_u \\ &= \int_{-\infty}^0 \int_0^t \psi(s-u) ds dW_u + \int_0^t \left[\int_u^t \psi(s-u)ds + \varphi(0) \right] dW_u \end{aligned}$$

By the stochastic version of Fubini's theorem (see e.g. Theorem 146 on page 160 of Protter (1990)), we can change the order of integration. Hence, the above equals

$$\begin{aligned} & \int_0^t \int_{-\infty}^0 \psi(s-u)dW_u ds + \int_0^t \int_0^s \psi(s-u) dW_u ds + \varphi(0)W_t \\ = & \int_0^t \int_{-\infty}^s \psi(s-u)dW_u ds + \varphi(0)W_t\,, \end{aligned}$$

which proves (3.9) and shows that $(Y^\varphi_t)_{t\geq 0}$ is a continuous semimartingale on $(\Omega, (\bar{\mathcal{F}}^W_t)_{t\geq 0}, P)$ and a finite variation process if $\varphi(0) = 0$. The rest of statement a) follows from (3.9) and Proposition 3.7.

b) can be proved with the following argument borrowed from the proof of Proposition 15 in Jeulin and Yor (1993):

Let $T \in (0, \infty)$. By Théorème 1 of Stricker (1984), $(Y_t^{\varphi})_{t \in [0,T]}$ is a semimartingale on $(\Omega, \left(\bar{\mathcal{F}}_t^W\right)_{t \in [0,T]}, P)$ if and only if it is a quasimartingale on $(\Omega, \left(\bar{\mathcal{F}}_t^W\right)_{t \in [0,T]}, P)$. For $0 \le s < t$,

$$\mathrm{E}\left[X_t - X_s \,|\, \bar{\mathcal{F}}_s^W\right] = \int_{-\infty}^{s} \left[\varphi(t-u) - \varphi(s-u)\right] dW_u \,,$$

and therefore,

$$\mathrm{E}\left[\left|\mathrm{E}\left[X_t - X_s \,|\, \bar{\mathcal{F}}_s^W\right]\right|\right] = \left(\frac{2}{\pi} \int_{-\infty}^{s} \left[\varphi(t-u) - \varphi(s-u)\right]^2 du\right)^{\frac{1}{2}} .$$

Hence, $(Y_t^{\varphi})_{t \ge 0}$ is a quasimartingale on $(\Omega, \left(\bar{\mathcal{F}}_t^W\right)_{t \in [0,T]}, P)$ if and only if

$$\int_0^\infty \left[\varphi(t+u) - \varphi(u)\right]^2 du = O(t^2) \quad \text{as } t \searrow 0 \,. \tag{3.11}$$

If φ satisfies (3.11), then the sequence of functions $\{\varphi_n\}_{n=1}^\infty$ given by

$$\varphi_n(t) := n\left(\varphi(t + \frac{1}{n}) - \varphi(t)\right) \,,\ t \ge 0 \,,\ n \ge 1 \,,$$

is bounded in $L^2(\mathbb{R}_+)$ and therefore by Alaoglu's theorem relatively compact in the weak topology. Therefore there exists a subsequence $\{\varphi_{n_l}\}_{l=1}^\infty$ that converges weakly to a limit $\psi \in L^2(\mathbb{R}_+)$. Since there exists a set $N \subset \mathbb{R}_+$ of Lebesgue measure zero such that for all $t \in \mathbb{R}_+ \setminus N$,

$$\lim_{n \to \infty} n \int_t^{t+\frac{1}{n}} \varphi(u) du = \varphi(t) \,,$$

one obtains for all $t, s \in \mathbb{R}_+ \setminus N$, such that $s < t$,

$$\begin{aligned} \int_s^t \psi(u) du &= \lim_{l \to \infty} \int_s^t \varphi_{n_l}(u) du \\ = \quad \lim_{l \to \infty} n_l &\left(\int_t^{t+\frac{1}{n_l}} \varphi(u) du - \int_s^{s+\frac{1}{n_l}} \varphi(u) du\right) = \varphi(t) - \varphi(s) \,. \end{aligned}$$

Let

$$v := \lim_{t \searrow 0 \,;\, t \in \mathbb{R}_+ \setminus N} \varphi(t) \,.$$

Then, for all $t \in \mathbb{R}_+ \setminus N$,

$$\varphi(t) = v + \int_0^t \psi(s) ds \,,$$

which shows that $\varphi \in \Phi_{\mathrm{SI}}^{\mathrm{I}}$. $\square$

Further results on general Gaussian semimartingales, similar to Théorème 1 of Stricker (1984), which is used in the proof of Theorem 3.9 b), can be found in Jain and Monrad (1982), Emery (1982), Stricker (1983) and Galchouk (1984).

It follows from Theorem 3.9 a) by Stricker's theorem (see Théorème 3.1 in Stricker (1977) or Theorem 4 on page 45 of Protter (1990) for an alternative proof) that for $\varphi \in \Phi_{\mathrm{SI}}^{\mathrm{I}}$, the process $(Y_t^{\varphi})_{t\geq 0}$ is also a semimartingale in its own filtration. However, the following example, given in Cherny (2001), shows that the condition $\varphi \in \Phi_{\mathrm{SI}}^{\mathrm{I}}$ is only sufficient and not necessary for $(Y_t^{\varphi})_{t\geq 0}$ to be a semimartingale in its own filtration.

Example 3.10 (Cherny (2001)) The function

$$g(z) := \frac{2z-1}{2-z}$$

is analytic on $\{z \in \mathbb{C} : |z| < 2\}$. Define the sequence $\{a_n\}_{n=0}^{\infty}$ by

$$\sum_{n=0}^{\infty} a_n z^n = g(z)\,, \quad |z| < 2\,.$$

Since $g(\bar{z}) = \bar{g}(z)$, all a_n are real. Furthermore, the fact that g is continuous on the circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ implies that $\sum_{n=0}^{\infty} a_n^2 < \infty$. It follows that the function

$$\varphi(t) := \sum_{n=0}^{\infty} a_n 1_{[n,\infty)}(t)\,, \quad t \in \mathbb{R}\,,$$

is in $\Phi_{\mathrm{SI}} \setminus \Phi_{\mathrm{SI}}^{\mathrm{I}}$. It can easily be checked that for $t_1 < t_2 \leq t_3 < t_4$ such that $t_2 - t_1 \leq \frac{1}{2}$ and $t_4 - t_3 \leq \frac{1}{2}$, there exists a constant $c \in [0, \frac{1}{2}]$ and an $l \geq 1$, such that

$$\begin{aligned} & \operatorname{Cov}\left(Y_{t_2}^{\varphi} - Y_{t_1}^{\varphi}, Y_{t_4}^{\varphi} - Y_{t_3}^{\varphi}\right) \\ = \; & \int_{\mathbb{R}} [\varphi(t_2 - u) - \varphi(t_1 - u)]\,[\varphi(t_4 - u) - \varphi(t_3 - u)]\,du \\ = \; & c \sum_{n=0}^{\infty} a_n a_{n+l}\,. \end{aligned} \tag{3.12}$$

Let μ be the normalized uniform measure on S^1. Since $|g(z)| = 1$ for $z \in S^1$, we have for all $l \geq 1$,

$$\sum_{n=0}^{\infty} a_n a_{n+l} = \int_{S^1} \left(\sum_{n=0}^{\infty} a_n z^{n+l}\right)\left(\sum_{n=0}^{\infty} a_n z^{-n}\right) d\mu(z) = \int_{S^1} z^l \, |g(z)|^2 \, d\mu(z) = 0\,.$$

Hence, (3.12) is zero for all $t_1 < t_2 \leq t_3 < t_4$ such that $t_2 - t_1 \leq \frac{1}{2}$ and $t_4 - t_3 \leq \frac{1}{2}$. By bilinearity, (3.12) is also zero for arbitrary $t_1 < t_2 \leq t_3 < t_4$. This shows that $(Y_t^{\varphi})_{t\geq 0}$ is a centred Gaussian process with independent stationary increments and therefore a multiple of a Brownian motion. In particular, it is a semimartingale in its own filtration.

4 The Masani transformation

In this section we use results of Masani (1972) on the representation of helices in Hilbert spaces to prove an analogue of Theorem 2.5 for centred Gaussian processes with stationary increments and to show that every process of the form (3.3) can be approximated by Gaussian semimartingales of the same form. Furthermore, we translate Theorem 3.9 and Example 3.10 to the framework of stationary centred Gaussian processes, which will lead to an extension of Theorem 6.5 in Knight (1992).

Let H be a Hilbert space with scalar product $\langle ., . \rangle$. In Masani (1972) a mapping x from $\mathbb{R}$ to H is called a stationary curve if it is continuous with respect to the norm of H and for all $a, t, s \in \mathbb{R}$,

$$\langle x(t+a), x(s+a) \rangle = \langle x(t), x(s) \rangle \ .$$

A mapping $y : \mathbb{R} \to H$ is called a helix if it is continuous and for all $a, t_1, t_2, t_3, t_4 \in \mathbb{R}$,

$$\begin{aligned} & \langle y(t_2+a) - y(t_1+a), y(t_4+a) - y(t_3+a) \rangle \\ = \ & \langle y(t_2) - y(t_1), y(t_4) - y(t_3) \rangle \ . \end{aligned}$$

It can easily be checked that for a stationary curve x in H and $t \in \mathbb{R}$, the integral $\int_0^t x(u)du$ exists as a limit of Riemann sums and the curve

$$Fx(t) := x(t) - x(0) + \int_0^t x(u)du \,, \quad t \in \mathbb{R} \,, \tag{4.1}$$

is a helix. The following theorem is a consequence of Lemma 2.10 and Theorem 2.22 in Masani (1972).

Theorem 4.1 (Masani (1972)) *Let y be a helix in a Hilbert space H. Then for all $t \in \mathbb{R}$,*

$$x(t) = \int_{-\infty}^0 e^u \left[y(t) - y(t+u) \right] du$$

exists as limit of Riemann sums, and $(x(t))_{t \in \mathbb{R}}$ is the unique stationary curve in H such that

$$y(t) - y(0) = Fx(t) \,.$$

It follows from Theorem 4.1 that the map F given in (4.1) is a linear bijection from the space of stationary curves in a Hilbert space H to the space of helices y in H that satisfy $y(0) = 0$. We call it Masani transformation.

Let $\xi \in \Phi_{\mathrm{S}}$. Then $t \mapsto \xi(t - .)$ is a stationary curve in $L^2(\mathbb{R})$ which is isometric to the stationary curve $(X_t^\xi)_{t \in \mathbb{R}}$ in $L^2(\Omega)$. It can easily be checked that F maps the curve $(\xi(t-.)_{t \in \mathbb{R}}$ to the helix $(f\xi(t-.) - f\xi(-.))_{t \in \mathbb{R}}$, where the function $f\xi$ is given by

$$f\xi(t) := \begin{cases} \xi(t) + \int_0^t \xi(u)du \,, & t \geq 0 \\ 0 & t < 0 \end{cases} \ .$$

Obviously, $f\xi$ satisfies condition (3.1). On the other hand, the fact that $(f\xi(t-.) - f\xi(-.))_{t\in\mathbb{R}}$ is a helix in $L^2(\mathbb{R})$ implies that $f\xi$ also satisfies (3.2). Hence,

$$f\xi \in \Phi_{\mathrm{SI}} \quad \text{and} \quad FX^{\xi} = Y^{f\xi}\,.$$

If $\varphi \in \Phi_{\mathrm{SI}}$, then $(\varphi(t-.) - \varphi(-.))_{t\in\mathbb{R}}$ is a helix in $L^2(\mathbb{R})$ which is isometric to the helix $(Y_t^{\varphi})_{t\in\mathbb{R}}$ in $L^2(\Omega)$. The map F^{-1} takes $(\varphi(t-.) - \varphi(-.))_{t\in\mathbb{R}}$ to the stationary curve $(f^{-1}\varphi(t-.))_{t\in\mathbb{R}}$, where

$$f^{-1}\varphi(t) = \begin{cases} \varphi(t) - \int_0^t e^{-u}\varphi(t-u)du\,, & t \geq 0 \\ 0 & t < 0 \end{cases}\,.$$

Hence,

$$f^{-1}\varphi \in \Phi_{\mathrm{S}} \quad \text{and} \quad F^{-1}Y^{\varphi} = X^{f^{-1}\varphi}\,.$$

Theorem 4.2 *Let $(Y_t)_{t\in\mathbb{R}}$ be a centred Gaussian process with stationary increments that satisfies* (i) *and* (ii) *of Proposition 3.5. Then there exists a $\varphi \in \Phi_{\mathrm{SI}}$ such that*

$$(Y_t - Y_0)_{t\in\mathbb{R}} \stackrel{(\mathrm{d})}{=} (Y_t^{\varphi})_{t\in\mathbb{R}}\,.$$

Proof. The process $(Y_t)_{t\in\mathbb{R}}$ is a helix in $L^2(\Omega)$. It follows from Theorem 4.1 that the centred Gaussian process

$$X_t = \int_{-\infty}^0 e^u(Y_t - Y_{t+u})du\,, \quad t \in \mathbb{R}\,,$$

is a stationary curve in $L^2(\Omega)$, and for $-\infty < s < t < \infty$,

$$Y_t - Y_s = X_t - X_s + \int_s^t X_u du\,.$$

Since

$$\bigcap_{t\in\mathbb{R}} \overline{\mathrm{sp}}\,\{X_s \,:\, -\infty < s \leq t\} = \bigcap_{t\in\mathbb{R}} \overline{\mathrm{sp}}\,\{Y_{s_2} - Y_{s_1} \,:\, -\infty < s_1, s_2 \leq t\} = \{0\}\,,$$

it follows from Theorem 2.5 that there exists a $\xi \in \Phi_{\mathrm{S}}$, such that

$$(X_t)_{t\in\mathbb{R}} \stackrel{(\mathrm{d})}{=} (X_t^{\xi})_{t\in\mathbb{R}}\,.$$

This implies that

$$\begin{aligned} (Y_t - Y_0)_{t\in\mathbb{R}} &= (FX_t - FX_0)_{t\in\mathbb{R}} \\ \stackrel{(\mathrm{d})}{=} & (FX_t^{\xi} - FX_0^{\xi})_{t\in\mathbb{R}} = (Y_t^{\varphi} - Y_0^{\varphi})_{t\in\mathbb{R}}\,, \end{aligned}$$

where $\varphi = f\xi \in \Phi_{\mathrm{SI}}$, and the theorem is proved. □

In analogy to $\Phi_{\mathrm{SI}}^{\mathrm{I}}$ we define $\Phi_{\mathrm{S}}^{\mathrm{I}} := \Phi_{\mathrm{S}} \cap \Phi_{\mathrm{SI}}^{\mathrm{I}}$.

Remarks 4.3

1. For all $\xi \in \Phi_{\rm S}$, $v \in \mathbb{R}$ and $\varepsilon > 0$, there exists a continuously differentiable function $\eta : \mathbb{R}_+ \to \mathbb{R}$ with compact support in $\mathbb{R}_+$ such that $\eta(0) = v$ and

$$\left(\int_0^\infty [\eta(u) - \xi(u)]^2 \, du \right)^{\frac{1}{2}} \leq 1 \wedge \frac{\varepsilon}{2\,(||\xi||_2 + 1)} \,.$$

The function $\tilde{\xi}$ given by

$$\tilde{\xi}(t) = \begin{cases} \eta(t)\,, & t \geq 0 \\ 0\,, & t < 0 \end{cases} \,,$$

is in $\Phi_{\rm S}^{\rm I}$, and for all $t \in \mathbb{R}$,

$$||X_t^{\tilde{\xi}} - X_t^{\xi}||_2 = ||\tilde{\xi} - \xi||_2 \leq 1 \wedge \frac{\varepsilon}{2\,(1 + ||\xi||_2)} \,. \tag{4.2}$$

This implies that for all $t \in \mathbb{R}$,

$$||X_t^{\tilde{\xi}}||_2 \leq ||X_t^{\tilde{\xi}} - X_t^{\xi}||_2 + ||X_t^{\xi}||_2 \leq 1 + ||\xi||_2 \,. \tag{4.3}$$

It follows from (4.2) and (4.3) that for all $t, s \in \mathbb{R}$,

$$\begin{aligned} & \left| \mathrm{Cov}\left(X_t^{\tilde{\xi}}, X_s^{\tilde{\xi}} \right) - \mathrm{Cov}\left(X_t^{\xi}, X_s^{\xi} \right) \right| \\ = & \left| \mathrm{Cov}\left(X_t^{\tilde{\xi}}, X_s^{\tilde{\xi}} - X_s^{\xi} \right) + \mathrm{Cov}\left(X_t^{\tilde{\xi}} - X_t^{\xi}, X_s^{\xi} \right) \right| \\ \leq & \ ||X_t^{\tilde{\xi}}||_2 ||X_s^{\tilde{\xi}} - X_s^{\xi}||_2 + ||X_t^{\tilde{\xi}} - X_t^{\xi}||_2 ||X_s^{\xi}||_2 \leq \varepsilon \,. \end{aligned}$$

2. If $\xi \in \Phi_{\rm S}^{\rm I}$, then there exists a $v \in \mathbb{R}$ and a $\zeta \in \Phi_{\rm S}$ such that for all $t \geq 0$,

$$\xi(t) = v + \int_0^t \zeta(u) du \,.$$

It can easily be checked that for all $t \geq 0$,

$$f\xi(t) = v + \int_0^t \psi(u) du \,, \qquad \text{where} \qquad \psi = \xi + \zeta \,.$$

On the other hand, if $\varphi \in \Phi_{\rm SI}^{\rm I}$, then there exists a $v \in \mathbb{R}$ and a $\psi \in \Phi_{\rm S}$ such that for all $t \geq 0$,

$$\varphi(t) = v + \int_0^t \psi(u) du \,,$$

and for all $t \geq 0$,

$$f^{-1}\varphi(t) = v + \int_0^t \zeta(u) du \,, \qquad \text{where} \qquad \zeta(t) = f^{-1}(\psi - v 1_{[0,\infty)}) \,.$$

This shows that

$$f(\Phi_{\rm S}^{\rm I}) = \Phi_{\rm SI}^{\rm I} \,. \tag{4.4}$$

Proposition 4.4 *Let $\varphi \in \Phi_{\mathrm{SI}}$, $v \in \mathbb{R}$ and $\delta, T, \varepsilon > 0$. Then there exists a $\tilde{\varphi} \in \Phi_{\mathrm{SI}}^{\mathrm{I}}$ such that $\tilde{\varphi}(0) = v$,*

$$\sup_{t \in \mathbb{R}} ||Y_t^{\tilde{\varphi}} - Y_t^{\varphi}||_2 \leq \delta \tag{4.5}$$

and

$$\sup_{t,s \in [-T,T]} \left| \operatorname{Cov}\left(Y_t^{\tilde{\varphi}}, Y_s^{\tilde{\varphi}}\right) - \operatorname{Cov}\left(Y_t^{\varphi}, Y_s^{\varphi}\right) \right| \leq \varepsilon\,. \tag{4.6}$$

Proof. Let

$$N := \sup_{t \in [-T,T]} ||Y_t^{\varphi}||_2 \quad \text{and} \quad \tilde{\delta} := \delta \wedge \frac{\varepsilon}{2(\delta + N)}\,. \tag{4.7}$$

We set $\xi := f^{-1}\varphi \in \Phi_{\mathrm{S}}$. By Remark 4.3.1, there exists a $\tilde{\xi} \in \Phi_{\mathrm{S}}^{\mathrm{I}}$ such that $\tilde{\xi}(0) = v$ and $||\tilde{\xi} - \xi||_2 \leq \frac{\tilde{\delta}}{2}$. Obviously, the function

$$\tilde{\varphi}(t) := \begin{cases} \tilde{\xi} + \int_0^t \xi(s)ds\,, & t \geq 0 \\ 0\,, & t < 0 \end{cases}\,,$$

belongs to $\Phi_{\mathrm{SI}}^{\mathrm{I}}$, $\tilde{\varphi}(0) = v$, and

$$||\tilde{\varphi} - \varphi||_2 = ||\tilde{\varphi} - f\xi||_2 = ||\tilde{\xi} - \xi||_2 \leq \frac{\tilde{\delta}}{2}\,.$$

This implies that for all $t \in \mathbb{R}$,

$$\begin{aligned} & ||Y_t^{\tilde{\varphi}} - Y_t^{\varphi}||_2 = ||\tilde{\varphi}(t - .) - \tilde{\varphi}(-.) - \varphi(t - .) + \varphi(-.)||_2 \\ \leq\ & ||\tilde{\varphi}(t - .) - \varphi(t - .)||_2 + ||\varphi(-.) - \tilde{\varphi}(-.)||_2 \leq \tilde{\delta}\,. \end{aligned}$$

By (4.7), this proves (4.5) and implies that for all $t \in [-T,T]$,

$$||Y_t^{\tilde{\varphi}}||_2 \leq ||Y_t^{\tilde{\varphi}} - Y_t^{\varphi}||_2 + ||Y_t^{\varphi}||_2 \leq \delta + N\,.$$

Hence, for all $t, s \in [-T,T]$,

$$\begin{aligned} & \left| \operatorname{Cov}\left(Y_t^{\tilde{\varphi}}, Y_s^{\tilde{\varphi}}\right) - \operatorname{Cov}\left(Y_t^{\varphi}, Y_s^{\varphi}\right) \right| \\ =\ & \left| \operatorname{Cov}\left(Y_t^{\tilde{\varphi}}, Y_s^{\tilde{\varphi}} - Y_s^{\varphi}\right) + \operatorname{Cov}\left(Y_t^{\tilde{\varphi}} - Y_t^{\varphi}, Y_s^{\varphi}\right) \right| \\ \leq\ & ||Y_t^{\tilde{\varphi}}||_2 ||Y_s^{\tilde{\varphi}} - Y_s^{\varphi}||_2 + ||Y_t^{\tilde{\varphi}} - Y_t^{\varphi}||_2 ||Y_s^{\varphi}||_2 \leq \varepsilon\,, \end{aligned}$$

which proves (4.6). □

The Masani transformation also allows us to derive from Theorem 3.9 the following extension of Theorem 6.5 in Knight (1992).

Theorem 4.5
a) *Let* $\xi \in \Phi_{\mathrm{S}}^{\mathrm{I}}$. *Then for all* $t \geq 0$,

$$X_t^\xi - X_0^\xi = \xi(0)W_t + \int_0^t \int_{-\infty}^s \zeta(s-u)dW_u ds\,, \quad \textit{where} \;\; \xi(t) = \xi(0) + \int_0^t \zeta(s)ds\,, \, t > 0\,.$$

In particular, $(X_t^\xi)_{t\geq 0}$ *is a finite variation process if* $\xi(0) = 0$.
If $\xi(0) \neq 0$, *then for all* $T \in (0,\infty)$, *the process* $(\frac{1}{\xi(0)}(X_t^\xi - X_0^\xi))_{t\in[0,T]}$ *is a Brownian motion on* $(\Omega, (\bar{\mathcal{F}}_t^W)_{t\in[0,T]}, Q_T^\xi)$, *and* $(X_t^\xi)_{t\in[0,T]}$ *is an Ornstein-Uhlenbeck process on* $(\Omega, (\bar{\mathcal{F}}_t^W)_{t\in[0,T]}, Q_T^{f\xi})$, *where* Q_T^ξ *and* $Q_T^{f\xi}$ *are defined as in (3.10).*
b) *Let* $\xi \in \Phi_{\mathrm{S}}$. *If there exists a* $T \in (0,\infty)$ *such that* $(X_t^\xi)_{t\in[0,T]}$ *is a semimartingale on* $(\Omega, (\bar{\mathcal{F}}_t^W)_{t\in[0,T]}, P)$, *then* $\xi \in \Phi_{\mathrm{S}}^{\mathrm{I}}$.

Proof.
a) If $\xi \in \Phi_{\mathrm{S}}^{\mathrm{I}}$, then $\xi \in \Phi_{\mathrm{SI}}^{\mathrm{I}}$ as well. Therefore, all statements of a) except the last one follow immediately from Theorem 3.9 a). To prove the last statement of a), we first note that by (4.4), $f\xi \in \Phi_{\mathrm{SI}}^{\mathrm{I}}$. It follows from Theorem 3.9 a) that for all $T \in (0,\infty)$, $(\frac{1}{f\xi(0)} Y_t^{f\xi})_{t\in[0,T]}$ is a Brownian motion on $(\Omega, (\bar{\mathcal{F}}_t^W)_{t\in[0,T]}, Q_T^{f\xi})$. By (4.1), $(X_t^\xi)_{t\in[0,T]}$ solves the stochastic differential equation

$$dX_t^\xi = -X_t^\xi dt + dY_t^{f\xi}\,,$$

which show that it is an Ornstein-Uhlenbeck process on $(\Omega, (\bar{\mathcal{F}}_t^W)_{t\in[0,T]}, Q_T^{f\xi})$.
b) Let $\xi \in \Phi_{\mathrm{S}} \subset \Phi_{\mathrm{SI}}$. If the process $(X_t^\xi)_{t\geq 0}$ is an $\bar{\mathcal{F}}^W$-semimartingale, then so is the process

$$Y_t^\xi = X_t^\xi - X_0^\xi\,, \quad t \geq 0\,.$$

Therefore, it follows from Theorem 3.9 b) that $\xi \in \Phi_{\mathrm{SI}}^{\mathrm{I}} \cap \Phi_{\mathrm{S}} = \Phi_{\mathrm{S}}^{\mathrm{I}}$. □

Examples 4.6
a) Let φ be the function from Example 3.10. Then, the function

$$\xi(t) := f^{-1}\varphi(t) = \begin{cases} \varphi(t) - \int_0^t e^{-u}\varphi(t-u)du\,, & t \geq 0 \\ 0 & t < 0 \end{cases}\,,$$

is in $\Phi_{\mathrm{S}} \setminus \Phi_{\mathrm{S}}^{\mathrm{I}}$. However, by (4.1), the process $(X_t^\xi)_{t\geq 0}$ solves the stochastic differential equation

$$dX_t^\xi = -X_t^\xi dt + dY_t^\varphi\,,$$

and $(Y_t^\varphi)_{t\in\mathbb{R}}$ is a multiple of a Brownian motion. Therefore, $(X_t^\xi)_{t\in\mathbb{R}}$ is a stationary Ornstein-Uhlenbeck process. In particular, it is a semimartingale in its own filtration.
b) Let $\xi(t) = 1_{[0,1]}(t) \in \Phi_{\mathrm{S}} \setminus \Phi_{\mathrm{S}}^{\mathrm{I}}$. Then,

$$X_t^\xi = W_t - W_{t-1}\,, \quad t \in \mathbb{R}\,.$$

Fix $T \in (0, \infty)$. It can be checked by calculation that $(X_t^\xi)_{t\in[0,T]}$ is a quasimartingale on $(\Omega, (\bar{\mathcal{F}}_t^{X^\xi})_{t\in[0,T]}, P)$ if and only if $T \in [0,1]$. Therefore, it follows from Théorème 1 of Stricker (1984) that $(X_t^\xi)_{t\in[0,T]}$ is a semimartingale on $(\Omega, (\bar{\mathcal{F}}_t^{X^\xi})_{t\in[0,T]}, P)$ if and only if $T \in [0,1]$.

5 Option pricing

Let us consider a financial market consisting of two securities whose prices evolve according to

$$\left.\begin{array}{l} S_t^0 = \exp(r(t)) \quad \text{and} \\ S_t = S_0 \exp\left(r(t) + \nu(t) + Y_t^\varphi\right) \end{array}\right\}, \quad t \in [0,T], \tag{5.1}$$

for $T, S_0 > 0$, $r, \nu \in C^1[0,T]$ such that $r(0) = \nu(0) = 0$ and $\varphi \in \Phi_{\mathrm{SI}}$. We assume that it is possible to trade continuously in time, that short-selling is allowed and that there exist no transaction costs. But trading strategies must be adapted to the filtration $(\bar{\mathcal{F}}_t^S)_{t\in[0,T]}$.

If $\varphi(t) = \sigma 1_{[0,\infty)}(t)$, for a positive constant σ then, by Example 3.3 a), $Y_t^\varphi = \sigma W_t$, $t \in \mathbb{R}$. If in addition, the functions r and ν are of the form

$$r(t) = Rt \quad \text{and } \nu(t) = Nt,$$

for constants R and N, then (5.1) is the standard Black–Scholes model (see Black and Scholes (1973)), also called Samuelson model (see Samuelson (1965)). Under the P-equivalent probability measure

$$Q := \exp\left\{-\left(\frac{N}{\sigma} + \frac{\sigma}{2}\right) W_T - \frac{1}{2}\left(\frac{N}{\sigma} + \frac{\sigma}{2}\right)^2 T\right\} \cdot P,$$

the process

$$W_t^Q := W_t + \left(\frac{N}{\sigma} + \frac{\sigma}{2}\right) t, \quad t \in [0,T],$$

is a Brownian motion, and the discounted price $\tilde{S} := S/S^0$ can be written as

$$\tilde{S}_t = S_0 \exp\left\{\sigma W_t^Q - \frac{1}{2}\sigma^2 t\right\}, \quad t \in [0,T].$$

Since $\tilde{S}$ is a martingale under Q, the Black–Scholes model is arbitrage-free. From the fact that Brownian motion has the predictable representation property (see e.g. Theorem V.3.4 in Revuz and Yor (1999)) it can be deduced that it is also complete, that is, every contingent claim with a time T pay-off that is given by a non-negative random variable $C \in L^1(\Omega, \bar{\mathcal{F}}_T^{Y^\varphi}, Q)$ can be replicated by trading in S^0 and S, and its unique fair time 0 price is given by

$$\mathrm{E}_Q\left[e^{-RT} C\right].$$

In particular, the time 0 value of a European call option with maturity T and strike price K, whose time T pay-off is given by $(S_T - K)^+$, is

$$\mathrm{BS}(\sigma, S_0, T, R, K) := \mathrm{E}\left[\left(S_0 \exp\left\{\sigma\sqrt{T}Z - \frac{1}{2}\sigma^2 T\right\} - e^{-RT}K\right)^+\right],$$

where Z is a standard normal random variable.

For general functions $r, \nu \in C^1[0,T]$ and $\varphi \in \Phi_{\mathrm{S}}$, we distinguish between the following three cases:

(i) $\varphi \in \Phi_{\mathrm{SI}} \setminus \Phi^{\mathrm{I}}_{\mathrm{SI}}$:
By Theorem 3.9 b), the process $(Y^\varphi_t)_{t\in[0,T]}$ and therefore, also the discounted price

$$\tilde{S}_t = \frac{S_t}{S^0_t}, \quad t \in [0,T],$$

is not a semimartingale on $(\Omega, (\bar{\mathcal{F}}^W_t)_{t\in[0,T]}, P)$. Therefore, it follows immediately from Theorem 7.2 of Delbaen and Schachermayer (1994) that there exists a free lunch with vanishing risk consisting of a sequence of simple $\bar{\mathcal{F}}^W$-predictable integrands.

(ii) $\varphi \in \Phi^{\mathrm{I}}_{\mathrm{SI}}$ and $\varphi(0) = 0$:
It follows from Theorem 3.9 a) that $(Y^\varphi_t)_{t\in[0,T]}$ is a finite variation process. Therefore there exist $\bar{\mathcal{F}}^{Y^\varphi}$-predictable arbitrage strategies (see Harrison, Pitbladdo and Schaefer (1984) or Section 4 of Cheridito (2002)).

(iii) $\varphi \in \Phi^{\mathrm{I}}_{\mathrm{SI}}$ and $\varphi(0) \neq 0$:
By Theorem 3.9 a), there exists an equivalent probability measure $Q^\varphi_T \sim P$ under which the process

$$B_t := \frac{1}{\varphi(0)} Y^\varphi_t, \quad t \in [0,T],$$

is a Brownian motion. By Girsanov's theorem, the process

$$B^Q_t := B_t + \frac{\nu(t)}{\varphi(0)} + \frac{\varphi(0)}{2} t, \quad t \in [0,T],$$

is a Brownian motion under the probability measure

$$Q := \exp\left\{-\int_0^T \left(\frac{\nu'(u)}{\varphi(0)} + \frac{\varphi(0)}{2}\right) dB_u - \frac{1}{2}\int_0^T \left(\frac{\nu'(u)}{\varphi(0)} + \frac{\varphi(0)}{2}\right)^2 du\right\} \cdot P^\varphi,$$

and

$$\tilde{S}_t = \frac{S_t}{S^0_t} = S_0 \exp\left\{\varphi(0) B^Q_t - \frac{1}{2}\varphi^2(0) t\right\}, \quad t \in [0,T].$$

Hence, exactly the same arguments that show that the standard Black–Scholes model is arbitrage-free and complete can be used to prove that the same is true for the model (5.1).

In particular, the unique fair price of a European call option with maturity T and strike price K is given by

$$\mathrm{BS}(\varphi(0), S_0, T, \frac{r(T)}{T}, K)\,. \tag{5.2}$$

If φ is of the form (i) or (ii), then it can easily be regularized: Choose an arbitrary volatility $v > 0$. By Proposition 4.4, there exists for all $\varepsilon > 0$ a function $\tilde{\varphi}$ of the form (iii) such that $\tilde{\varphi}(0) = v$ and

$$\sup_{t,s\in[0,T]} \left|\mathrm{Cov}\left(Y_t^{\tilde{\varphi}}, Y_s^{\tilde{\varphi}}\right) - \mathrm{Cov}\left(Y_t^{\varphi}, Y_s^{\varphi}\right)\right| \leq \varepsilon\,.$$

Remarks 5.1
1. Let $\varphi \in \Phi^{\mathrm{I}}_{\mathrm{SI}}$ with $\varphi(0) \neq 0$. Obviously, the distribution of the process $(Y_t^{\varphi})_{t\in[0,T]}$ depends on the whole function φ. On the other hand, the option price (5.2) depends only on $\varphi(0)$. The reason for this is that the option price given by (5.2) is the minimal amount of initial wealth needed to replicate the option's pay-off with a trading strategy that can be adjusted continuously in time, and it can be seen from (3.9) that the volatility of the model (5.1) is given by $\varphi(0)$.
2. By replacing the function $\varphi \in \Phi_{\mathrm{SI}}$ in the representation (3.3) by a suitable stochastic process $(\varphi_t)_{t\in[0,T]}$ with values in Φ_{SI}, it should be possible to extend models of the form (5.1) to models with stochastic volatility.

Example 5.2 (Regularized fractional Black–Scholes model)
Let

$$\varphi(t) = \frac{\sigma}{c_H} 1_{(0,\infty)}(t)\, t^{H-\frac{1}{2}} \tag{5.3}$$

for a positive constant σ, $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and c_H as in Example 3.3 b). Then the process $(Y_t^{\varphi})_{t\in\mathbb{R}}$ is equal to $(\sigma B_t^H)_{t\in\mathbb{R}}$, where $(B_t^H)_{t\in\mathbb{R}}$ is a standard fBm, and the corresponding model (5.1) is a fractional version of the Black–Scholes model. For a discussion of the empirical evidence of correlation in stock price returns see e.g Cutland, Kopp and Willinger (1995) or Willinger, Taqqu and Teverovsky (1999) and the references therein. In Klüppelberg and Kühn (2002) fractional asset price models are motivated by a demonstration that fBm can be seen as a limit of Poisson shot noise processes. However, it follows from Theorem 3.9 b) that $(B_t^H)_{t\in[0,T]}$ is not a semimartingale with respect to the filtration $(\bar{\mathcal{F}}_t^W)_{t\in[0,T]}$, and it is well known that it is not a semimartingale in its own filtration either (for a proof in the case $H \in (\frac{1}{2}, 1)$ see Example 4.9.2 in Liptser and Shiryaev (1989), for a general proof see Maheswaran and Sims (1993) or Rogers (1997)). It follows from Theorem 7.2 in Delbaen and Schachermayer (1994) that there exists a free lunch with vanishing risk consisting of simple $\bar{\mathcal{F}}^S$-predictable trading strategies. An early discussion about the existence of arbitrage in fBm models can be found in Maheswaran and Sims (1993). In Rogers (1997) an arbitrage for a linear fBm model is constructed, and it is shown that fBm can be turned into a semimartingale by modifying the function φ near zero. The arbitrage strategies given in Shiryaev (1998) and Salopek (1998) work for linear and exponential fBm models with $H \in (\frac{1}{2}, 1)$. In Cheridito (2003) arbitrage for linear and exponential fBm models is constructed for all $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$.

To regularize the fractional Black–Scholes model, we can modify the function (5.3) as follows: For $v > 0$ and $d > 0$, define

$$\varphi^{v,d}(t) := \begin{cases} v + \frac{\varphi(d) - v}{d} t & \text{if } t \in [0, d] \\ \varphi(t) & \text{if } t \in (-\infty, 0) \cup (d, \infty) \end{cases} .$$

It is clear that for given $v > 0$,

$$\lim_{d \searrow 0} ||\varphi^{v,d} - \varphi||_2 = 0 .$$

Hence, it can be shown as in the proof of Proposition 4.4 that for all $\varepsilon > 0$ there exists a $d > 0$ such that

$$\sup_{t,s \in [0,T]} \left| \operatorname{Cov}\left(Y_t^{\varphi^{v,d}}, Y_s^{\varphi^{v,d}}\right) - \operatorname{Cov}\left(Y_t^{\varphi}, Y_s^{\varphi}\right) \right| \leq \varepsilon .$$

On the other hand, since the function $\varphi^{v,d}$ is of the form (iii), the corresponding model (5.1) is arbitrage-free and complete, and the price of a European call option is given by (5.2).

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